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Game k-Domination Number of Graphs

Rana Khoeilar, Mustapha Chellali, Hossein Karami and Seyed Mahmoud Sheikholeslami

Abstract. For a positive integer k, a subset D of vertices in a digraph $\overrightarrow{G}$ is a k-dominating set if every vertex not in D has at least k direct predecessors in D. The k-domination number is the minimum cardinality among all k-dominating sets of $\overrightarrow{G}$. The game k-domination number of a simple and undirected graph is defined by the following game. Two players, $\mathcal{A}$ and $\mathcal{D}$, orient the edges of the graph alternately until all edges are oriented. Player $\mathcal{D}$ starts the game, and his goal is to decrease the k-domination number of the resulting digraph, while $\mathcal{A}$ is trying to increase it. The game k-domination number of the graph G is the k-domination number of the directed graph resulting from this game. This is well defined if we suppose that both players follow their optimal strategies. We are mainly interested in the study of the game 2-domination number, where some upper bounds will be presented. We also establish a Nordhaus-Gaddum bound for the game 2-domination number of a graph and its complement.

1 Introduction

For notation and graph theory terminology we in general follow Haynes, Hedetniemi and Slater [6]. Specifically, let G be a graph with vertex set V and order $|V| = n$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the *open neighborhood* of set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$. The *degree* of a vertex $v \in V$ is $d_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex $v \in V$ is said to *dominate* all the vertices in its closed neighborhood $N[v]$. For a positive integer k, a subset D of V is a *k-dominating set* of G if D dominates every vertex of $V \setminus D$ at least k times. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among all k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A *matching* in a graph G is a subset of pairwise non-adjacent edges. The *matching number* $\alpha'(G)$ (α' for short) is the size of a largest matching in G. A *perfect*

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Corresponding author: Seyed Mahmoud Sheikholeslami.

matching of G is a matching which matches (or covers) all vertices of the graph. A *near-perfect matching* is a matching in which exactly one vertex is unmatched. A graph is *factor-critical* if the deletion of any vertex leaves a graph with a perfect matching. Clearly, a factor-critical graph has an odd order.

A *separator* of a connected graph G is a set of vertices of the graph whose removal makes the graph disconnected. If S is a separator of a graph G, then let $o(G-S)$ denote the number of odd components of $G-S$, i.e., components of odd orders. A *barrier* of G is a separator S such that $o(G-S) = |S|+t$, where $t = n - 2\alpha'$ is the number of vertices of G which are not covered by a maximum matching. By Tutte-Berge's Theorem every connected graph admits barriers. Moreover (see for example exercise 3.3.18 in [7]) if S is a maximal barrier, then all the components $G_1, G_2, \cdots, G_{|S|+t}$ of $G-S$ are factor-critical (hence odd) and every maximum matching of G is formed by a matching pairing S with $|S|$ different components of $G-S$ and a near perfect matching in each component. Therefore, with the notation $|S|+t = \ell$ and $|V(G_i)| = n_i$,

$$\alpha'(G) = |S| + \sum_{i=1}^{\ell} \frac{n_i - 1}{2}. \tag{1.1}$$

For a positive integer k, a k-dominating set of a digraph $\overrightarrow{G}$ is a set D of vertices such that for every vertex $v \notin D$ there exist k-vertices $u_1, \ldots, u_k \in D$ with $\overrightarrow{u_1 v}, \ldots, \overrightarrow{u_k v} \in E(\overrightarrow{G})$. The k-domination number $\gamma_k(\overrightarrow{G})$ is defined as the minimum cardinality among all k-dominating sets of $\overrightarrow{G}$.

Following Alon, Balogh, Bollobás and Szabó, we introduce the game k-domination number of an undirected graph G as follows. Let $k \geq 1$ be an integer, players $\mathcal{A}$ and $\mathcal{D}$ orient the edges of the graph G alternately with $\mathcal{D}$ playing first, until all edges are oriented. Player $\mathcal{D}$ (frequently called Dominator) tries to minimize the k-domination number of the resulting digraph, while player $\mathcal{A}$ (Avoider) tries to maximize it. This game gives a unique number depending only on G, if we suppose that both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies. We call this number the *game k-domination number* of G and we denote it by $\gamma_g^k(G)$. Clearly, the game 1-domination number $\gamma_g^1(G)$ corresponds to the game domination number introduced by Alon, Balogh, Bollobás and Szabó in [2] and also studied in [3, 5]. Since the k-domination number of any orientation of a graph G is at least as large as the k-domination number of the graph itself, and for any positive integer k, $V(G)$ itself is a k-dominating set of any orientation of G, we get

$$\gamma_k(G) \leq \gamma_g^k(G) \leq n. \tag{1.2}$$

Our purpose in this paper is to initiate the study of the game k-domination number, especially when $k \in \{1, 2\}$. We list below some results obtained in [2] that will be useful in our investigation.

