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ON COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION IN COMMON FIXED POINT CONSIDERATION

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Abstract. In this paper, we prove some common fixed point theorems for compatible mappings satisfying an implicit relation.

1. Introduction

Wang, Li, Gao and Iseki [14] proved some fixed point theorems on expansion mappings which correspond to some contractive mappings. In a paper Rhoades [9] generalized the results for pairs of mappings. Some theorems on unique fixed point for expansion mappings are proved by Popa [6]. Popa [7] further extended results [6], [9] for compatible mappings.

In 1999, Popa [8] proved some fixed point theorems for compatible mappings satisfying an implicit relation.

Let S and T be two self mappings of a metric space (X, d). Sessa [10] defines S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in X. Jungck [1] defines S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some x in X. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but implications are not reversible [11, Ex 1] and [1, Ex 2.2].

Many authors have proved common fixed point theorems for compatible mappings for this we refer to Jungck [1], [2] and [3], Sessa, Rhoades and Khan [12], Kang, Cho and Jungck [4], Kang and Ray [5] and Sharma and Patidar [13].

In this paper, we prove common fixed point theorems for compatible mappings in Banach spaces, satisfying an implicit relation. We improve and generalize the results of Popa [6], [7] and [8].

Lemma 1.([1]) *Let S and T be compatible self mappings on a metric space (X, d). If $S(t) = T(t)$ for some $t \in X$ then $ST(t) = TS(t)$.*

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2. Implicit Relations

Let Φ be the set of all real continuous functions $\phi(t_1, t_2, \ldots, t_6) : R_+^6 \to R$ satifying the following conditions:

ϕ_1 : ϕ is non-increasing in variable t_6,

ϕ_2 : there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

$$(\phi_a) : \phi(u, v, v, u, (1/2)(u+v), 0) \leq 0$$

or

$$(\phi_b) : \phi(u, v, u, v, (1/2)(u+v), u+v) \leq 0$$

we have $u \leq hv$.

ϕ_3 : $\phi(u, u, 0, 0, 0, u) > 0$ for all $u > 0$.

Example 1. $\phi(t_1, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, (1/2)t_6\}$, where $k \in (0,1)$.

ϕ_1: Obviously.

ϕ_2: Let $u > 0$, $\phi(u, v, v, u, (1/2)(u+v), 0) = u - k \max\{v, v, u, (1/2)(u+v), 0\} \leq 0$. If $u \geq v$ then $u \leq ku < u$, a contradiction. Thus $u < v$ and $u \leq kv = hv$, where $h = k \in (0,1)$.
Similarly, if $u > 0$ then $\phi(u, v, u, v, (1/2)(u+v), u+v) \leq 0$ imply $u \leq hv$. If $u = 0$, then $u \leq hv$.

ϕ_3: $\phi(u, u, 0, 0, 0, u) = (1-k)u > 0$, for all $u > 0$.

Example 2. $\phi(t_1, t_2, \ldots, t_6) = t_1^2 - a\{t_2^2 - t_6((1/2)(t_3+t_4) - t_5)\}$, where $a \in (0,1)$.

ϕ_1: Obviously.

ϕ_2: Let $u > 0$, $\phi(u, v, v, u, (1/2)(u+v), 0) = u^2 - av^2 \leq 0$, which implies $u \leq a^{1/2}v = hv$, where $h = a^{1/2} < 1$.
Similarly, if $u > 0$ then $\phi(u, v, u, v, (1/2)(u+v), u+v) \leq 0$ imply $u \leq hv$. If $u = 0$, then $u \leq hv$.

ϕ_3 : $\phi(u, u, 0, 0, 0, u) = u^2(1-a) > 0$, for all $u > 0$.

Example 3. $\phi(t_1, \ldots, t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, (1/2)t_4 t_6\}$, where $c_1 > 0$, $c_2 \geq 0$, $c_1 + c_2 < 1$.

ϕ_1: Obviously.

