Approximation of Functions in Besov Space

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Abstract. In the present paper, we establish a theorem on best approximation of a function \( g \in B^{p,q}_{λ}(L^r) \) of its Fourier series. Our main theorem generalizes some known results of this direction of work. Thus, the results of [10], [26] and [27] become the particular case of our main Theorem 3.1.

1 Introduction

The degree of approximation of the functions in Lipschitz spaces and Hölder spaces using single and product summability means has been studied by the authors [7, 8, 9, 15, 16, 18, 19, 20, 21, 22, 23, 25, 28]. This motivates us to study the degree of approximation of a function in more generalized function space. Therefore, in this paper, we study the degree of approximation of a function \( g \) in Besov space using Hausdorff-generalized Nörlund means of its Fourier series. It can be noted that Besov space generalizes different Sobolev spaces, Lipschitz spaces and generalized Hölder spaces [13]. Besov space can also be used to study regularity properties of the functions.

Let \( C_{2π} := C[0, 2π] \) denote the Banach space of all \( 2π \)-periodic continuous functions \( g \) defined on \([0, 2π]\) under the supremum norm.

Let

\[
L^r := L^r[0, 2π] := \left\{ g : [0, 2π] \mapsto \mathbb{R} : \int_0^{2π} |g(z)|^r dz < \infty \right\}, \quad r \geq 1
\]

be the space of all \( 2π \)-periodic, integrable functions \( L^r \)-norm of the function \( g \) is given by

\[
\|g\|_r := \begin{cases} \frac{1}{2π} \int_0^{2π} |g(z)|^r dz \quad & \text{for } 1 \leq r < \infty \\ \text{ess sup}_{0<z<2π} |g(z)| & \text{for } r = \infty. \end{cases}
\]

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The modulus of continuity of a function \( g \in L^r \) is defined by
\[
w(g; l) := \sup_{z, z+h \in [0,2\pi], |h|<l} |g(z + h) - g(z)|.
\]
(1.1)

The integral modulus of continuity of the first order of a function \( g \in L^r \) is defined \([3]\) by
\[
w_1(g; l)_r := \sup_{|h|<l, z \in \mathbb{R}} \|g(z + h) - g(z)\|_r.
\]
(1.2)

The integral moduli of continuity of the second order (modulus of smoothness) of a function \( g \in L^r \) is defined \([2]\) by
\[
w_2(g; l)_r = \sup_{0<h \leq l, z \in \mathbb{R}} \|g(z + h) + g(z - h) - 2g(z)\|_r.
\]
(1.3)

The \( j^{th} \) order modulus of smoothness of a function \( g \in L^r \) is defined \([1]\) by
\[
w_j(g; l)_r := \sup_{0<h \leq l} \|\Delta_h^j(g, \cdot )\|_r, \quad l > 0,
\]
(1.4)
where
\[
\Delta_h^j(g, z) := \sum_{\rho=0}^{j} (-1)^{j-\rho} \binom{j}{\rho} g(z + \rho h), \quad j \in \mathbb{N}.
\]
(1.5)

Remark 1.

(i) If \( r = \infty, j = 1 \) and \( g \) being a continuous function, then \( w_j(g; l)_r \) reduces to \( w(g; l) \).

(ii) If \( 0 < r < \infty, j = 1 \) and \( g \) being a continuous function, then \( w_j(g; l)_r \) reduces to \( w_1(g; l)_r \).

(iii) If \( g \in C_{2\pi} \) and \( w(g, l) = O(l^{\lambda}), 0 < \lambda \leq 1 \), then \( g \in \text{Lip } \lambda \).

(iv) If \( g \in L^r, 0 < r < \infty \) and \( w(g, l)_r = O(l^{\lambda}), 0 < \lambda \leq 1 \), then \( g \in \text{Lip}(\lambda, r) \).

(v) If \( r = \infty \), then \( \text{Lip}(\lambda, r) \) class reduces to \( \text{Lip } \lambda \).

Note 1. From Remark 1(iv) and 1(v), we write
\[
\text{Lip}(\lambda) \subseteq \text{Lip}(\lambda, r).
\]

Let \( \lambda > 0, j > \lambda \) i.e. \( j = [\lambda] + 1 \), where \( j \) being smallest integer. For \( g \in L^r \), if
\[
w_j(g, l)_r = O(l^{\lambda}), \quad l > 0,
\]
(1.6)
then \( g \in \text{Lip}^*(\lambda, r) \) and its semi-norm is given by

\[
|g|_{\text{Lip}^*(\lambda, r)} = \sup_{l > 0} (l^{-\lambda} w_j(g, l)r),
\]

where \( \text{Lip}^*(\lambda, r) \) is a generalized Lipschitz class of function \( g \).

Thus,

\[
\text{Lip}(\lambda, r) \subseteq \text{Lip}^*(\lambda, r).
\]

For \( 0 < \lambda \leq 1 \), let

\[
\text{H}_\lambda := \{ g \in C_{2\pi} : w(g, l) = O(l^\lambda) \},
\]

where \( \text{H}_\lambda \) is a Banach space with the norms

\[
\|g\|_\lambda = \|g\|_C + \sup_{l > 0} (l^{-\lambda} w(l)) \quad \text{for} \quad 0 < \delta \leq \lambda < 1
\]

and

\[
\|g\|_0 = \|g\|_C.
\]

Thus, we observe that

\[
\text{H}_\lambda \subseteq \text{H}_\delta \subseteq C_{2\pi} \quad \text{for} \quad 0 < \delta \leq \alpha < 1([28]).
\]

The metric induced by the norm \( \| \cdot \|_\lambda \) on \( \text{H}_\lambda \) is called the Hölder metric.