Theorem 1. *For a path P_n on n vertices we have $\gamma_g(P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 2. *For a cycle C_n on n vertices we have $\gamma_g(C_n) = \lfloor \frac{n}{2} \rfloor$.*

Theorem 3. *Let G be a "lollipop" on n vertices formed by an even cycle with a tail (a single path) attached to one of its vertices. Then $\gamma_g(G) = \lfloor \frac{n}{2} \rfloor$.*

Theorem 4. *For any nontrivial connected graph G, $\gamma_g(G) \leq \lfloor \frac{2n}{3} \rfloor$.*

Theorem 5. *If a graph G has minimum degree at least two, then $\gamma_g(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Theorem 6. *Let G be a graph on $n \geq 2$ vertices.*

1. *If G has a perfect matching M, then $\gamma_g(G) \leq \frac{n}{2}$.*
2. *If G is connected with at most one vertex of degree 1 and G has a matching of size $\frac{n-1}{2}$, then $\gamma_g(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Since the proof of second item is not explicitly appeared in [2], we recall its proof (see the second paragraph of the proof of Theorem 4.5). By assumption, we have $n = 2k + 1$ for some positive integer k, and there is a matching of size k, containing the edges $(v_1, u_1), \ldots, (v_k, u_k)$, leaving only one more vertex for v to dominate. Since deg$(v) \geq 1$, v is connected to a vertex of the matching, say v_1. We may assume without loss of generality that deg$(u_1) \geq$ deg(v), for otherwise we can replace v by u_1. By assumption we have deg$(u_1) \geq 2$. If u_1 is also connected to v, then the proof is done, as the resulting triangle vv_1u_1 can easily be dominated by one vertex if D starts the game. Otherwise u_1 is connected to another vertex of the matching, say v_2. Following this algorithm we build an alternating path $vv_1u_1v_2u_2 \ldots v_iu_i$ until u_i is connected to a previous vertex on this alternating path. If this vertex is v or u_j , then we end up with an odd cycle and a matching (we might need to change the matching edges along the alternating path up to the cycle) and finish with a dominating set of size at most $n/2$ applying Theorem 2. Finally, if u_i is attached to a vertex v_j on the path, then we have an even cycle with an odd path attached to it, and some independent edges of the original matching, and we can easily get the desired dominating set by Theorem 3. □

2 Preliminary results on game domination

In this section we consider a variant of game domination, where $\mathcal{A}$ starts the game which improves in some sense the results given in the previous section. Our proofs are mainly similar to those given in [2].

Proposition 2.1. Let G be a graph on n vertices.

1. If G has a perfect matching M, then $\gamma_g(G) \leq \frac{n}{2}$ even if $\mathcal{A}$ orients all edges of G.

2. If G has a near perfect matching, then $\gamma_g(G) \le \lceil \frac{n}{2} \rceil$ even if $\mathcal{A}$ orients all edges of G.

Proof. 1. Each vertex of M can be dominated by one of its vertices regardless of the orientation of the edges, implying $\gamma_g(G) \le \frac{n}{2}$.

2. The vertices of G can be partitioned into $(n-1)/2$ sets of disjoint edges and one single vertex. Each of these sets can be dominated by one of its vertices regardless of the orientation of the edges, implying $\gamma_g(G) \le \lceil \frac{n}{2} \rceil$.

□

As an immediate consequence of Proposition 2.1, we obtain the following corollaries that improve those in [2].

Corollary 2.1. For a path P_n on n vertices we have $\gamma_g(P_n) \le \lceil \frac{n}{2} \rceil$ even if $\mathcal{A}$ starts the orientation.

Corollary 2.2. For a cycle C_n on n vertices we have $\gamma_g(C_n) \le \lfloor \frac{n}{2} \rfloor$ even if $\mathcal{A}$ starts the orientation.

Proof. If n is even, then the result follows by Proposition 2.1 (Item 1). Hence let n be odd and assume that $\mathcal{A}$ starts the game. Then $\mathcal{D}$ in his first move, can make sure that there is a vertex dominating its two neighbors. The remaining $n-3$ vertices can be partitioned into $(n-3)/2$ independent edges, where for each of these edges, an endvertex will dominate the other endvertex regardless of the orientation. Therefore, $\gamma_g(C_n) \le 1 + (n-3)/2 = \lfloor \frac{n}{2} \rfloor$. □

Corollary 2.3. Let G be a "lollipop" on n vertices. Then $\gamma_g(G) \le \lceil \frac{n}{2} \rceil$ even if $\mathcal{A}$ starts the orientation.

Proposition 2.2. Let G be a connected graph of order n with minimum degree at least two. If $\mathcal{A}$ starts the orientation, then $\gamma_g(G) \le \lceil \frac{n}{2} \rceil$.

Proof. Our aim is to find a maximum matching in the graph and use those edges to dominate their endvertices regardless of their orientation. Clearly, if G has a perfect matching, then the result follows from Proposition 2.1. Now we suppose that a maximum matching of G covers all but t $(t \ge 1)$ vertices.

Suppose S is a maximal barrier of G and let $H_1, H_2, \ldots, H_\ell$ be the components of $G - S$. Recall that $\ell = |S| + t$. If $S = \emptyset$, then G is a factor critical graph and it follows from Proposition 2.1 that $\gamma_g(G) \le \lceil \frac{n}{2} \rceil$. Henceforth, we assume that $S \ne \emptyset$. Then $\ell \ge 2$.

The strategy of $\mathcal{D}$ is simple by ensuring that vertices in the ℓ components of $G - S$ will be dominated 'efficiently' by less than half of their orders plus the set of vertices of S and one other vertex. Now, to see how this can be done we distinguish between two types of odd components in $G - S$ depending on the number of edges joining them to S.

First, if H_i for some i is a component attached to S with at least two edges, then vertices of H_i can be dominated as follows: we orient out of S one of the edges between S and H_i which leads to dominate some vertex y of $V(H_i)$, and then one can use the perfect matching in the subgraph induced by $V(H_i) - \{y\}$ (as seen in the proof of item 1 of Proposition 2.1).