ϕ_2: Let $u > 0$, $\phi(u, v, v, u, (1/2)(u+v), 0) = u^2 - c_1 \max\{v^2, v^2, u^2\} - c_2 \max\{v(1/2)(u+v), 0\} \leq 0$. If $u \geq v$ then $u^2(1 - c_1 - c_2) \leq 0$, which implies $c_1 + c_2 \geq 1$, a contradiction. Thus $u < v$ and $u \leq (c_1 + c_2)^{1/2} v = hv$, where $h = (c_1 + c_2)^{1/2} < 1$.
Similarly, if $u > 0$ then $\phi(u, v, u, v, (1/2)(u+v), u+v) \leq 0$ imply $u \leq hv$. If $u = 0$, then $u \leq hv$.

ϕ_3 : $\phi(u, u, 0, 0, 0, u) = u^2(1 - c_1) > 0$, for all $u > 0$.

3. Main Results

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and $A, B, S, T : X \to X$ be four mappings satisfying the conditions:*

$$\phi\,(\|Ax - By\|, \|Sx - Ty\|, \|Sx - Ax\|, \|Ty - By\|, (1/2)(\|Sx - Ax\| + \|Ty - By\|), \|Ty - Ax\|) \leq 0 \tag{3.1}$$

for all x, y in X, where $\phi \in \Phi$,

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.2}$$

$$\text{one of } A, B, S, T \text{ is continuous}, \tag{3.3}$$

$$\{A, S\} \text{ and } \{B, T\} \text{ are compatible pairs}. \tag{3.4}$$

Then A, B, S and T have a unique common fixed point.

Proof. By (3.2), since $A(X) \subset T(X)$, for an arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point $x_1 \in X$, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for every $n = 0, 1, 2, \ldots$.
By (3.1), we have

$$\phi\,(\|Ax_{2n} - Bx_{2n+1}\|, \|Sx_{2n} - Tx_{2n+1}\|, \|Sx_{2n} - Ax_{2n}\|, \|Tx_{2n+1} - Bx_{2n+1}\|,$$
$$(1/2)(\|Sx_{2n} - Ax_{2n}\| + \|Tx_{2n+1} - Bx_{2n+1}\|), \|Tx_{2n+1} - Ax_{2n}\|) \le 0,$$

$$\phi\,(\|Ax_{2n} - Bx_{2n+1}\|, \|Bx_{2n-1} - Ax_{2n}\|, \|Bx_{2n-1} - Ax_{2n}\|,$$
$$\|Ax_{2n} - Bx_{2n+1}\|, (1/2)(\|Bx_{2n-1} - Ax_{2n}\| + \|Ax_{2n} - Bx_{2n+1}\|), 0) \le 0.$$

By (ϕ_a), we have

$$\|Ax_{2n} - Bx_{2n+1}\| \le h\|Bx_{2n-1} - Ax_{2n}\|.$$

Similarly by (ϕ_b) and ϕ_1, we have

$$\|Ax_{2n} - Bx_{2n-1}\| \le h\|Ax_{2n-2} - Bx_{2n-1}\|.$$

and so

$$\|Ax_{2n} - Bx_{2n-1}\| \le h^{2n}\|Ax_0 - Bx_1\| \text{ for } n = 0, 1, 2, \ldots$$

.

By a routine calculation it follows that $\{y_n\}$ is a Cauchy sequence in X and hence it converges to a point z in X. Consequently, subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point z.

Let us now suppose that S is continuous, so the sequence $\{SAx_{2n}\}$ converges to $\{Sz\}$. We have

$$\|ASx_{2n} - Sz\| \le \|ASx_{2n} - SAx_{2n}\| + \|SAx_{2n} - Sz\|.$$

Since S is continuous and A and S are compatible, letting n tends to infinity, we state that the sequence $\{ASx_{2n}\}$ also converges to Sz. Using (3.1), we have

$$\phi\,(\|ASx_{2n} - Bx_{2n+1}\|, \|SSx_{2n} - Tx_{2n+1}\|, \|SSx_{2n} - ASx_{2n}\|, \|Tx_{2n+1} - Bx_{2n+1}\|,$$
$$(1/2)(\|SSx_{2n} - ASx_{2n}\| + \|Tx_{2n+1} - Bx_{2n+1}\|), \|Tx_{2n+1} - ASx_{2n}\|) \le 0.$$

