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Approximation of Functions in Besov Space

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Abstract. In the present paper, we establish a theorem on best approximation of a function $g \in B_q^\lambda(L^r)$ of its Fourier series. Our main theorem generalizes some known results of this direction of work. Thus, the results of [10], [26] and [27] become the particular case of our main Theorem 3.1.

1 Introduction

The degree of approximation of the functions in Lipschitz spaces and Hölder spaces using single and product summability means has been studied by the authors [7, 8, 9, 15, 16, 18, 19, 20, 21, 22, 23, 25, 28]. This motivates us to study the degree of approximation of a function in more generalized function space. Therefore, in this paper, we study the degree of approximation of a function g in Besov space using Hausdorff-generalized Nörlund means of its Fourier series. It can be noted that Besov space generalizes different Sobolev spaces, Lipschitz spaces and generalized Hölder spaces [13]. Besov space can also be used to study regularity properties of the functions.

Let $C_{2\pi} := C[0, 2\pi]$ denote the Banach space of all 2π- periodic continuous functions g defined on $[0, 2\pi]$ under the supremum norm.

Let

$$L^r := L^r[0,2\pi] := \left\{ g : [0,2\pi] \mapsto \mathbb{R} : \int_0^{2\pi} |g(z)|^r dz < \infty \right\}, r \geq 1$$

be the space of all 2π-periodic, integrable functions L^r-norm of the function g is given by

$$\|g\|_r := \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(z)|^r dz \right\}^{1/r} & \text{for} \quad 1 \leq r < \infty \\ ess\sup_{0<z<2\pi} |g(z)| & \text{for} \quad r = \infty. \end{cases}$$

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The modulus of continuity of a function $g \in L^r$ is defined by

$$w(g;l) := \sup_{\substack{z,\ z+h\in[0,2\pi] \\ |h|<l}} |g(z+h)-g(z)|. \tag{1.1}$$

The integral modulus of continuity of the first order of a function $g \in L^r$ is defined [3] by

$$w_1(g;l)_r := \sup_{|h|<l, z\in\mathbb{R}} \|g(z+h)-g(z)\|_r. \tag{1.2}$$

The integral moduli of continuity of the second order (modulus of smoothness) of a function $g \in L^r$ is defined [2] by

$$w_2(g;l)_r = \sup_{0<h\leq l, z\in\mathbb{R}} \|g(z+h)+g(z-h)-2g(z)\|_r. \tag{1.3}$$

The j^{th} order modulus of smoothness of a function $g \in L^r$ is defined [1] by

$$w_j(g,l)_r := \sup_{0<h\leq l} \|\Delta_h^j(g,\cdot)\|_r, \quad l>0, \tag{1.4}$$

where

$$\Delta_h^j(g,z) := \sum_{\rho=0}^{j} (-1)^{j-\rho} \binom{j}{\rho} g(z+\rho h), \quad j\in\mathbb{N}. \tag{1.5}$$

Remark 1.

(i) If $r=\infty$, $j=1$ and g being a continuous function, then $w_j(g,l)_r$ reduces to $w(g,l)$.

(ii) If $0<r<\infty$, $j=1$ and g being a continuous function, then $w_j(g,l)_r$ reduces to $w_1(g,l)_r$.

(iii) If $g\in C_{2\pi}$ and $w(g,l)=O(l^\lambda)$, $0<\lambda\leq 1$, then $g\in Lip\,\lambda$.

(iv) If $g\in L^r$, $0<r<\infty$ and $w(g,l)_r=O(l^\lambda)$, $0<\lambda\leq 1$, then $g\in Lip(\lambda,r)$.

(v) If $r=\infty$, then $Lip(\lambda,r)$ class reduces to $Lip\,\lambda$.