Second, there are components connected to S by only one edge. Without loss of generality, let $H_1, \dots, H_m$ be such components of $G - S$ and let v_i be the only vertex of H_i such that $|N(v_i) \cap S| = 1$. If $\mathcal{D}$ can orient the bridge from S toward v_i, then one can use the perfect matching in the subgraph induced by $V(H_i) - \{v_i\}$ and dominate the vertices of $V(H_i)$ by $\frac{n(H_i)-1}{2}$ vertices. If $\mathcal{A}$ orients the bridge from v_i toward S, then $\mathcal{D}$ starts the game in the components H_i and by Theorem 6 (item 2), he can dominate the vertices of $V(H_i)$ by $\frac{n(H_i)-1}{2}$ vertices. Now, if we have to select every vertex in S into the dominating set, it can not be larger than $|S| + \sum_{i=1}^{\ell} \frac{n(H_i)-1}{2} = |S| + \frac{n-|S|-|S|-t}{2} \leq \lfloor \frac{n}{2} \rfloor$. □

By a closer look at the proof of proposition 2.2, we can see the next result.

Corollary 2.4. Let G be a connected graph of order n with $\delta(G) \geq 2$ such that G has a non-empty maximal barrier. If $\mathcal{A}$ starts the orientation, then $\gamma_g(G) \leq \lfloor \frac{n}{2} \rfloor$.

3 Bounds on the game 2-domination number

In the aim to provide strategy for $\mathcal{D}$ to obtain upper bounds for the game k-domination number, one can try to find a small set S of vertices that dominates every vertex not in S at least $2k$ times. This guarantees to orient at least k edges by $\mathcal{D}$ out of S toward any vertex not in S and hence making S a k-dominating set of the resulting digraph. Therefore for every positive integer k, we have

$$\gamma_g^k(G) \leq \gamma_{2k}(G). \tag{3.1}$$

Using (3.1) and a known upper bound on the k-domination number due to [4] we obtain the following upper bound on the game 2-domination number of a graph in terms of the order and minimum degree.

Proposition 3.1. *If G is a graph of order n and minimum degree $\delta \geq 2k$, then*

$$\gamma_g^k(G) \leq \gamma_{2k}(G) \leq \frac{\delta}{2\delta + 1 - 2k} n.$$

For the particular case $k = 2$, the following corollary is an immediate consequence of Proposition 3.1, since $\frac{\delta}{2\delta - 3} \leq 3/4$ for every $\delta \geq 5$.

Corollary 3.1. *If G is a connected graph of order n and minimum degree $\delta \geq 5$, then*

$$\gamma_g^2(G) \leq \frac{3n}{4}.$$

Proposition 3.2. *For any graph G of order n,*

$$\gamma_g^2(G) \leq n - \delta + 3.$$

Proof. The result is immediate when $\delta \leq 3$. Thus assume that $\delta \geq 4$. Let x be a vertex with minimum degree. Let $y_1, y_2, y_3 \in N(x)$ and $X = V(G) - N[x]$. Clearly $X \cup \{x, y_1, y_2, y_3\}$ is a 4-dominating set of G and the result follows by (3.1). □

Restricted to graphs G with minimum degree at least three, we present an upper bound for the game 2-domination number in terms of the order of G. Before presenting this result, we need to recall the unfriendly partition. As defined in [1] and elsewhere, a bipartition $\pi = (X, Y)$ of the vertex set V of a graph G is called an *unfriendly partition* if every vertex $u \in X$ has at least as many neighbors in Y as it does in X, and every vertex $v \in Y$ has at least as many neighbors in X as it does in Y. It was shown in [1] that every finite connected graph G of order $n \geq 2$ has an unfriendly partition.

Theorem 3.2. *If G is a connected graph of order n and minimum degree $\delta \geq 3$, then*

$$\gamma_g^2(G) \leq \frac{5n}{6}.$$

Proof. Let $\pi = (X, Y)$ be an unfriendly partition of G, where without loss of generality $|X| \leq |Y|$. For any vertex z, let $d_X(z) = N(z) \cap X$ and $d_Y(z) = N(z) \cap Y$. Since $\pi = (X, Y)$ is an unfriendly partition of G, we have $d_X(z) \geq d_Y(z)$ for each $z \in Y$ and $d_Y(z) \geq d_X(z)$ for each $z \in X$. Since $\delta(G) \geq 3$, each vertex in Y has at least two neighbors in X. Let $G_1, \ldots, G_t$ be the components of $G[Y]$ of order at least two and $K_1, \ldots, K_\ell$ the trivial components of $G[Y]$. Player $\mathcal{D}$ can easily orient the edges to dominate each vertex $y \in Y$ by a vertex of X as well as to 2-dominate $\lfloor \ell/2 \rfloor$ vertices belonging to K_i's by the vertices of X even if $\mathcal{A}$ starts the orientation. Now, let $T_1, \ldots, T_t$ be the spanning trees of $G_1, \ldots, G_t$, respectively. For each T_i, we partition its vertices into stars each of order at least two (for instance, consider a maximum matching M in T_i where the endvertices of each edge in M initially form a star to which we can add the vertices uncovered by M). For a star $K_{1,r}$ $(r \geq 4)$, clearly $\mathcal{D}$ can orient at least $\lfloor \frac{r}{2} \rfloor$ edges from its center toward the leaves. Thus $\mathcal{D}$ can easily dominate the vertices of such a star $K_{1,r}$ with $\lceil \frac{r}{2} \rceil + 1 \leq \lfloor \frac{2}{3}(r+1) \rfloor$ vertices, even if he does not start the orientation. Moreover, in a star $K_{1,3}$ clearly $\mathcal{D}$ can orient at least 2 edges from its center toward the leaves if he plays first on it while he can orient at least 1 edge from its center toward the leaves if $\mathcal{A}$ starts the orientation. Thus two vertices dominate vertices of such a star $K_{1,3}$ if $\mathcal{D}$ plays first and three vertices if $\mathcal{A}$ does. Thus, the strategy of D is to orient the edges from X to vertices in Y and make use of a star-partition: he starts in a trivial component (if any) and orients an edge from X to it, otherwise he starts in a $K_{1,3}$ (if any) and orients an edge from the center to a leaf, then plays on the same star as $\mathcal{A}$, except if he chooses another $K_{1,3}$. Then $\mathcal{D}$ does the same, ensuring that at least half of the stars of order four will be dominated by two vertices, which in average gives a dominating set of size at most $\frac{5}{8} < \frac{2}{3}$ of the vertices forming these stars (note that for every two stars $K_{1,3}$, player $\mathcal{D}$ can starts the game in at least one of these stars and so two vertices will dominate its vertices and for other star we need three vertices to dominate its vertices. Hence 5 vertices will dominate the 8 vertices of these two stars. Hence to dominate the vertices of all stars $K_{1,3}$ we need 5/8 of their vertices). This strategy provides a 2-domination number equals to at most $\frac{5n}{6}$ in the resulting digraph. $\square$