For \( 0 < \lambda \leq 1 \), \( 0 < r \leq \infty \), let

\[
\text{H}_{\lambda, r} := \text{H}_{\lambda, [0, 2\pi]} = \{ g \in L^r : w(g, l)r = O(l^\lambda) \},
\]

where \( \text{H}_{\lambda, r} \) is also a Banach space with the norm \( \| \cdot \|_{\lambda, r} \) defined by

\[
\|g\|_{\lambda, r} = \|g\|_r + \sup_{l > 0} (l^{-\lambda} w(g, l)r) \quad \text{for} \quad 0 < \lambda \leq 1
\]

and

\[
\|g\|_{0, r} = \|g\|_r.
\]

Then \( \text{H}_{\lambda, r} \) is a Banach space for \( r \geq 1 \) and a complete \( r \)-normed space ([14], p. 87) for \( 0 < r < 1 \).

Thus,

\[
\text{H}_{\lambda, r} \subseteq \text{H}_{\delta, r} \subseteq L^r \quad \text{for} \quad 0 < \delta \leq \lambda \leq 1([9]).
\]
For $\lambda > 0$ and let $j > \lambda$ i.e. $j = [\lambda] + 1$. For $0 < r, q \leq \infty$, the Besov space $B^\lambda_q(L^r)$ is a collection of all $2\pi$-periodic function $g \in L^r$ such that

$$|g|_{B^\lambda_q(L^r)} := \|w_j(g, \cdot)\|_{\lambda,q} = \begin{cases} \left( \int_0^{2\pi} [l^{-\lambda} w_j(g, l)]^q \frac{dl}{l} \right)^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup_{l>0} (l^{-\lambda} w_j(g, l)), & q = \infty, \end{cases}$$

(1.7)

is finite ([24], p. 237) for $2\pi$-periodic function $g$ ([1], p. 54).

It is further observed that (1.7) is a semi-norm if $1 \leq r, q \leq \infty$ and a quasi semi-norm in other cases ([1], p. 55).

The quasi-norm for Besov space is given by

$$\|g\|_{B^\lambda_q(L^r)} := \|g\|_r + |g|_{B^\lambda_q(L^r)} = \|g\|_r + \|w_j(g, \cdot)\|_{\lambda,q}.$$  

**Note 2.**

(i) If $0 < \lambda < 1$, then the Besov space $B^\lambda_\infty(L^r)$ reduces to the $H_{\lambda,r}$ ([9]).

(ii) If $r = \infty = q$ and $0 < \lambda < 1$, the Besov space $B^\lambda_\infty(L^r)$ reduces to the space $H_\lambda$ ([28]).

The $m$-order error approximation of a function $g \in C_{2\pi}$ is defined by $E_m(g) := \inf_{t_m} \|g - t_m\|$ where $t_m$ is a trigonometric polynomial of degree $m$ ([3]).

If $E_m(g) \to 0$ as $m \to \infty$, $E_m(g)$ is said to be the best approximation of $g$ ([3]).

## 2 Definitions

The Hausdorff matrix $H \equiv (h_{m,j})$ is an infinite lower triangular matrix defined by,

$$h_{m,j} = \begin{cases} \binom{m}{j} \Delta^{m-j} \mu_j, & 0 \leq j \leq m, \\ 0, & j > m, \end{cases}$$

where $\Delta$ is a forward operator defined by $\Delta \mu_m = \mu_m - \mu_{m+1}$ and $\Delta^{j+1} \mu_m = \Delta(\Delta^j \mu_m)$ ([6]).

A Hausdorff matrix $H$ is regular iff $\int_0^1 |d\gamma(y)| < \infty$, where the mass function $\gamma(y)$ is continuous at $y = 0$ and belongs to $BV[0,1]$ such that $\gamma(0+) = 0, \gamma(1) = 1$; and for $0 < y < 1$, $\gamma(y) = [\gamma(y+0) + \gamma(y-0)]/2$ [4, 11]. Thus $\{\mu_m\}$, known as moment sequence, has the representation

$$\mu_m = \int_0^1 y^m d\gamma(y).$$
The Hausdorff means of a trigonometric Fourier series of \( g \) is defined by

\[
H_m(g; z) = \sum_{j=0}^{m} h_{m,j} s_j(g; z), \quad \forall \ m \geq 0.
\]

The series is said to be summable to \( s \) by Hausdorff means, if \( H_m(g; z) \to s \) as \( m \to \infty \) and we denote Hausdorff means by \( \Delta_H \) throughout the paper.

**Example 1.**

(i) If

\[
h_{m,j} = \begin{cases} 
\binom{m}{j} \frac{q^{m-j}}{(1+q)^m}, & 0 \leq j \leq m, \\
0, & j > m,
\end{cases}
\]

then the Hausdorff matrix \( H \equiv (h_{m,j}) \) reduces to \( (E, q) \) matrix (Euler matrix of order \( q > 0 \)) and defines the corresponding \( (E, q) \) means by

\[
E_m^q(g; z) := \frac{1}{(1+q)^m} \sum_{j=0}^{m} \binom{m}{j} q^{m-j} s_j(g; z).
\]

(ii) If \( \mu_m = \frac{1}{m+1} \) then the Hausdorff matrix \( H \equiv (h_{m,j}) \) reduces to \( (C, 1) \) matrix (Cesàro matrix of order 1) and defines the corresponding means by

\[
H_m(g; z) := \frac{1}{(m+1)} \sum_{j=0}^{m} s_j(g; z).
\]

Let \( \{p_m\} \) and \( \{q_m\} \) be the sequence of constants, real or complex, such that

\[
P_m = p_0 + p_1 + p_2 + \cdots + p_m = \sum_{v=0}^{m} p_v \to \infty, \text{ as } m \to \infty
\]

\[
Q_m = q_0 + q_1 + q_2 + \cdots + q_m = \sum_{v=0}^{m} q_v \to \infty, \text{ as } m \to \infty
\]

\[
R_m = p_0 q_m + p_1 q_{m-1} + p_2 q_{m-2} + \cdots + p_m q_0 = \sum_{v=0}^{m} p_v q_{m-v} \to \infty, \text{ as } m \to \infty.
\]

Given two sequences \( \{p_m\} \) and \( \{q_m\} \) convolution \( (p \ast q) \) is defined as

\[
R_m = (p \ast q)_m = \sum_{j=0}^{m} p_{m-j} q_j.
\]

We write

\[
\mu_m^{p,q} = \frac{1}{R_m} \sum_{j=0}^{m} p_{m-j} q_j s_j.
\]
If $R_m \neq 0$, for all $m$, the generalized Nörlund transform of the sequence $\{s_m\}$ is the sequence $\{t_{m}^{p,q}\}$. If $t_{m}^{p,q} \rightarrow s$, as $m \rightarrow \infty$, then the series $\sum_{m=0}^{\infty} a_m$ or sequence $\{s_m\}$ is summable to $s$ by generalized Nörlund method and is denoted by $s_m \rightarrow s(N^{p,q})$.