Note 1. From Remark 1(iv) and 1(v), we write

$$Lip(\lambda)\subseteq Lip(\lambda,r).$$

Let $\lambda>0$, $j>\lambda$ i.e. $j=[\lambda]+1$, where j being smallest integer. For $g\in L^r$, if

$$w_j(g,l)_r = O(l^\lambda), \quad l>0, \tag{1.6}$$

then $g \in Lip^*(\lambda, r)$ and its semi-norm is given by

$$|g|_{Lip^*(\lambda,r)} = \sup_{l>0}(l^{-\lambda} w_j(g,l)_r),$$

where $Lip^*(\lambda, r)$ is a generalized Lipschitz class of function g.

Thus,

$$Lip(\lambda, r) \subseteq Lip^*(\lambda, r).$$

For $0 < \lambda \leq 1$, let

$$H_\lambda := \{g \in C_{2\pi} : w(g,l) = O(l^\lambda)\},$$

where H_λ is a Banach space with the norms

$$\|g\|_\lambda = \|g\|_C + \sup_{l>0}(l^{-\lambda} w(l)) \quad \text{for} \quad 0 < \delta \leq \lambda < 1$$

and

$$\|g\|_0 = \|g\|_C.$$

Thus, we observe that

$$H_\lambda \subseteq H_\delta \subseteq C_{2\pi} \quad \text{for} \quad 0 < \delta \leq \alpha < 1([28]).$$

The metric induced by the norm $\|\cdot\|_\lambda$ on H_λ is called the Hölder metric.

For $0 < \lambda \leq 1, \;\; 0 < r \leq \infty$, let

$$H_{\lambda,r} := H_{\lambda,r}[0, 2\pi] = \left\{g \in L^r : w(g,l)_r = O(l^\lambda)\right\},$$

where $H_{\lambda,r}$ is also a Banach space with the norm $\|\cdot\|_{\lambda,r}$ defined by

$$\|g\|_{\lambda,r} = \|g\|_r + \sup_{l>0}(l^{-\lambda} w(g,l)_r) \quad \text{for} \quad 0 < \lambda \leq 1$$

and

$$\|g\|_{0,r} = \|g\|_r.$$

Then $H_{\lambda,r}$ is a Banach space for $r \geq 1$ and a complete r-normed space ([14], p. 87) for $0 < r < 1$.

Thus,

$$H_{\lambda,r} \subseteq H_{\delta,r} \subseteq L^r \quad \text{for} \quad 0 < \delta \leq \lambda \leq 1([9]).$$

For $\lambda > 0$ and let $j > \lambda$ i.e. $j = [\lambda] + 1$. For $0 < r,\ q \leq \infty$, the Besov space $B_q^\lambda(L^r)$ is a collection of all 2π-periodic function $g \in L^r$ such that

$$|g|_{B_q^\lambda(L^r)} := \|w_j(g,\cdot)\|_{\lambda,q} = \begin{cases} \left(\int_0^\pi [l^{-\lambda} w_j(g,l)_r]^q \frac{dl}{l}\right)^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup_{l>0}(l^{-\lambda} w_j(g,l)_r), & q = \infty, \end{cases} \tag{1.7}$$

is finite ([24], p. 237) for 2π - periodic function g ([1], p. 54).

It is further observed that (1.7) is a semi-norm if $1 \leq r, q \leq \infty$ and a quasi semi-norm in other cases ([1], p. 55).

The quasi-norm for Besov space is given by

$$\begin{aligned} \|g\|_{B_q^\lambda(L^r)} &:= \|g\|_r + |g|_{B_q^\lambda(L^r)} \\ &= \|g\|_r + \|w_j(g,\cdot)\|_{\lambda,q}. \end{aligned}$$

Note 2.

(i) If $0 < \lambda < 1$, then the Besov space $B_\infty^\lambda(L^r)$ reduces to the $H_{\lambda,r}$ ([9]).

(ii) If $r = \infty = q$ and $0 < \lambda < 1$, the Besov space $B_\infty^\lambda(L^r)$ reduces to the space H_λ ([28]).

The m-order error approximation of a function $g \in C_{2\pi}$ is defined by $E_m(g) := \inf_{t_m} \|g - t_m\|$ where t_m is a trigonometric polynomial of degree m ([3]).
If $E_m(g) \to 0$ as $m \to \infty$, $E_m(g)$ is said to be the best approximation of g ([3]).