Our next result improves Corollary 3.1 by showing that the bound remains valid for graphs with minimum degree $\delta \geq 4$. We first need to prove the following useful lemma.

Lemma 3.1. For any nontrivial connected graph G, there is a partition $V(G) = L \cup S \cup V_1 \cup V_2 \cup \ldots \cup V_t$ such that every vertex of L (if any) is a leaf, every vertex of S (if any) has degree at least three and either $\delta(G[V_i]) \geq 2$ or $G[V_i]$ contains a Hamiltonian path of even order.

Proof. Assume that the result does not hold and let G be the smallest nontrivial connected graph for which the above partition does not exist. Then $\delta(G) = 1$ and G is not an even path, otherwise G satisfies the partition. Let $v \in V(G)$ be a vertex of degree one. If $\Delta(G) = 2$, then G is an odd path and thus $\{v\} \cup (V(G) - \{v\})$ is the desired partition, a contradiction with our assumption. Hence we assume $\Delta(G) \geq 3$. Let $B = \{u \in V \mid \deg(u) \geq 3\}$ and let vPy be a shortest path between v and B, where $y \in B$ and P is an induced path (possibly empty) between v and y. Let $G' = G - (V(P) \cup \{v\})$. Clearly G' satisfies the result and so $V(G') = L \cup S \cup V_1 \cup V_2 \cup \ldots \cup V_s$ such that $\deg_{G'}(x) = 1$ for each $x \in L$, $\deg_{G'}(x) \geq 3$ for each $x \in S$ and $\delta(G'[V_i]) \geq 2$ or $G'[V_i]$ contains a Hamiltonian path of even order. Now, if P is an odd path, then $V(G) = L \cup S \cup V_1 \cup V_2 \cup \ldots \cup V_s \cup (V(P) \cup \{v\})$ is a desired partition, a contradiction with our assumption. Thus P is an even path, but then $V(G) = (L \cup \{v\}) \cup S \cup V_1 \cup V_2 \cup \ldots \cup V_s \cup V(P)$ is a desired partition too, a contradiction. This completes the proof. $\square$

Theorem 3.3. *If G is a connected graph of order n and minimum degree $\delta \geq 4$, then*

$$\gamma_g^2(G) \leq \frac{3n}{4}.$$

Proof. Let $\pi = (X, Y)$ be an unfriendly partition of G, where without loss of generality $|X| \leq |Y|$. As in the proof Theorem 3.2, $d_X(z) \geq d_Y(z)$ for each $z \in Y$ and $d_Y(z) \geq d_X(z)$ for each $z \in X$. Also each vertex in Y has at least two neighbors in X.

Let $y_1, \ldots, y_\ell$ be the isolated vertices of $G[Y]$ and let G' be the subgraph induced by all nontrivial component of $G[Y]$. Applying Lemma 3.1 to each nontrivial component of G', we obtain a partition $V(G') = L \cup S \cup V_1 \cup V_2 \cup \ldots \cup V_t$ such that $\deg_{G'}(x) = 1$ for each $x \in L$, $\deg_{G'}(x) \geq 3$ for each $x \in S$ and $\delta(G'[V_i]) \geq 2$ or $G'[V_i]$ contains a Hamiltonian path of even order for each i. Suppose $G'[V_1], \ldots, G'[V_s]$ are graphs which has neither a perfect matching nor a non-empty maximal barrier, if any. Note that because of $\delta \geq 4$ and the fact that $\pi = (X, Y)$ is an unfriendly partition of G, each vertex in $L \cup S$ has least three neighbors in X.