Letting n tends to infinity we have by the continuity of ϕ,

$$\phi(\|Sz - z\|, \|Sz - z\|, 0, 0, 0, \|z - Sz\|) \leq 0,$$

which is a contradiction to ϕ_3 if $\|z - Sz\| \neq 0$. Thus $Sz = z$. Further by (3.1), we have

$$\phi\ (\|Az - Bx_{2n+1}\|, \|Sz - Tx_{2n+1}\|, \|Sz - Az\|, \|Tx_{2n+1} - Bx_{2n+1}\|,$$
$$(1/2)(\|Sz - Az\| + \|Tx_{2n+1} - Bx_{2n+1}\|), \|Tx_{2n+1} - Az\|) \leq 0.$$

Letting n tends to infinity we have by the continuity of ϕ,

$$\phi(\|Az - z\|, 0, \|z - Az\|, 0, (1/2)\|z - Az\|, \|z - Az\|) \leq 0,$$

which implies by (ϕ_b), that $Az = z$. This means that z is in the range of A and since $A(X) \subset T(X)$, there exists a point u in X such that $Tu = z$.

Using (3.1), we have

$$\begin{aligned}&\phi\ (\|Az - Bu\|, \|Sz - Tu\|, \|Sz - Az\|, \|Tu - Bu\|,\\ &\quad (1/2)(\|Sz - Az\| + \|Tu - Bu\|), \|Tu - Az\|) \leq 0\\ &= \phi(\|z - Bu\|, 0, 0, \|z - Bu\|, (1/2)\|z - Bu\|, 0) \leq 0,\end{aligned}$$

which implies by (ϕ_a), that $z = Bu$.

Since $Tu = Bu = z$, by Lemma 1, it follows that $BTu = TBu$ and so $Bz = BTu = TBu = Tz$.

Thus from (3.1), we have

$$\begin{aligned}&\phi\ (\|Az - Bz\|, \|Sz - Tz\|, \|Sz - Az\|, \|Tz - Bz\|,\\ &\quad (1/2)(\|Sz - Az\| + \|Tz - Bz\|), \|Tz - Az\|) \leq 0\\ &= \phi(\|z - Tz\|, \|z - Tz\|, 0, 0, 0, \|Tz - z\|) \leq 0,\end{aligned}$$

which is contradiction to ϕ_3 if $||Tz - z|| \neq 0$. Thus $Tz = z = Bz$.

We have therefore, proved that z is a common fixed point of A, B, S and T. The same result holds if T is continuous instead of S. Now suppose that A is continuous. Then the sequence $\{ASx_{2n}\}$ converges to Az we have

$$\|SAx_{2n} - Az\| \leq \|SAx_{2n} - ASx_{2n}\| + \|ASx_{2n} - Az\|.$$

Since A is continuous and A and S are compatible, letting n tends to infinity we obtain that $\{SAx_{2n}\}$ converges to Az. Using (3.1), we have

$$\phi\ (\|AAx_{2n} - Bx_{2n+1}\|, \|SAx_{2n} - Tx_{2n+1}\|, \|SAx_{2n} - AAx_{2n}\|, \|Tx_{2n+1} - Bx_{2n+1}\|,$$
$$(1/2)(\|SAx_{2n} - AAx_{2n}\| + \|Tx_{2n+1} - Bx_{2n+1}\|), \|Tx_{2n+1} - AAx_{2n}) \leq 0.$$

Letting n tends to infinity, we have by continuity of ϕ

$$\phi(\|Az - z\|, \|Az - z\|, 0, 0, 0, \|z - Az\|) \leq 0,$$

a contradiction to ϕ_3 if $z \neq Az$. Thus $z = Az$.

