



Characterization of a New Family of Distribution Through Upper Record Values

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Abstract. In this paper, a new class of distribution has been characterized through the conditional expectations, conditioned on a non- adjacent upper record value. Also an equivalence between the unconditional and conditional expectation is used to characterize the new class of distribution.

1 Introduction

An observation is called a record if its value is greater than (or analogously, less than) all the preceding observations. Records arise naturally in many fields of studies such as climatology, sports, science, engineering, medicine, traffic and industry among others. The development of the general theory of statistical analysis of record values began with the work of Chandler [4]. For more details and applications, see Arnold et al. [2], Ahsanullah [6] and Nevzorov [10].

Suppose $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (*iid*) random variables (*r v*) with distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x)$. Let $Y_n = \max(\min)\{X_j | 1 \leq j \leq n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of $\{X_n | n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times. $\{U(n), n \geq 1\}$, where $U(n) = \min\{j | j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. The record times of the sequence $(X_n)_{n \geq 1}$ are the same as those for the sequence $(F(X_n))_{n \geq 1}$.

The joint *pdf* $f(x_1, x_2, \dots, x_r)$ of the r record values $X_{U(1)}, X_{U(2)}, \dots, X_{U(r)}$ is given by

$$f(x_1, x_2, \dots, x_r) = \prod_{i=1}^{r-1} f(x_i) \frac{f(x_i)}{\overline{F}(x_i)}, \quad -\infty < x_1 < x_2 < \dots < x_r < \infty, \quad (1.1)$$

where $\overline{F}(x) = 1 - F(x)$.

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The joint *pdf* of $X_{U(r)}$ and $X_{U(s)}$ is

$$f_{U(r), U(s)}(x, y) = \frac{1}{\Gamma(r)\Gamma(s-r)} [-\ln(\bar{F}(x))]^{r-1} [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-1} \\ \times \frac{f(x)}{\bar{F}(x)} f(y), \quad -\infty < x_r < x_s < \infty \quad (1.2)$$

The conditional *pdf* of $X_{U(s)}$ given $X_{U(r)} = x$, $1 \leq r < s$ is

$$f(X_{U(s)}|X_{U(r)} = x) = \frac{1}{\Gamma(s-r)} [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-1} \frac{f(y)}{\bar{F}(x)} \quad (1.3)$$

Many authors utilized the concepts of record values in their works. Lee [8] characterized the exponential distribution by conditional expectations of record values. Lee et al. [9] characterized the exponential distribution by order statistics and conditional expectations of record values. Athar et al. [3] characterized the distribution through the linear regression of record values and order statistics. Characterization of continuous distribution through record values presented by Khan et al. [1] and Khan and Khan [5] characterized the modified Makeham distribution through the generalized upper record values among others.

A random variable X is said to have shape- scale family (Maswadah and Faheem, [7]) if its *pdf* is of the form

$$f(x) = \alpha\beta g^{\alpha-1}(x)g'(x)\exp(-\beta g^\alpha(x)), \quad \alpha, \beta, x > 0 \quad (1.4)$$

with the corresponding *df*

$$F(x) = 1 - \exp(-\beta g^\alpha(x)), \quad \alpha, \beta, x > 0. \quad (1.5)$$

The parameters α and β are shape and scale respectively.

For convenience we assume $g(x)$ to be differentiable as well as strictly increasing function of x , $g(0^+) = 0$ and $g(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Note that $f(x)$ and $F(x)$ satisfy the relation,

$$f(x) = \alpha\beta g^{\alpha-1}(x)g'(x)\bar{F}(x). \quad (1.6)$$

This family includes among others the most popular parametric models in lifetime distributions such as the Weibull extension model, modified Weibull model, Weibull distribution, Pareto distribution, Burr-type XII distribution, Lomax distribution and the Generalized Pareto distribution according to the values of $g^\alpha(x)$. Some important members of this family are shown in Table 1.

2 Characterization Theorems

Theorem 2.1. *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \leq F(x) \leq 1$ for all $x > 0$. Then for $1 \leq r < s$*

$$E [g^\alpha(X_{L(s)}|X_{L(r)} = x)] = g^\alpha(x) + \frac{s-r}{\beta} \tag{2.1}$$

if and only if

$$F(x) = 1 - \exp(-\beta g^\alpha(x)), \quad \alpha, \beta, x > 0. \tag{2.2}$$

Proof. From (1.3), we have

$$\begin{aligned} E [g^\alpha(X_{L(s)}|X_{L(r)} = x)] &= \frac{1}{\Gamma(s-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \\ &= \frac{1}{\Gamma(s-r)} \int_x^\infty g^\alpha(y) [\beta g^\alpha(y) - \beta g^\alpha(x)]^{s-r-1} \alpha \beta g^{\alpha-1}(y) g'(y) e^{-[\beta g^\alpha(y) - \beta g^\alpha(x)]} dy. \end{aligned} \tag{2.3}$$