The necessary and sufficient condition for $(N^{p,q})$ method to be regular are

$$\sum_{j=0}^{m} |p_{m-j}q_j| = O(|R_m|)$$

and $p_{m-j} = o(|R_m|)$, as $m \rightarrow \infty$ for every fixed $j \geq 0$, for which $q_j \neq 0$ ([17]).

If the method $\Delta_H$ is superimposed on the $N^{p,q}$ method, another new method of summability $\Delta_H N^{p,q}$ is obtained.

The Hausdorff transform of $N^{p,q}$ transform is defined as $\Delta_H N^{p,q}$ product transform of the partial sum $s_m$, which can be given by

$$t_{m}^{\Delta_H N^{p,q}} = \sum_{j=0}^{m} h_{m,j} t_{j}^{p,q} = \sum_{j=0}^{m} h_{m,j} \frac{1}{R_j} \sum_{v=0}^{j} p_{j-v} q_v s_v.$$  

If $t_{m}^{\Delta_H N^{p,q}} \rightarrow s$ as $m \rightarrow \infty$ then the series $\sum_{m=0}^{\infty} a_m$ or the sequence $\{s_m\}$ is summable to $s$ by $\Delta_H N^{p,q}$ means.

Now, we define the regularity of $\Delta_H N^{p,q}$ method.

$$s_m \rightarrow s \implies t_{m}^{p,q} \rightarrow s, \quad \text{as} \quad m \rightarrow \infty \quad \text{so} \quad N^{p,q} \quad \text{method is regular},$$

$$\implies \Delta_H (t_{m}^{p,q}) = t_{m}^{\Delta_H N^{p,q}} \rightarrow s \quad \text{as} \quad m \rightarrow \infty \quad \text{so} \quad \Delta_H \quad \text{method is regular},$$

$$\implies (\Delta_H N^{p,q}) \quad \text{method is regular}.$$

**Remark 2.**

(i) $\Delta_H N^{p,q}$ means reduces to $E_q N^{p,q}$ means if $h_{m,j} = \begin{cases} \binom{m}{j} \frac{q^{m-j}}{(1+q)^m}, & \text{if } 0 \leq j \leq m, \\ 0, & \text{if } j > m. \end{cases}$

(ii) $\Delta_H N^{p,q}$ means reduces to $C_1 N^{p,q}$ means if $h_{m,j} = \begin{cases} \frac{1}{m+1}, & \text{if } 0 \leq j \leq m, \\ 0, & \text{if } j > m. \end{cases}$

(iii) $\Delta_H N^{p,q}$ means reduces to $\Delta_H N^{p_m}$ means if $q_m = 1, \forall m$.

(iv) $\Delta_H N^{p,q}$ means reduces to $\Delta_H \tilde{N}_m^{p_m}$ means if $p_m = 1, \forall m$. 
\(\Delta_H N^{p,q}\) means reduces to \(\Delta_H C_\alpha\) means if \(p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right), \alpha > 0\) and \(q_m = 1, \forall m\).

**Remark 3.** The above cases (i) and (ii) of Remark 2 can be further reduced as

(i) \(E_q N^{p,q}\) means reduces to \(E_q N^{pm}\) means if \(q_m = 1, \forall m\).

(ii) \(E_q N^{p,q}\) means reduces to \(E_q N^{pm}\) means if \(q_m = 1, \forall m\).

(iii) \(E_q N^{p,q}\) means reduces to \(E_q \tilde{N}^{q_m}\) means if \(p_m = 1, \forall m\).

(iv) \(E_q N^{p,q}\) means reduces to \(E_q C_\alpha\) means if \(p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right), \alpha > 0\) and \(q_m = 1, \forall m\).

(v) \(C_1 N^{p,q}\) means reduces to \(C_1 N^{pm}\) means if \(p_m = 1, \forall m\).

(vi) \(C_1 N^{p,q}\) means reduces to \(C_1 \tilde{N}^{q_m}\) means if \(p_m = 1, \forall m\).

### 3 Main Theorems

**Theorem 3.1.** If \(g\) is a 2\(\pi\)-periodic and Lebesgue integrable function, then for \(0 \leq \delta < \lambda < 2\), the best approximation of \(g\) in \(B_\lambda^q(L^r), r \geq 1, 1 < q \leq \infty\) space using \(\Delta_H N^{p,q}\) means, is given by

\[
\|t_m^{\Delta_H N^{p,q}}(z) - g(z)\| = \begin{cases} 
O(m+1)^{-\frac{\lambda}{(m+1)^{\lambda-s\frac{1}{1/q}}}} + O\left(\frac{1}{(m+1)^{\lambda-s}}\right) & ; 1 < q < \infty \\
O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-s}}\right) & ; q = \infty.
\end{cases}
\]