This means that z is in the range of A and since $A(X) \subset T(X)$, there exists a point v in X such that $Tv = z$. Thus by (3.1), we have

$$\phi\,(\|AAx_{2n} - Bv\|, \|SAx_{2n} - Tv\|, \|SAx_{2n} - AAx_{2n}\|, \|Tv - Bv\|, \\ (1/2)(\|SAx_{2n} - AAx_{2n}\| + \|Tv - Bv\|), \|Tv - AAx_{2n}\|) \leq 0.$$

Letting n tends to infinity we get

$$\phi(\|z - Bv\|, 0, 0, \|z - Bv\|, (1/2)\|z - Bv\|, 0) \leq 0$$

and by (ϕ_a) it follows that $z = Bv$. Since $Tv = Bv = z$, by Lemma 1, it follows that $Bz = BTv = TBv = Tz$. Thus from (3.1) we have

$$\phi\,(\|Ax_{2n} - Bz\|, \|Sx_{2n} - Tz\|, \|Sx_{2n} - Ax_{2n}\|, \|Tz - Bz\|, \\ (1/2)(Sx_{2n} - Ax_{2n}\| + \|Tz - Bz\|), \|Tz - Ax_{2n}) \leq 0.$$

Letting n tends to infinity, we obtain

$$\phi(\|z - Bz\|, \|z - Bz\|, 0, 0, 0, \|Bz - z\|) \leq 0,$$

which is contradiction to ϕ_3, it follows that $z = Bz = Tz$. This means that z is in the range of B and since $B(X) \subset S(X)$ there exists $w \in X$ such that $Sw = z$. Thus from (3.1) we have

$$\phi\,(\|Aw - Bz\|, \|Sw - Tz\|, \|Sw - Aw\|, \|Tz - Bz\|, \\ (1/2)(\|Sw - Aw\| + \|Tz - Bz\|), \|Tz - Aw\|) \leq 0. \\ \phi\,(\|Aw - z\|, 0, \|z - Aw\|, 0, (1/2)\|z - Aw\|, \|z - Aw\|) \leq 0,$$

and by (ϕ_b), we have $z = Aw = Sw$. Since $Aw = Sw = z$, by Lemma 1, it follows that $SAw = ASw$ and so $Sz = SAw = ASw = Az = z$. We have therefore proved that z is a common fixed point of A, B, S and T.

The same result holds, if we assume that B is continuous instead of A. By (3.1) and ϕ_3 it follows that z is unique.

Theorem 3.1 and Examples 1 to 3 imply the following:

Corollary 3.2. *Let* $(X, \|\cdot\|)$ *be a Banach space and* A, B, S *and* $T : X \to X$ *be four mappings satisfying the conditions* (3.2) *to* (3.4) *and the following :*

$$\|Ax - By\| \leq k \max\{\|Sx - Ty\|, \|Sx - Ax\|, \|Ty - By\|, \\ (1/2)(\|Sx - Ax\| + \|Ty - By\|), (1/2)\|Ty - Ax\|\} \qquad (3.5)$$

for all x, y *in* X, *where* $k \in (0, 1)$,
or

$$\|Ax - By\|^2 \leq a\|Sx - Ty\|^2 \qquad (3.6)$$

for all x, y in X, where $a \in (0,1)$,
or

$$
\begin{aligned}
\|Ax - By\|^2 \le c_1 \max\{&\|Sx - Ty\|^2, \|Sx - Ax\|^2, \|Ty - By\|^2\} \\
&+ c_2 \max\{\|Sx - Ax\|(1/2)(\|Sx - Ax\| + \|Ty - By\|), \\
&(1/2)\|Ty - By\| \, \|Ty - Ax\|\} \qquad (3.7)
\end{aligned}
$$

for all x, y in X, where $c_1 > 0,\ c_2 \ge 0,\ c_1 + c_2 < 1$.

Then A, B, S and T have a unique common fixed point.