Clearly, Player D has a strategy to orient the edges of G to ensure that each vertex y_i is dominated by at least two vertices in X and each vertex in $Y \setminus \{y_1, \ldots, y_k\}$ is dominated by at least one vertex in X. Also, if $\mathcal{D}$ starts the orientation of edges, then either at least $\left\lceil \frac{|L \cup S|}{2} \right\rceil$ vertices of $L \cup S$ can be dominated twice by X and $\mathcal{D}$ can start to orient $\lfloor \frac{s}{2} \rfloor$ graphs of $G'[V_1], \ldots, G'[V_s]$, or either at least $\left\lfloor \frac{|L \cup S|}{2} \right\rfloor$ vertices of $L \cup S$ can be dominated twice by X and $\mathcal{D}$ can start to orient $\lceil \frac{s}{2} \rceil$ graphs of $G'[V_1], \ldots, G'[V_s]$. Thus, the strategy of $\mathcal{D}$ is to orient the edges from X to vertices in Y and make use of the partition of $V(G')$. More precisely, player $\mathcal{D}$ begins by orienting an edge having an endvertex in X and the other endvertex in $L \cup S$ or starting to orient the edges of graphs $G'[V_1], \ldots, G'[V_s]$. Moreover, if $\mathcal{A}$ starts the orientation of edges of $G'[V_i]$ for $1 \leq i \leq s$, then $\mathcal{D}$ does the same. By Corollaries 2.1, 2.4 and Proposition 2.2, this strategy provides in the resulting digraph a 2-domination number equals to at most $|X| + \frac{|Y|}{2}$. Using the facts that $|X| \leq \frac{n}{2}$ and $|Y| + |X| = n$, we obtain $\gamma_g^2(G) \leq \frac{3n}{4}$. □

4 Nordhaus-Gaddum type result for game 2-domination

In [2], Alon et al. gave the following Nordhaus-Gaddum bound for game domination number of a graph and its complement.

Theorem 7. *For a graph G with n vertices, $\gamma_g(G) + \gamma_g(\overline{G}) \leq n + 2$.*

For the game 2-domination number, we establish the following inequality.

Theorem 4.1. *For any graph G of order n,*

$$\gamma_g^2(G) + \gamma_g^2(\overline{G}) \leq n + 7.$$

Proof. The result is trivial for $n \leq 7$. Hence assume that $n \geq 8$. Let S be a 5-subset of $V(G)$, $Y = N(S) - S$ and $T = V(G) \setminus (S \cup Y)$. For $1 \leq i \leq 5$, define $X_i = \{x \in Y : |N(x) \cap S| = i\}$ and let $X_{2,3} = X_2 \cup X_3$. Player $\mathcal{D}$ can easily dominate (once) each vertex of $X_{2,3}$ by S and since each vertex in $X_4 \cup X_5$ has at least four neighbors in S, he can dominate twice each vertex in $X_4 \cup X_5$ by S (even if $\mathcal{A}$ starts the orientation). Player $\mathcal{D}$ starts the game from the induced subgraph $G[X_{2,3}]$. Now for any $\gamma_g(G[X_{2,3}])$-set D, set $S \cup T \cup X_1 \cup D$ is clearly a 2-dominating set in the resulting digraph of G. Therefore

$$\gamma_g^2(G) \leq |S| + |T| + |X_1| + |D|. \tag{4.1}$$

On the other hand in $\overline{G}$, Player $\mathcal{D}$ can easily dominate (once) each vertex of $X_{2,3}$ by S and since each vertex in $X_1 \cup T$ has at least four neighbors in S, he can dominate twice each vertex in $X_1 \cup T$ by S (even if $\mathcal{A}$ starts the orientation). Player $\mathcal{D}$ starts the game in $\overline{G}$ from the induced subgraph $\overline{G}[X_{2,3}]$. Hence for any $\gamma_g(\overline{G}[X_{2,3}])$-set $\overline{D}$, set $S \cup X_4 \cup X_5 \cup \overline{D}$ is a 2-dominating set in the resulting digraph of $\overline{G}$ yielding

$$\gamma_g^2(\overline{G}) \leq |S| + |X_4| + |X_5| + |\overline{D}|. \tag{4.2}$$

By Theorem 7, we have $|D| + |\overline{D}| \leq |X_2| + |X_3| + 2$. Combining this with inequalities (4.1) and (4.2), we obtain $\gamma_g^2(G) + \gamma_g^2(G) \leq n + 7$. □

5 Exact values of the game 2-domination number

In this section, we establish the exact values of the game 2-domination number for some elementary classes of graphs.

Theorem 5.1. $\gamma_g^2(K_{2,n}) = \begin{cases} 3 & \text{if} \quad n = 2 \\ 4 & \text{if} \quad n = 3 \\ n & \text{if} \quad n \geq 4 \end{cases}$

Proof. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{2,n}$. Let $\overrightarrow{K_{2,n}}$ denote the resulting digraph.

First let $n = 2$. Player $\mathcal{D}$ orients the edge x_1y_1 from x_1 toward y_1 in first step. If $\mathcal{A}$ orients the edge x_2y_1 from x_2 to y_1, then $\{x_1, x_2, y_2\}$ is a 2-dominating set of $\overrightarrow{K_{2,2}}$. If $\mathcal{A}$ orients the edge x_2y_1 from y_1 to x_2, then $\mathcal{D}$ orients the edge x_2y_2 from y_2 toward x_2 in the second step and clearly $\{x_1, y_1, y_2\}$ is a 2-dominating set of $\overrightarrow{K_{2,2}}$. Finally, if $\mathcal{A}$ does not orient the edge x_2y_1, then $\mathcal{D}$ can orient the edge x_2y_1 from x_2 toward y_1 in the second step and clearly $\{x_1, x_2, y_2\}$ is a 2-dominating set of $\overrightarrow{K_{2,2}}$. In either case, there exists a 2-dominating set of size 3 of the resulting digraph, hence $\gamma_g^2(K_{2,2}) \leq 3$. To prove that $\gamma_g^2(K_{2,2}) \geq 3$, suppose without loss of generality that D first orients the edge x_1y_1 from x_1 toward y_1. Player $\mathcal{A}$ orients the edge x_2y_1 from y_1 to x_2, and thus any 2-dominating set in $\overrightarrow{K_{2,2}}$ must contain x_1, y_1 and either x_2 or y_2. Hence $\gamma_g^2(K_{2,2}) \geq 3$, and the equality follows.