2 Definitions

The Hausdorff matrix $H \equiv (h_{m,j})$ is an infinite lower triangular matrix defined by,

$$h_{m,j} = \begin{cases} \binom{m}{j} \Delta^{m-j} \mu_j, & 0 \leq j \leq m, \\ 0, & j > m, \end{cases}$$

where Δ is a forward operator defined by $\Delta \mu_m = \mu_m - \mu_{m+1}$ and $\Delta^{j+1}\mu_m = \Delta(\Delta^j \mu_m)$ ([6]).

A Hausdorff matrix H is regular iff $\int_0^1 |d\gamma(y)| < \infty$, where the mass function $\gamma(y)$ is continuous at $y = 0$ and belongs to BV$[0,1]$ such that $\gamma(0+) = 0, \gamma(1) = 1$; and for $0 < y < 1$, $\gamma(y) = [\gamma(y+0) + \gamma(y-0)]/2$ [4, 11]. Thus $\{\mu_m\}$, known as moment sequence, has the representation

$$\mu_m = \int_0^1 y^m d\gamma(y).$$

The Hausdorff means of a trigonometric Fourier series of g is defined by

$$H_m(g;z) = \sum_{j=0}^{m} h_{m,j} s_j(g;z), \quad \forall \quad m \geq 0.$$

The series is said to be summable to s by Hausdorff means, if $H_m(g;z) \to s$ as $m \to \infty$ and we denote Hausdorff means by Δ_H through out the paper.

Example 1. (i) If

$$h_{m,j} = \begin{cases} \binom{m}{j} \frac{q^{m-j}}{(1+q)^m}, & 0 \leq j \leq m, \\ 0, & j > m, \end{cases}$$

then the Hausdorff matrix $H \equiv (h_{m,j})$ reduces to (E,q) matrix (Euler matrix of order $q > 0$) and defines the corresponding (E,q) means by

$$E_m^q(g;z) := \frac{1}{(1+q)^m} \sum_{j=0}^{m} \binom{m}{j} q^{m-j} s_j(g;z).$$

(ii) If $\mu_m = \dfrac{1}{m+1}$ then the Hausdorff matrix $H \equiv (h_{m,j})$ reduces to $(C,1)$ matrix (Cesàro matrix of order 1) and defines the corresponding means by

$$H_m(g;z) := \frac{1}{(m+1)} \sum_{j=0}^{m} s_j(g;z).$$

Let $\{p_m\}$ and $\{q_m\}$ be the sequence of constants, real or complex, such that

$$P_m = p_0 + p_1 + p_2 + \cdots + p_m = \sum_{v=0}^{m} p_v \to \infty, \quad \text{as} \quad m \to \infty$$

$$Q_m = q_0 + q_1 + q_2 + \cdots + q_m = \sum_{v=0}^{m} q_v \to \infty, \quad \text{as} \quad m \to \infty$$

$$R_m = p_0 q_m + p_1 q_{m-1} + p_2 q_{m-2} + \cdots + p_m q_0 = \sum_{v=0}^{m} p_v q_{m-v} \to \infty, \quad \text{as} \quad m \to \infty.$$

Given two sequences $\{p_m\}$ and $\{q_m\}$ convolution $(p * q)$ is defined as

$$R_m = (p * q)_m = \sum_{j=0}^{m} p_{m-j} q_j.$$

We write

$$t_m^{p,q} = \frac{1}{R_m} \sum_{j=0}^{m} p_{m-j} q_j s_j.$$

If $R_m \neq 0$, for all m, the generalized Nörlund transform of the sequence $\{s_m\}$ is the sequence $\{t_m^{p,q}\}$. If $t_m^{p,q} \to s$, as $m \to \infty$, then the series $\sum_{m=0}^{\infty} a_m$ or sequence $\{s_m\}$ is summable to s by generalized Nörlund method and is denoted by $s_m \to s(N^{p,q})$.