Let

$$[\beta g^\alpha(y) - \beta g^\alpha(x)] = t.$$

Then (2.3) reduces to:

$$E [g^\alpha(X_{L(s)}|X_{L(r)} = x)] = \frac{1}{\Gamma(s-r)} \int_0^\infty [g^\alpha(x) + \frac{t}{\beta}] [t]^{s-r-1} e^{-t} dt. \tag{2.4}$$

After simplification (2.4), we get

$$E [g^\alpha(X_{L(s)}|X_{L(r)} = x)] = g^\alpha(x) + \frac{s-r}{\beta}. \tag{2.5}$$

This proves the necessary part.

For the sufficiency part, we consider:

$$g_{s|r}(x) = g^\alpha(x) + \frac{s-r}{\beta}$$

$$\frac{1}{\Gamma(s-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-1} f(y) dy = g_{s|r}(x) \bar{F}(x). \tag{2.6}$$

Differentiating (2.6) both sides with respect to x

$$-\frac{1}{\Gamma(s-r-1)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-2} \frac{f(x)}{\bar{F}(x)} f(y) dy$$

$$\begin{aligned}
 &= g'_{s|r}(x) \bar{F}(x) - f(x)g_{s|r}(x) \\
 -g_{s|r+1}(x) &= g'_{s|r}(x) \frac{\bar{F}(x)}{f(x)} - g_{s|r}(x) \\
 g'_{s|r}(x) \frac{\bar{F}(x)}{f(x)} &= g_{s|r}(x) - g_{s|r+1}(x) \\
 \frac{f(x)}{\bar{F}(x)} &= \frac{g'_{s|r}(x)}{[g_{s|r}(x) - g_{s|r+1}(x)]} = \frac{\alpha g^{\alpha-1}(x)g'(x)}{\left[\left(g^\alpha(x) + \frac{s-r}{\beta} \right) - \left(g^\alpha(x) + \frac{s-r-1}{\beta} \right) \right]} \\
 \frac{f(x)}{\bar{F}(x)} &= \alpha \beta g^{\alpha-1}(x)g'(x) \\
 f(x) &= \alpha \beta g^{\alpha-1}(x)g'(x) \bar{F}(x)
 \end{aligned}$$

and hence the Theorem. □

Theorem 2.2. Under the conditions as given in Theorem 2.1 and for $1 \leq r \leq s < t$

$$E [g^\alpha(X_{L(t)} - X_{L(s)} | X_{L(r)} = x)] = \frac{(t-s)}{\beta} \tag{2.7}$$

if and only if

$$F(x) = 1 - \exp(-\beta g^\alpha(x)), \quad \alpha, \beta, x > 0. \tag{2.8}$$

Proof. Necessary part can be proved on the line of Theorem 2.1.

Now, for the sufficiency part, let $c = \frac{(t-s)}{\beta}$

$$\begin{aligned}
 &\frac{1}{\Gamma(t-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-r-1} f(y) dy \\
 &- \frac{1}{\Gamma(s-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-1} f(y) dy = c\bar{F}(x). \tag{2.9}
 \end{aligned}$$

Differentiating both sides of the equation (2.9) with respect to x ,

$$\begin{aligned}
 &\frac{(t-r-1)}{\Gamma(t-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-r-2} \frac{f(y)}{\bar{F}(x)} f(x) dy \\
 &- \frac{(s-r-1)}{\Gamma(s-r)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-2} \frac{f(y)}{\bar{F}(x)} f(x) dy = cf(x), \quad f(x) > 0, \forall x
 \end{aligned}$$

That is,

$$\begin{aligned}
 &\frac{1}{\Gamma(t-r-1)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-r-2} f(y) dy \\
 &- \frac{1}{\Gamma(s-r-1)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-2} f(y) dy = c\bar{F}(x).
 \end{aligned}$$

Again differentiating both sides of the equation (2.9) with respect to x ,

$$\frac{1}{\Gamma(t-r-2)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-r-3} f(y) dy - \frac{1}{\Gamma(s-r-2)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{s-r-3} f(y) dy = c\bar{F}(x).$$

Similarly differentiating both sides of the equation (2.9) with respect to x upto $(s-r)$ times, we get,

$$\frac{1}{\Gamma(t-s)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \frac{f(y)}{\bar{F}(x)} dy = g^\alpha(x) + c. \tag{2.10}$$

Integrating left hand side of (2.10) by parts and simplifying, we have

$$\frac{1}{\Gamma(t-s-1)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-2} \frac{f(y)}{\bar{F}(x)} dy + \frac{1}{\Gamma(t-s)} \int_x^\infty \alpha g^{\alpha-1}(y) g'(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \frac{\bar{F}(y)}{\bar{F}(x)} dy = g^\alpha(x) + c. \tag{2.11}$$