Now let $n = 3$. Since $\mathcal{D}$ stars the game, where he can orient two edges from Y toward x_1 and so $Y \cup \{x_2\}$ is a 2-dominating set in $\overrightarrow{K_{2,3}}$, yielding $\gamma_g^2(K_{2,3}) \leq 4$. The strategy of $\mathcal{A}$ is as follows: when $\mathcal{D}$ orients the edge x_1y_i (resp. x_2y_i) from x_1 (resp. x_2) toward y_i, then $\mathcal{A}$ orients the x_2y_i (resp. x_1y_i) from y_i toward x_2 (resp. x_1) and vice versa. Hence in the resulting digraph, each y_i has in-degree one and so belongs to any 2-dominating set. On the other hand, there are exactly three edges oriented from Y toward X, at most one vertex of X can be dominated by Y, and hence $\gamma_g^2(K_{2,3}) \geq 4$. Therefore $\gamma_g^2(K_{2,3}) = 4$.

Finally let $n \geq 4$. Clearly, Y is a 4-dominating set of $K_{2,n}$ and thus by (3.1) we have $\gamma_g^2(K_{2,n}) \leq n$. Next we shall prove that $\gamma_g^2(K_{2,n}) \geq n$. The strategy of $\mathcal{A}$ is as follows: when $\mathcal{D}$ orients the edge x_1y_i (resp. x_2y_i) from x_1 (resp. x_2) toward y_i, then he orients the x_2y_i (resp. x_1y_i) from y_i toward x_2 (resp. x_1) and if $\mathcal{D}$ orients the edge x_1y_i (resp. x_2y_i) from y_i toward x_1 (resp. x_2), then he orients the x_2y_i (resp. x_1y_i) from x_2 (resp. x_1) toward y_i. In the resulting digraph each vertex y_i has in-degree one and thus must belong to any 2-dominating set. It follows that $\gamma_g^2(K_{2,n}) \geq n$, and the equality is obtained. □

Theorem 5.2. $\gamma_g^2(K_{3,n}) = \begin{cases} 4 & \text{if} \quad n = 3, 4, \\ \lfloor \frac{n}{2} \rfloor + 3 & \text{if} \quad n \geq 5. \end{cases}$

Proof. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{3,n}$. Let $\overrightarrow{K_{3,n}}$ denote the resulting digraph.

First let $n = 3$. Player $\mathcal{D}$ first orients the edge x_1y_1 from x_1 toward y_1. Next he can easily orients one of the edges in each of the sets $\{x_1y_2, x_1y_3\}$, $\{x_2y_1, x_3y_1\}$, $\{x_2y_2, x_2y_3\}$ and $\{x_3y_2, x_3y_3\}$ from X toward Y. This implies that y_1, and at least one of y_2 or y_3, say y_2, are dominated twice by X and so $X \cup \{y_3\}$ is a 2-dominating set of $\overrightarrow{K_{3,3}}$. Therefore $\gamma_g^2(K_{3,3}) \leq 4$. Next we show that $\gamma_g^2(K_{3,3}) \geq 4$. The strategy of $\mathcal{A}$ is as follows: If $\mathcal{D}$ orients an edge from X to Y (resp. from Y to X), then $\mathcal{A}$ orients an edge from Y to X (resp. from X to Y). Under this strategy, we may assume that in $\overrightarrow{K_{3,3}}$ there are five edges from X to Y and four edges from Y to X. Thus X can not 2-dominate Y and vice versa. Now, let S be a $\gamma_2(\overrightarrow{K_{3,3}})$-set. Clearly $S \cap X \neq \emptyset$ and $S \cap Y \neq \emptyset$. If $X \subseteq S$ or $Y \subseteq S$, then $|S| \geq 4$ as desired. Hence we assume, without loss of generality, that $x_1 \notin S$ and $y_1 \notin S$. Then to 2-dominate x_1, y_1, we must have $y_2, y_3, x_2, x_3 \in S$ and so $|S| \geq 4$. Therefore $\gamma_g^2(K_{3,3}) \geq 4$ and thus $\gamma_g^2(K_{3,3}) = 4$.

Now let $n = 4$. Obviously Y is a 4-dominating set of $K_{3,4}$ and by (3.1), we have $\gamma_g^2(K_{3,4}) \leq 4$. Next we shall show that $\gamma_g^2(K_{3,4}) \geq 4$. Let S be a $\gamma_2(\overrightarrow{K_{3,4}})$-set. We note that $\mathcal{A}$ can easily orient two edges incident to some y_i from y_i toward X as well as two other edges incident to another vertex, say y_j. Hence $y_i, y_j \in S$. If $Y \subseteq S$, then $|S| \geq 4$ and we are done. Thus we assume, without loss of generality, that $y_1 \notin S$. To 2-dominate y_1, we must have $|S \cap X| \geq 2$ and so $|S| \geq 4$ as desired. Therefore $\gamma_g^2(K_{3,4}) = 4$.