We now give an example to illustrate the above results:

Example 4. Consider $X = [1, 15]$ with the usual norm.
Define A, B, S and T by

$$Ax = 1 \text{ if } x \in [1, 15]$$

$$Bx = \begin{cases} 1 & \text{if } x = 1,\ x > 3 \\ 2 & \text{if } 1 < x \le 3 \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } x = 1,\ x > 3, \quad x \ne 12 \\ 2 & \text{if } x = 12 \\ 15 & \text{if } 1 < x \le 3 \end{cases}$$

$$Tx = \begin{cases} 1 & \text{if } x = 1,\ x > 3 \\ 5 & \text{if } 1 < x \le 3 \end{cases}$$

Then A, B, S and T satisfy condition (3.2) of Theorem 3.1. In this example only mapping A is continuous and so condition (3.3) is satisfied.

Let us consider a decreasing sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 3$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$ and $\lim_{n\to\infty} \|ASx_n - SAx_n\| = 0$. Thus the pair $\{A, S\}$ is compatible. Similarly the pair $\{B, T\}$ is also compatible.

We see that condition (3.1) is satisfied if ϕ is similar to that of Example 1 or Example 2 or Example 3. Here we take $k \in [0.5, 1)$ in Example 1, $a \in [0.3, 1)$ in Example 2 and $c_1 = 0.8,\ c_2 = 0.1$ in Example 3.

Thus A, B, S and T satisfy all the conditions of Theorem 3.1 and Corollary 3.2 and have a unique common fixed point $x = 1$.

For a function $f : (X, \|\cdot\|) \to (X, \|\cdot\|)$ we denote $F_f = \{x \in X : x = f(x)\}$.

Theorem 3.3. *Let A, B, S and T be mappings from a normed linear space $(X, \|\cdot\|)$ into itself. If the inequality (3.1) holds for x, y in X then $(F_S \cap F_T) \cap F_A = (F_S \cap F_T) \cap F_B$.*

Proof. Let $x \in (F_S \cap F_T) \cap F_A$. Then by (3.1), we have

$$
\begin{aligned}
&\phi\,(\|Ax - Bx\|, \|Sx - Tx\|, \|Sx - Ax\|, \|Tx - Bx\|, \\
&\quad (1/2)(\|Sx - Ax\| + \|Tx - Bx\|), \|Tx - Ax\|) \le 0. \\
&= \phi(\|x - Bx\|, 0, 0, \|x - Bx\|, (1/2)\|x - Bx\|, 0) \le 0,
\end{aligned}
$$

which implies, by (ϕ_a) that $x = Bx$. Thus

$$(F_S \cap F_T) \cap F_A \subset (F_S \cap F_T) \cap F_B.$$

Similarly, we have by (ϕ_b), that

$$(F_S \cap F_T) \cap F_B \subset (F_S \cap F_T) \cap F_A.$$

Remark. Theorem 3.3 is true if we replace the condition (3.1) by (3.5) or (3.6) or (3.7).

Theorem 3.1 implies following one.

Theorem 3.4. *Let S, T and $\{A_i\}_{i \in N}$ be mappings from a Banach space $(X, \|\cdot\|)$ into itself such that*

$$A_2(X) \subset S(X) \text{ and } A_1(X) \subset T(X), \tag{3.8}$$

$$\text{one of } S, T, A_1 \text{ and } A_2 \text{ is continuous}, \tag{3.9}$$

$$\text{the pairs } (A_1, S) \text{ and } (A_2, T) \text{ are compatible}, \tag{3.10}$$

the inequality

$$\begin{aligned} &\phi\left(\|A_i x - A_{i+1} y\|, \|Sx - Ty\|, \|Sx - A_i x\|, (\|Ty - A_{i+1} y\|,\right. \\ &\quad (1/2)((\|Sx - A_i x\| + \|Ty - A_{i+1} y\|), \|Ay - A_i x\|) \leq 0 \end{aligned} \tag{3.11}$$

holds for each $x, y \in X$, for all $i \in N$ and $\phi \in \Phi$. Then S, T and $\{A_i\}_{i \in N}$ have a unique common fixed point.

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