### 4 Lemmas

**Lemma 4.1.** Let \(K_m^{\Delta_H N^{p,q}}(\eta) := \int_0^1 M_m(\eta) d\gamma(y)\) where

\[
M_m(\eta) := \left[\sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{\frac{1}{R_j} \sum_{\nu=0}^j p_{j-\nu} q_{\nu} \sin \left(\nu + \frac{1}{2}\right) \eta \right\}\right],
\]

then

\[
K_m^{\Delta_H N^{p,q}}(\eta) = \begin{cases} 
O(m+1), & 0 \leq \eta \leq \frac{1}{(m+1)}; \\
O\left(\frac{1}{\eta}\right), & \frac{1}{(m+1)} \leq \eta \leq \pi.
\end{cases}
\]
Proof. For $0 \leq \eta \leq \frac{1}{m+1}$, we have $m\eta \leq m \sin \eta$, then

\[ M_m(\eta) \]

\[ = \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin(\nu + \frac{1}{2})\eta}{2 \sin \frac{\eta}{2}} \right\} \]

\[ \leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{(2\nu + 1) \sin \frac{\eta}{2}}{\sin \frac{\eta}{2}} \right\} \]

\[ \leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu}(2\nu + 1) \right\} \]

\[ = \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} \left\{ p_j q_0 (2\cdot 0 + 1) + p_{j-1} q_1 (2\cdot 1 + 1) + \cdots + p_0 q_j (2\cdot j + 1) \right\} \]

\[ \leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} \left\{ p_j q_0 (2\cdot j + 1) + p_{j-1} q_1 (2\cdot j + 1) + \cdots + p_0 q_j (2\cdot j + 1) \right\} \]

\[ = \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} (2j + 1) \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \]

\[ = \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} (2j + 1) O(\vert R_j \vert) \]

\[ = O \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} (2j + 1) \right] \]

\[ = O \left[ 2 \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j + \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \right]. \] \hspace{1cm} (4.1)

Now, solving first term of (4.1),

\[ \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j = (1-y)^m \sum_{j=0}^{m} \binom{m}{j} \left( \frac{y}{1-y} \right)^j \cdot j \]

\[ = (1-y)^m \sum_{j=0}^{m} \binom{m}{j} p^j \cdot j, \]

where \( \frac{y}{1-y} = p \Rightarrow \frac{1}{1+p} = \frac{1}{1-y}. \)

\[ \sum_{j=0}^{m} \binom{m}{j} p^j \cdot j = \binom{m}{0} p^0 \cdot 0 + \binom{m}{1} p^1 \cdot 1 + \binom{m}{2} p^2 \cdot 2 + \cdots + \binom{m}{m} p^m \cdot m. \]
\[ = \binom{m}{1} p + 2 \binom{m}{2} p^2 + 3 \binom{m}{3} p^3 + \cdots + m \binom{m}{m} p^m. \quad (4.2) \]

We know that

\[ (1 + p)^m = \binom{m}{0} + \binom{m}{1} \cdot p + \binom{m}{2} \cdot p^2 + \cdots + \binom{m}{m} \cdot p^m. \]

By differentiating with respect to \( p \), we have

\[ m(1 + p)^{m-1} = 0 + \binom{m}{1} \cdot 2p + \binom{m}{2} \cdot 3p^2 + \cdots + \binom{m}{m} \cdot mp^{m-1}. \]

Multiplying above equation by \( p \) on both side, we have

\[ mp(1 + p)^{m-1} = \binom{m}{1} p + 2 \binom{m}{2} p^2 + 3 \binom{m}{3} p^3 + \cdots + m \binom{m}{m} p^m. \quad (4.3) \]

Now, from (4.2) and (4.3), we have

\[ \sum_{j=0}^{m} \binom{m}{j} \frac{y^j}{1 - y} \cdot j = mp(1 + p)^{m-1} \]

\[ (1 - y)^m \sum_{j=0}^{m} \binom{m}{j} \frac{y^j}{1 - y} \cdot j = (1 - y)^m \left\{ mp(1 + p)^{m-1} \right\} \]

\[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \cdot j = (1 - y)^m \left\{ m \cdot \frac{y}{1 - y} \cdot \frac{1}{(1 - y)^{m-1}} \right\} \]

\[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \cdot j = my. \quad (4.4) \]

Now, solving second term of (4.1),

\[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \]

\[ = \binom{m}{0} y^0 (1 - y)^m + \binom{m}{1} y^1 (1 - y)^{m-1} + \cdots + \binom{m}{m} y^m (1 - y)^{m-m} \]

\[ = (1 - y + y)^m = 1. \quad (4.5) \]

Now, from (4.1), (4.4) and (4.5), we get

\[ M_m(\eta) = O (2my + 1). \]
Thus,

\[ K_m^{\Delta H^{N_p,q}}(\eta) = \int_0^1 M_m(\eta)d\gamma(y) \]

\[ = O(1) \int_0^1 (2my + 1) \, dy \]

\[ = O(m + 1). \]

For \( \frac{1}{m+1} \leq \eta \leq \pi \), by Jordan's lemma we have, \( \sin \frac{\eta}{2} \geq \frac{n \eta}{\pi} \) and \( \sin n \eta \leq 1 \). Thus,

\[ M_m(\eta) = \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^j p_{j-\nu} q_{\nu} \frac{\sin(\nu + \frac{1}{2})\eta}{2 \sin \frac{\eta}{2}} \right\} \right] \]

\[ \leq \frac{1}{2} \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^j p_{j-\nu} q_{\nu} \frac{1}{\frac{\eta}{\pi}} \right\} \right] \]

\[ \leq \frac{\pi}{2\eta} \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^j p_{j-\nu} q_{\nu} \right\} \right] \]

\[ = \frac{\pi}{2\eta} \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} O(|R_j|) \right\} \right] \]

\[ = \frac{\pi}{2\eta} \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \{ O(1) \} \right] \]

\[ = O \left( \frac{\pi}{2\eta} \right) \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \right] \]

\[ = O \left( \frac{1}{\eta} \right) \text{ since } \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} = 1. \]

Thus,

\[ K_m^{\Delta H^{N_p,q}}(\eta) = \int_0^1 M_m(\eta)d\gamma(y) \]

\[ = \int_0^1 O \left( \frac{1}{\eta} \right) \, dy \]

\[ = O \left( \frac{1}{\eta} \right) \int_0^1 \, dy \]

\[ = O \left( \frac{1}{\eta} \right). \]
Lemma 4.2. ([12]) Let \(1 \leq r \leq \infty\) and \(0 < \lambda < 2\). If \(g \in L^r\) then for \(0 < l, \eta \leq \pi\):

(i) \(\|\Phi(\cdot, l, \eta)\|_r \leq 4w_j(g, l)_r\),

(ii) \(\|\Phi(\cdot, l, \eta)\|_r \leq 4w_j(g, \eta)_r\),

(iii) \(\|\Phi(\eta)\|_r \leq 2w_j(g, \eta)_r\),

where \(j = [\lambda] + 1\).