Next assume that $n = 5$. Clearly, $\gamma_g^2(K_{3,5}) \leq 5$. To prove the inverse inequality, we observe, without loss of generality, that $\mathcal{A}$ will have the opportunity to begin the orientation of the edges incident to two vertices of Y, say y_4 and y_5 allowing him to orient two edges incident to y_4 and two edges incident to y_5 from Y toward X. Now, assume that $\mathcal{D}$ orients some edge from a vertex in $Y - \{y_4, y_5\}$ toward X in his first three moves or $\mathcal{A}$ can start orienting the edges incident to one of vertices in $Y - \{y_4, y_5\}$, say y_3. In this case, we can have two edges incident to y_3 oriented from y_3 to X. It follows that y_3, y_4, y_5 belong to any γ_2-set S of $\overrightarrow{K_{3,5}}$ and as in the case $n = 4$

we can see that $\gamma_g^2(K_{3,5}) \geq 5$. Hence we assume that the previous assumption does not occur. This means that $\mathcal{D}$ orients one edge incident to y_1, y_2, y_3 in his first three moves from X toward these vertices. Then $\mathcal{A}$ can easily orient two edges from y_1, y_2, y_3 toward two different vertices in X. Let S be a γ_2-set of $\overrightarrow{K_{3,5}}$. Recall that $y_4, y_5 \in S$. If $Y \subseteq S$, then $|S| \geq 5$ and we are done. Assume without loss of generality that $y_1 \notin S$. To 2-dominate y_1, we must have $|S \cap X| \geq 2$. If $S \cap \{y_2, y_3\} \neq \emptyset$, then $|S| \geq 5$ as desired. Hence let $S \cap \{y_2, y_3\} = \emptyset$. Since $\mathcal{A}$ oriented two edges from the vertices y_1, y_2, y_3 toward two different vertices in X, two vertices of X can not dominate y_1, y_2, y_3 and so $X \subseteq S$. Thus $|S| \geq 5$ as desired. Therefore $\gamma_g^2(K_{3,5}) = 5$.

Finally let $n \geq 6$. Player $\mathcal{D}$ can easily dominate twice at least $\lceil \frac{n}{2} \rceil$ vertices of Y by X, say $y_1, \ldots, y_{\lceil \frac{n}{2} \rceil}$. Hence $X \cup \{y_{\lceil \frac{n}{2} \rceil + 1}, \ldots, y_n\}$ is a 2-dominating set of resulting digraph yielding $\gamma_g^2(K_{3,n}) \leq \lfloor \frac{n}{2} \rfloor + 3$.

To prove that $\gamma_g^2(K_{3,n}) \geq \lfloor \frac{n}{2} \rfloor + 3$, consider the set $\{(y_i, y_{i+\lfloor \frac{n}{2} \rfloor}) \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. Without loss of generality, we may assume that in the same position $\mathcal{D}$ orients the edge $x_i y_j$ before the edge $x_i y_k$ when $j < k$ and the edge $x_r y_j$ before the edge $x_s y_j$ when $r < s$. The strategy of $\mathcal{A}$ is simple by trying to orient the edges from Y toward X as long as he can. In fact:

- If $\mathcal{D}$ orients the edge $x_i y_j$ from y_j to x_i, then $\mathcal{A}$ orients the edge $x_i y_{j+\lceil \frac{n}{2} \rceil}$ from $y_{j+\lceil \frac{n}{2} \rceil}$ to x_i,
- If for some $y_i \in Y$, there is exactly one edge incident to y_i oriented from y_i toward X, and $\mathcal{D}$ orients the second edge incident to y_i from a vertex of X toward y_i, then $\mathcal{A}$ orients the third edge incident to y_i from y_i toward X.
- If $\mathcal{D}$ orients the edge $x_i y_j$ from x_i to y_j, then $\mathcal{A}$ orients the edge $x_{i+1} y_j$ from y_j to x_{i+1},
- If $y_i \in Y$ and exactly one edge incident to y_i is oriented from X toward y_i and one edge incident to y_i is oriented from y_i toward X and $\mathcal{D}$ orients the third edge incident to y_i from a vertex $x \in X$ toward y_i, then $\mathcal{A}$ orients the edge $x y_{i+\lceil \frac{n}{2} \rceil}$ from $y_{i+\lceil \frac{n}{2} \rceil}$ to x.

Following above strategy, $\mathcal{A}$ ensures that at least $\lfloor \frac{n}{2} \rfloor$ vertices of Y has in-neighbor one and must belong to any 2-dominating set of resulting digraph. Also to 2-dominate the remaining vertices we need clearly at least three vertices and so $\gamma_g^2(K_{3,n}) \geq \lfloor \frac{n}{2} \rfloor + 3$. Therefore, $\gamma_g^2(K_{3,n}) = \lfloor \frac{n}{2} \rfloor + 3$. □

Theorem 5.3. *For $n \geq 4$, $\gamma_g^2(K_{4,n}) = 4$.*

Proof. Let X and Y be the bipartite sets of $K_{4,n}$, with $|X| = 4$ and $|Y| = n$. Clearly X is a 4-dominating set of $K_{4,n}$ and by (3.1) we have $\gamma_g^2(K_{4,n}) \leq 4$. Next we prove that $\gamma_g^2(K_{4,n}) \geq 4$. Let $\overrightarrow{K_{4,n}}$ be an arbitrary orientation of $K_{4,n}$ and let D be a $\gamma_2(\overrightarrow{K_{4,n}})$-set. If $X \subseteq D$ or $Y \subseteq D$, then clearly $|D| \geq 4$ and the result follows. Hence we assume that $X - D \neq \emptyset$ and $Y - D \neq \emptyset$. Let $y \in Y - D$ and $x \in X - D$. To 2-dominate x and y we must have $|D \cap Y| \geq 2$ and $|D \cap X| \geq 2$, respectively. Thus $|D| \geq 4$ regardless the orientation we have. Therefore, $\gamma_g^2(K_{4,n}) = 4$. $\square$

We close this section by presenting an upper bound on the game 2-domination number of paths.