In view of (2.10), reduces to

$$\frac{1}{\Gamma(t-s)} \int_x^\infty \alpha g^{\alpha-1}(y) g'(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \bar{F}(y) dy = \frac{\bar{F}(x)}{\beta}. \tag{2.12}$$

Differentiating above equation with respect to x and simplifying for $(t-s)$ times, we get

$$\alpha g^{\alpha-1}(x) g'(x) \bar{F}(x) = \frac{f(x)}{\beta}$$

$$f(x) = \alpha\beta g^{\alpha-1}(x) g'(x) \bar{F}(x).$$

Hence the result is proved. □

Remark 1. At $r = s$ Theorem 2.2 reduces to Theorem 2.1.

Theorem 2.3. Under the conditions as given in Theorem 2.1 and for $1 \leq r \leq s < t$,

$$E [g^\alpha(X_{L(t)}) - g^\alpha(X_{L(s)})] + g^\alpha(x) = E [g^\alpha(X_{L(t)}) | X_{L(s)} = x] \tag{2.13}$$

if and only if

$$F(x) = 1 - \exp(-\beta g^\alpha(x)), \alpha, \beta, x > 0. \tag{2.14}$$

Proof. It is easy to see that (2.14) implies (2.13) and hence the necessary part is proved.

For the sufficiency part, we have:

$$\begin{aligned} & E [g^\alpha(X_{L(t)}) - g^\alpha(X_{L(s)})] + g^\alpha(x) \\ &= \frac{1}{\Gamma(t-s)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \frac{f(y)}{\bar{F}(x)} dy. \end{aligned} \quad (2.15)$$

Integrating the right hand side of (2.15) by parts, we have

$$\begin{aligned} & \frac{1}{\Gamma(t-s-1)} \int_x^\infty g^\alpha(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-2} \frac{f(y)}{\bar{F}(x)} dy \\ &+ \frac{1}{\Gamma(t-s)} \int_x^\infty \alpha g^{\alpha-1}(y) g'(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \frac{\bar{F}(y)}{\bar{F}(x)} dy. \end{aligned} \quad (2.16)$$

In view of (2.15) and (2.16), we have

$$\begin{aligned} & E [g^\alpha(X_{L(t)}) - g^\alpha(X_{L(t-1)})] \bar{F}(x) \\ &= \frac{1}{\Gamma(t-s)} \int_x^\infty \alpha g^{\alpha-1}(y) g'(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \bar{F}(y) dy. \end{aligned} \quad (2.17)$$

Since

$E [g^\alpha(X_{L(t)}) - g^\alpha(X_{L(t-1)})] = \frac{1}{\beta}$, is independent of x , (2.17) can be written as

$$\frac{1}{\Gamma(t-s)} \int_x^\infty \alpha g^{\alpha-1}(y) g'(y) [-\ln(\bar{F}(y)) + \ln(\bar{F}(x))]^{t-s-1} \bar{F}(y) dy = \frac{\bar{F}(x)}{\beta}. \quad (2.18)$$

Differentiating (2.18) with respect to x and simplifying for $(t-s)$ times, we get

$$\alpha g^{\alpha-1}(x) g'(x) \bar{F}(x) = \frac{f(x)}{\beta}.$$

Implying that

$$f(x) = \alpha \beta g^{\alpha-1}(x) g'(x) \bar{F}(x).$$

This proves the Theorem. □

Table 1: Example based on $df F(x) = 1 - \exp(-\beta g^\alpha(x))$, $\alpha, \beta, x > 0$.

Distribution	$F(x)$	$g^\alpha(x)$
Weibull Extension	$1 - \exp(-\beta(\exp(x^\alpha) - 1))$	$\exp(x^\alpha) - 1$
Modified Weibull	$1 - \exp(-x^\alpha \beta \exp(\lambda x))$	$x^\alpha \exp(\lambda x)$
Weibull	$1 - \exp(-\beta x^\alpha)$	x^α
Burr-type XII	$1 - (1 + x^\alpha)^{-\beta}$	$\ln(1 + x^\alpha)$
Lomax	$1 - (1 + x/\alpha)^{-\beta}$	$\ln(1 + x/\alpha)$
Generalized Pareto	$1 - (1 - x/\alpha)^\beta$	$-\ln(1 - x/\alpha)$
Pareto-type I	$1 - (x/\alpha)^{-\beta}$	$\ln(x/\alpha)$

3 Conclusion

In this paper, we have derived the characterization results through the upper record values when a sample is available from a new family of continuous distribution. It has been seen that this family consists a list of continuous probability distributions as given in Table 1. Further, the results provided in this paper will be a useful reference for the researchers in the field of record value theory and its applications.

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