Lemma 4.3. Let \(0 \leq \delta < \lambda < 2\). If \(g \in B^{\lambda}_{1q}(L^r), r \geq 1, 1 < q < \infty\), then

\[
\begin{align*}
(i) & \quad \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| \left( \int_0^\pi \frac{||\Phi(\cdot, l, \eta)||_r^q}{l^{s_q}} \frac{dl}{l} \right)^{\frac{1}{q}} d\eta = \frac{O(1)}{\int_0^\pi (\eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)|)^{\frac{q}{q-1}} d\eta} \left( \int_0^\pi \frac{||\Phi(\cdot, l, \eta)||_r^q}{l^{s_q}} \frac{dl}{l} \right)^{\frac{1}{q}} \eta^{\delta}\, d\eta, \\
(ii) & \quad \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| \left( \int_\eta^\pi \frac{||\Phi(\cdot, l, \eta)||_r^q}{l^{s_q}} \frac{dl}{l} \right)^{\frac{1}{q}} d\eta = \frac{O(1)}{\int_0^\pi (\eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)|)^{\frac{q}{q-1}} d\eta} \left( \int_\eta^\pi \frac{||\Phi(\cdot, l, \eta)||_r^q}{l^{s_q}} \frac{dl}{l} \right)^{\frac{1}{q}} \eta^{\delta}\, d\eta.
\end{align*}
\]

Proof. This Lemma can be proved along the same lines of the proof of Lemma 1 of [12]. \(\square\)

Lemma 4.4. ([12]) Let \(0 \leq \delta < \lambda < 2\). If \(g \in B^{\lambda}_{1q}(L^r), r \geq 1, q = \infty\), then

\[
\sup_{0 < l, \eta \leq \pi} (l^{-\delta} \|\Phi(\cdot, l, \eta)\|_r) = O(\eta^{\lambda-\delta}). \quad (4.6)
\]

5 Proof of the Main theorem

Proof. Following [5], \(s_m(g, z)\) of Fourier series is given by

\[
s_m(g; z) - g(z) = \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \frac{\sin(m + \frac{1}{2})\eta}{\sin \frac{\eta}{2}} d\eta.
\]

Denoting the \(N^{p,q}\) summability transform of \(s_m(g; z)\) by \(t_{m,p,q}(z)\), we get

\[
t_{m,p,q}(z) - g(z) = \sum_{j=0}^m t_{j,p,q} \{s_j(g; z) - g(z)\}
\]
where given by

\[
\sum_{j=0}^{m} \frac{t_{p,q}^{j}}{H_{m}(\Delta \nu)} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \delta \eta
\]

\[
\left\{ \frac{1}{2\pi} \int_{0}^{\pi} \phi_{z}(\eta) \sum_{j=0}^{m} \left( \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right) d\eta \right\}
\]

The Hausdorff transform of \( t_{m}^{p,q}(z) \) i.e., \( \Delta_{H}N^{p,q} \) transform of \( s_{m}(g; z) \) denoted by \( t_{m}^{\Delta_{H}N^{p,q}} \), is given by

\[
t_{m}^{\Delta_{H}N^{p,q}}(z) = g(z)
\]

\[
= \sum_{j=0}^{m} h_{m,j} \{ t_{p,q}^{j}(z) - g(z) \}
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \Delta^{m-j} \mu_{j} \{ t_{p,q}^{j}(z) - g(z) \}
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \Delta^{m-j} \mu_{j} \left\{ \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right\}
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} \phi_{z}(\eta) \sum_{j=0}^{m} \binom{m}{j} \Delta^{m-j} \mu_{j} \left\{ \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right\} d\eta
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} \phi_{z}(\eta) \sum_{j=0}^{m} \binom{m}{j} \int_{0}^{1} y^{j} (1 - y)^{m-j} d\gamma(y) \left\{ \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right\} d\eta
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} \phi_{z}(\eta) \left\{ \int_{0}^{1} \sum_{j=0}^{m} \binom{m}{j} y^{j} (1 - y)^{m-j} \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) d\gamma(y) \right\} d\eta
\]

\[
= \int_{0}^{\pi} \phi_{z}(\eta) K_{m}^{\Delta_{H}N^{p,q}}(\eta) d\eta.
\]

Let

\[
l_{m}(z) := t_{m}^{\Delta_{H}N^{p,q}}(z) - g(z) = \frac{1}{\pi} \int_{0}^{\pi} \phi_{z}(\eta) K_{m}^{\Delta_{H}N^{p,q}}(\eta) d\eta,
\]

where

\[
K_{m}^{\Delta_{H}N^{p,q}}(\eta) = \int_{0}^{1} \sum_{j=0}^{m} \binom{m}{j} y^{j} (1 - y)^{m-j} \left\{ \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right\} d\gamma(y)
\]

\[
= \int_{0}^{1} M_{m}(\eta) d\gamma(y),
\]

where

\[
M_{m}(\eta) = \sum_{j=0}^{m} \binom{m}{j} y^{j} (1 - y)^{m-j} \left\{ \frac{1}{R_{j}} \sum_{\nu=0}^{j} p_{j-\nu} \sin \left( \frac{m + \frac{1}{2}}{2 \sin \frac{\eta}{2}} \right) \right\}.
\]
We write,
\[
\Phi(z, l, \eta) = \begin{cases} 
\phi_{z+l}(\eta) - \phi_z(\eta), & 0 < \lambda < 1, \\
\phi_{z+l}(\eta) + \phi_{z-l}(\eta) - 2\phi_z(\eta), & 1 \leq \lambda < 2.
\end{cases}
\]
and
\[
L_m(z, l) = \begin{cases} 
l_m(z + l) - l_m(z), & 0 < \lambda < 1, \\
l_m(z + l) + l_m(z - l) - 2l_m(z), & 1 \leq \lambda < 2.
\end{cases}
\]
Now, we have
\[
L_m(z, l) = \frac{1}{\pi} \int_0^\pi K_m^{\Delta H N^{p,q}}(\eta) \Phi(z, l, \eta) \, d\eta \quad \text{and} \quad \omega_j(l_m, l) r = \|L_m(\cdot, l)\|_r.
\]