Lemma 5.1. *1.* $\gamma_g^2(P_4) = 3$.

2. $\gamma_g^2(P_5) = 4$.

3. $\gamma_g^2(P_5) \leq 4$ *when $\mathcal{A}$ starts the game.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a path on n vertices and let $\overrightarrow{P_n}$ denote the resulting digraph.

1. Player $\mathcal{D}$ first orients the edge v_1v_2 from v_1 toward v_2. If $\mathcal{A}$ orients the edge v_2v_3 from v_3 toward v_2 or $\mathcal{D}$ orients such an edge from v_3 toward v_2 in his second move, then $\{v_1, v_3, v_4\}$ is a 2-dominating set of $\overrightarrow{P_4}$. Thus we can assume that $\mathcal{A}$ orients v_2v_3 from v_2 toward v_3. Then $\mathcal{D}$ will orients v_3v_4 from v_4 toward v_3 and so $\{v_1, v_2, v_4\}$ is a 2-dominating set of $\overrightarrow{P_4}$. In either case, we have $\gamma_g^2(P_4) \leq 3$. On the other hand, any 2-dominating set D of an arbitrary orientation of P_4, must contain v_1, v_4 and at least one of v_2 and v_3. Hence $\gamma_g^2(P_4) \geq 3$ and the equality follows.

2. Assume that $\mathcal{D}$ starts the game. As above, we can see that $\gamma_g^2(P_5) \leq 4$. To prove the inverse inequality, $\mathcal{A}$ plays as follows: if $\mathcal{D}$ first orients the edge v_1v_2 (resp. v_4v_5), then $\mathcal{A}$ orients in the same direction the edge v_2v_3 (resp. v_3v_4), and if $\mathcal{D}$ first orients the edge v_2v_3 (resp. v_3v_4), then $\mathcal{A}$ orients in the same direction the edge v_3v_4 (resp. v_2v_3). In either situation, any 2-dominating set of $\overrightarrow{P_5}$ contains at least four vertices and so $\gamma_g^2(P_5) = 4$.

3. Let $\mathcal{A}$ stars the game. The strategy of $\mathcal{D}$ is as follows: if $\mathcal{A}$ first orients the edge v_1v_2 (resp. v_4v_5), then $\mathcal{D}$ orients v_4v_5 from v_5 to v_4 (resp. v_1v_2 from v_1 to v_2) and continue the game as seen in Item 1 for the induced path P_4. If $\mathcal{A}$ first orients the edge v_2v_3 from v_2 to v_3, then $\mathcal{D}$ orients v_3v_4 from v_4 to v_3 and if $\mathcal{A}$ first orients the edge v_2v_3 from v_3 to v_2, then $\mathcal{D}$ orients v_1v_2 from v_1 to v_2. If $\mathcal{A}$ first orients the edge v_3v_4, then $\mathcal{D}$ plays similarly as above. In either situation, $\overrightarrow{P_n}$ has a 2-dominating set of size four and thus $\gamma_g^2(P_5) \leq 4$.

$\square$

Theorem 5.4.

$$\gamma_g^2(P_n) \leq \begin{cases} (4n-1)/5 & \text{if} \quad n \equiv 4 \pmod 5 \\ \\ (4n+r)/5 & \text{if} \quad n \equiv r \pmod 5 \text{ and } 0 \leq r \leq 3. \end{cases}$$

Proof. Assume that $n = 5k + r$ where $k \geq 1$ and $0 \leq r \leq 4$. Let $P_n = v_1 v_2 \ldots v_n$ be a path on n vertices and let $V_i = \{v_{5i+1}, v_{5i+2}, v_{5i+3}, v_{5i+4}, v_{5i+5}\}$ for $0 \leq i \leq k-1$, and $V_k = \{v_n, \ldots, v_{n-r+1}\}$ if $r \neq 0$. Player $\mathcal{D}$ follows the strategy described in Lemma 5.1 in each subpath induced by V_i. This leads to the desired bound. □

We conclude this paper with a list of open problems.

1. Determine the exact values of the game 2-domination number for paths, cycles, complete graphs and complete bipartite graphs.
2. Determine $\gamma_g^2(G)$ and $\gamma_g^3(G)$ for every grid graph $G = P_m \Box P_n$.
3. Does there exist a polynomial algorithm for computing $\gamma_g^2(T)$ for any tree T?
4. What can you say about the complexity result for the game 2-domination problem?

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Rana Khoeilar Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran

E-mail: khoeilar@azaruniv.ac.ir

Mustapha Chellali LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270, Blida, Algeria

E-mail: m_chellali@yahoo.com

Hossein Karami Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran

E-mail: h.karami@azaruniv.ac.ir

Seyed Mahmoud Sheikholeslami Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran

E-mail: s.m.sheikholeslami@azaruniv.ac.ir