**Case I:** For \(1 < q < \infty, r \geq 1, 0 \leq \delta < \lambda < 2\).

By definition, we have
\[
\|l_m(\cdot)\|_{B^\delta_r(L^r)} = \|l_m(\cdot)\|_r + \|w_j(l_m, \cdot)\|_{\delta, q}.
\]
Using generalized Minkowski’s inequality [3], Lemma 4.2 (iii) and (5.1), we have
\[
\|l_m(\cdot)\|_r \leq \frac{1}{\pi} \int_0^\pi \|\phi(\eta)\|_r |K_m^{\Delta H N^{p,q}}(\eta)| \, d\eta
\leq \frac{2}{\pi} \int_0^\pi w_j(g, \eta, \eta) r |K_m^{\Delta H N^{p,q}}(\eta)| \, d\eta.
\]
Using Hölder’s inequality and definition of Besov space, we get
\[
\|l_m(\cdot)\|_r \leq 2 \left\{ \int_0^\pi \left( |K_m^{\Delta H N^{p,q}}(\eta)| \eta^{\lambda + q - 1} \right)^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}} \times \left\{ \int_0^\pi \left( \frac{w_j(g, \eta, \eta)}{\eta^{\lambda + q - 1}} \right)^q \eta^q \, d\eta \right\}^{q^{-1}}
= O(1) \left\{ \int_0^\pi \left( |K_m^{\Delta H N^{p,q}}(\eta)| \eta^{\lambda + q - 1} \right)^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}}
= O(1) \left\{ \left( \int_0^{1/m+1} + \int_{1/m+1}^\pi \right) \left( |K_m^{\Delta H N^{p,q}}(\eta)| \eta^{\lambda + q - 1} \right)^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}}
:= O(1) [I_1 + I_2].
\]
Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we get
\[
I_1 = \left\{ \int_0^{1/m+1} \left( |K_m^{\Delta H N^{p,q}}(\eta)| \eta^{\lambda + q - 1} \right)^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}}
= O(m + 1) \left\{ \int_0^{1/m+1} \left( \eta^{\lambda + q - 1} \right)^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}}
\]
\[ I_2 = \left\{ \int_{\frac{1}{m+1}}^{\pi} \left( |K_m^{\Delta H_{NPq}}(\eta)| \eta^{\lambda+q-1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} = O\left(\frac{1}{(m + 1)^{-\lambda}}\right). \] (5.4)

By using Lemma 4.1 for \( \frac{1}{m+1} \leq \eta \leq \pi \), we get

\[ I_2 = \left\{ \int_{\frac{1}{m+1}}^{\pi} \left( |K_m^{\Delta H_{NPq}}(\eta)| \eta^{\lambda+q-1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} = O\left(\frac{1}{(m + 1)^{-\lambda}}\right). \] (5.5)

From (5.3), (5.4) and (5.5), we have

\[ \|l_m(\cdot)\|_r = O\left(\frac{1}{(m + 1)^{-\lambda}}\right). \] (5.6)

Now, using generalized Minkowski’s inequality and using Lemma 4.3, we have

\[ \|w_j(l_m, \cdot)\|_{\delta, q} \]

\[ = \left\{ \int_0^{\pi} \left( \frac{w_j(l_m, l) r}{l^{\delta+1}} \right)^q d\eta \right\}^{q^{-1}} = \left\{ \int_0^{\pi} \left( \frac{\|L_m(\cdot, l)\|_r}{l^{\delta+1}} \right)^q d\eta \right\}^{q^{-1}} \]

\[ = \left\{ \int_0^{\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} |L_m(z, l)|^r dz \right)^{q/r} d\eta \right\}^{q^{-1}} \]

\[ = \left\{ \int_0^{\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} \Phi(z, l, \eta) K_m^{\Delta H_{NPq}}(\eta) d\eta \right)^{q/r} d\eta \right\}^{1/r} \]

\[ \leq \frac{1}{\pi} \left[ \int_0^{\pi} \left( \frac{1}{2\pi} \right)^{q/r} \left\{ \int_0^{\pi} \Phi(z, l, \eta) K_m^{\Delta H_{NPq}}(\eta) d\eta \right\}^{1/r} d\eta \right]^{q^{-1}} \]

\[ = \frac{1}{\pi} \left[ \int_0^{\pi} \left\{ \int_0^{\pi} \Phi(z, l, \eta) K_m^{\Delta H_{NPq}}(\eta) d\eta \right\}^{1/r} d\eta \right]^{q^{-1}} \]
\[
\begin{align*}
\leq \frac{1}{\pi} \int_0^\pi |K_{m}^{\Delta H N^{p,q}}(\eta)| d\eta \left( \int_0^\pi \frac{\left\| \Phi (\cdot, l, \eta) \right\|^q}{l^{\delta q}} \frac{dl}{l} \right)^{q-1} \\
= \frac{1}{\pi} \int_0^\pi |K_{m}^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \left( \int_0^\eta + \int_\eta^\pi \right) \frac{\left\| \Phi (\cdot, l, \eta) \right\|^q}{l^{\delta q}} \frac{dl}{l} \right\}^{q-1} \\
\leq \frac{1}{\pi} \int_0^\pi |K_{m}^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \int_0^\eta \frac{\left\| \Phi (\cdot, l, \eta) \right\|^q}{l^{\delta q}} \frac{dl}{l} \right\}^{q-1} \\
+ \frac{1}{\pi} \int_0^\pi |K_{m}^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \int_\eta^\pi \frac{\left\| \Phi (\cdot, l, \eta) \right\|^q}{l^{\delta q}} \frac{dl}{l} \right\}^{q-1} \\
= O(1) \left\{ \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)} \\
+ O(1) \left\{ \left( \eta^{\lambda-\delta+(1/q)} |K_{m}^{\Delta H N^{p,q}}(\eta)|^{q/(q-1)} d\eta \right\}^{1-(1/q)} \\
:= O(1) (J_1 + J_2). \tag{5.7}
\end{align*}
\]

Since \((x + y)^r \leq x^r + y^r\) for positive \(x, y\) and \(0 < r \leq 1\), then

\[
J_1 = \left\{ \int_0^\pi \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)}
\]

\[
= \left\{ \left( \int_0^{1/(m+1)} + \int_{1/(m+1)}^\pi \right) \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} \right\}^{1-(1/q)}
\]

\[
\leq \left\{ \int_0^{1/(m+1)} \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)}
\]

\[
+ \left\{ \int_{1/(m+1)}^\pi \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)}
\]

\[
= I_{11} + I_{12}. \tag{5.8}
\]

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we have

\[
I_{11} = \left\{ \int_0^{1/(m+1)} \left( \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)}
\]

\[
= O(m + 1) \left\{ \int_0^{1/(m+1)} \eta^{q/(q-1)} (\lambda-\delta) d\eta \right\}^{1-(1/q)}
\]

\[
= O(m + 1) \left\{ \int_0^{1/(m+1)} \eta^{q/(q-1)} (\lambda-\delta+1-(1/q))^{-1} d\eta \right\}^{1-(1/q)}
\]

\[
= O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \tag{5.9}
\]
Using Lemma 4.1 for $0 < \eta \leq \frac{1}{m+1}$, we have

\[
I_{12} = \left\{ \int_{1/(m+1)}^{\pi} \left( \eta^{\lambda-\delta} |K^\Delta_m H^{N_p,q}(\eta)| \right)^{q/(q-1)} \frac{d\eta}{\eta^{q-1}} \right\}^{1-q^{-1}}
\]

\[
= O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right).
\] (5.10)

From (5.8), (5.9) and (5.10), we have

\[
J_1 := I_{11} + I_{12}
= O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \tag{5.11}
\]

Now,

\[
J_2 = \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K^\Delta_m H^{N_p,q}(\eta)| \right)^{q/(q-1)} \frac{d\eta}{\eta^{q-1}} \right\}^{1-q^{-1}}
\]

\[
= \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K^\Delta_m H^{N_p,q}(\eta)| \right)^{q/(q-1)} \frac{d\eta}{\eta^{q-1}} \right\}^{1-q^{-1}}
= J_{11} + J_{12}. \tag{5.12}
\]

Using Lemma 4.1 for $0 \leq \eta \leq \frac{1}{m+1}$, we have

\[
J_{11} = \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K^\Delta_m H^{N_p,q}(\eta)| \right)^{q/(q-1)} \frac{d\eta}{\eta^{q-1}} \right\}^{1-q^{-1}}
\]

\[
= O \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \tag{5.13}
\]
Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we have

$$J_{12} = \left\{ \int_0^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K_m^{\Delta_{H}^{N_p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}}$$

$$= O\left( \left( \frac{1}{(m+1)^{\lambda-\delta}} \right) \right).$$

From (5.12), (5.13) and (5.14), we have

$$J_2 := J_{11} + J_{12}$$

$$= O\left( \left( \frac{1}{(m+1)^{\lambda-\delta}} \right) \right).$$

From (5.7), (5.11) and (5.15), we get

$$\| w_j (l_m, \cdot) \|_{\delta,q} = O\left( \left( \frac{1}{(m+1)^{\lambda-\delta}} \right) \right) + O\left( \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right) \right).$$

From (5.2), (5.6) and (5.16), we get

$$\| l_m (\cdot) \|_{B^\delta_\infty (L^r)} = \| l_m (\cdot) \|_r + \| w_j (l_m, \cdot) \|_{\delta,q}$$

$$= O\left( (m+1)^{-\lambda} + \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right) \right).$$

This completes the proof of case I.

**Case II :** For $q = \infty$, $0 \leq \delta < \lambda < 2$.

We have

$$\| l_m (\cdot) \|_{B^\delta_\infty (L^r)} = \| l_m (\cdot) \|_r + \| w_j (l_m, \cdot) \|_{\delta,\infty}.$$  

Using (1.6), we have

$$\| l_m (\cdot) \|_r \leq 2 \int_0^\pi w_j (g, \eta) |K_m^{\Delta_{H}^{N_p,q}}(\eta)| d\eta$$

$$= O\left( \left( \int_0^{1/(m+1)} \eta^{\lambda} |K_m^{\Delta_{H}^{N_p,q}}(\eta)| d\eta + \int_{1/(m+1)}^\pi \eta^{\lambda} |K_m^{\Delta_{H}^{N_p,q}}(\eta)| d\eta \right) \right).$$
\[ := O \left( 1 \right) \left( I_2 + J_2 \right). \quad (5.18) \]

Using Lemma 4.1 for \( 0 \leq \eta \leq \frac{1}{m+1} \), we get

\[
I_2 = \int_0^{1/(m+1)} \eta^\lambda |K_m^{\Delta H_{N^p,q}}(\eta)| d\eta \\
= O \left( m + 1 \right) \int_0^{1/(m+1)} \eta^\lambda d\eta \\
= O \left( m + 1 \right)^{-\lambda}. \quad (5.19)
\]

Again, using Lemma 4.1 for \( \frac{1}{m+1} \leq \eta \leq \pi \), we get

\[
J_2 = \int_0^{\pi} \eta^\lambda |K_m^{\Delta H_{N^p,q}}(\eta)| d\eta \\
= O \left( 1 \right) \int_0^{\pi} \eta^{\lambda-1} d\eta \\
= O \left( m + 1 \right)^{1-\lambda}. \quad (5.20)
\]

From (5.18), (5.19) and (5.20), we get

\[
\|l_m(\cdot)\|_r = O \left( 1 \right) \left( I_2 + J_2 \right) \\
= O \left( m + 1 \right)^{-\alpha}. \quad (5.21)
\]

Using generalized Minkowski’s inequality and Lemma 4.4, we get

\[
\|w_j(l_m, \cdot)\|_{\delta,\infty} \\
= \sup_{l > 0} \left( l^{-\delta} w_j(l_m, l) \right) \\
= \sup_{l > 0} \left( l^{-\delta} \|L_m(\cdot, l)\|_r \right) \\
= \sup_{l > 0} \left[ l^{-\delta} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{\pi} K_m^{\Delta H_{N^p,q}}(\eta) \phi(z, l, \eta) d\eta \right|^r dz \right]^{1/r} \right] \\
\leq \sup_{l > 0} \left[ \frac{l^{-\delta}}{\pi} \left( \frac{1}{2\pi} \right)^{1/r} \int_0^{\pi} \left\{ \int_0^{2\pi} \left| K_m^{\Delta H_{N^p,q}}(\eta) \right|^r \phi(z, l, \eta) d\eta \right\}^{1/r} d\eta \right] \\
= \sup_{l > 0} \left[ \frac{l^{-\delta}}{\pi} \int_0^{\pi} \|\phi(\cdot, l, \eta)\|_r \left| K_m^{\Delta H_{N^p,q}}(\eta) \right| d\eta \right] \\
= \frac{1}{\pi} \int_0^{\pi} \left( \sup_{l > 0} l^{-\delta} \|\Phi(\cdot, l, \eta)\|_r \right) \left| K_m^{\Delta H_{N^p,q}}(\eta) \right| d\eta \\
= O \left( 1 \right) \int_0^{\pi} \eta^{\lambda-\delta} \left| K_m^{\Delta H_{N^p,q}}(\eta) \right| d\eta
\]
\[\approx \mathcal{O}(1) \left[ \int_0^{1/(m+1)} \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \, d\eta \right] + \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \, d\eta \]
\[= \mathcal{O}(1) [I_3 + J_3]. \] (5.22)

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we get
\[I_3 = \int_0^{1/(m+1)} \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \, d\eta \]
\[= \mathcal{O}(m+1) \int_0^{1/(m+1)} \eta^{\lambda-\delta} \, d\eta \]
\[= \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \] (5.23)

Using Lemma 4.1 for \(\frac{1}{m+1} \leq \eta \leq \pi\), we get
\[J_3 = \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta} |K_{m}^{\Delta H N^{p,q}}(\eta)| \, d\eta \]
\[= \mathcal{O}(1) \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta-1} \, d\eta \]
\[= \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \] (5.24)

From (5.22), (5.23) and (5.24), we get
\[\|w_j(l_m, \cdot)\|_{\delta, \infty} = \mathcal{O}(1) [I_3 + J_3] \]
\[= \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \] (5.25)

From (5.17), (5.21) and (5.25), we have
\[\|l_m(\cdot)\|_{B^{\alpha}_\infty(L^r)} = \|l_m(\cdot)\|_{r} + \|w_j(l_m, \cdot)\|_{\delta, \infty} \]
\[= \mathcal{O}(m+1)^{\lambda-\delta} + \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \]

This completes the proof of case II. \(\square\)

6 Corollary

The following corollary are derived from our main theorem.

**Corollary 6.1.** If \(q_m = 1 \forall m\), then \(\Delta H N^{p,q}\) means reduces to \(\Delta H N^{p,m}\) means and the best approximation of \(g \in B^\lambda_q(L^r)\) space by \(\Delta H N^{p,m}\) means of Fourier series is
\[\|l_m^{\Delta H N^{p,m}}(z) - g(z)\| = \begin{cases} O(m+1)^{\lambda} + \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta-(4/q)}} \right) + \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right) & ; 1 < q < \infty \\ O(m+1)^{\lambda} + \mathcal{O} \left( \frac{1}{(m+1)^{\lambda-\delta}} \right) & ; q = \infty. \end{cases} \]
Corollary 6.2. If \( p_m = 1 \forall m \), then \( \Delta_H N^{p,q} \) reduces to \( \Delta_H N^q \) means and the best approximation of \( g \in B^\lambda_q(L^r) \) space by \( \Delta_H N^q \) means of Fourier series is

\[
\| t_m \Delta_H N^q (z) - g(z) \| = \begin{cases} 
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty.
\end{cases}
\]

Corollary 6.3. If \( p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right) \alpha > 0 \), and \( q_m = 1 \forall m \), then \( \Delta_H N^{p,q} \) means reduces to \( \Delta_H C_\alpha \) means of and the best approximation of \( g \in B^\lambda_q(L^r) \) space by \( \Delta_H C_\alpha \) means of Fourier series is

\[
\| t_m \Delta_H C_\alpha (z) - g(z) \| = \begin{cases} 
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty.
\end{cases}
\]

Corollary 6.4. If \( h_{m,j} = \left(\frac{m-j}{(1+q)^m}\right) \), then \( \Delta_H N^{p,q} \) means and the best approximation of \( g \in B^\lambda_q(L^r) \) space by \( E_q N^{p,q} \) means of Fourier series is

\[
\| t_m E_q N^{p,q} (z) - g(z) \| = \begin{cases} 
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty.
\end{cases}
\]

Corollary 6.5. If \( h_{m,j} = \left(\frac{1}{m+1}\right) \), then \( \Delta_H N^{p,q} \) means and the best approximation of \( g \in B^\lambda_q(L^r) \) space by \( C_1 N^{p,q} \) means of Fourier series is

\[
\| t_m C_1 N^{p,q} (z) - g(z) \| = \begin{cases} 
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\
O((m+1)^{-\lambda}) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty.
\end{cases}
\]

7 Particular cases

(i) In view of Remark 2 (i) and 3 (ii), our result becomes a particular case of [10].

(ii) In view of Remark 2 (i) and 3 (iv), our result becomes a particular case of [26].

(iii) In view of Remark 2 (ii) and 3 (v), our result becomes a particular case of [27].

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