The necessary and sufficient condition for $(N^{p,q})$ method to be regular are

$$\sum_{j=0}^{m} |p_{m-j} q_j| = O\left(|R_m|\right)$$

and $p_{m-j} = o\left(|R_m|\right)$, as $m \to \infty$ for every fixed $j \geq 0$, for which $q_j \neq 0$ ([17]).

If the method Δ_H is superimposed on the $N^{p,q}$ method, another new method of summability $\Delta_H N^{p,q}$ is obtained.
The Hausdorff transform of $N^{p,q}$ transform is defined as $\Delta_H N^{p,q}$ product transform of the partial sum s_m, which can be given by

$$\begin{aligned} t_m^{\Delta_H N^{p,q}} &= \sum_{j=0}^{m} h_{m,j} t_j^{p,q} \\ &= \sum_{j=0}^{m} h_{m,j} \frac{1}{R_j} \sum_{v=0}^{j} p_{j-v} q_v s_v . \end{aligned}$$

If $t_m^{\Delta_H N^{p,q}} \to s$ as $m \to \infty$ then the series $\sum_{m=0}^{\infty} a_m$ or the sequence $\{s_m\}$ is summable to s by $\Delta_H N^{p,q}$ means.

Now, we define the regularity of $\Delta_H N^{p,q}$ method.

$$\begin{aligned} s_m \to s &\implies t_m^{p,q} \to s, \quad \text{as} \quad m \to \infty \quad \text{so } \; N^{p,q} \; \text{ method is regular,} \\ &\implies \Delta(t_m^{p,q}) = t_m^{\Delta_H N^{p,q}} \to s \quad \text{as} \quad m \to \infty \quad \text{so } \; \Delta_H \; \text{ method is regular,} \\ &\implies (\Delta_H N^{p,q}) \quad \text{method is regular.} \end{aligned}$$

Remark 2.

(i) $\Delta_H N^{p,q}$ means reduces to $E_q N^{p,q}$ means if $h_{m,j} = \begin{cases} \binom{m}{j} \frac{q^{m-j}}{(1+q)^m}, & if \;\; 0 \leq j \leq m, \\ 0, & if \;\; j > m. \end{cases}$

(ii) $\Delta_H N^{p,q}$ means reduces to $C_1 N^{p,q}$ means if $h_{m,j} = \begin{cases} \frac{1}{m+1}, & if \;\; 0 \leq j \leq m, \\ 0, & if \;\; j > m. \end{cases}$

(iii) $\Delta_H N^{p,q}$ means reduces to $\Delta_H N^{p_m}$ means if $q_m = 1, \forall\, m$.

(iv) $\Delta_H N^{p,q}$ means reduces to $\Delta_H \tilde{N}^{q_m}$ means if $p_m = 1, \forall\, m$.

(v) $\Delta_H N^{p,q}$ means reduces to $\Delta_H C_\alpha$ means if $p_m = \binom{m+\alpha-1}{\alpha-1}$, $\alpha > 0$ and $q_m = 1, \forall\, m$.

Remark 3. The above cases (i) and (ii) of Remark 2 can be further reduced as

(i) $E_q N^{p,q}$ means reduces to $E_q N^{p_m}$ means if $q_m = 1, \forall\, m$.

(ii) $E_q N^{p,q}$ means reduces to $E_1 N^{p_m}$ means if $q_m = 1, \forall\, m$.

(iii) $E_q N^{p,q}$ means reduces to $E_q \tilde{N}^{q_m}$ means if $p_m = 1, \forall\, m$.

(iv) $E_q N^{p,q}$ means reduces to $E_q C_\alpha$ means if $p_m = \binom{m+\alpha-1}{\alpha-1}$, $\alpha > 0$ and $q_m = 1, \forall\, m$.

(v) $C_1 N^{p,q}$ means reduces to $C_1 N^{p_m}$ means if $q_m = 1, \forall\, m$.

(vi) $C_1 N^{p,q}$ means reduces to $C_1 \tilde{N}^{q_m}$ means if $p_m = 1, \forall\, m$.

3 Main Theorems

Theorem 3.1. *If g is a 2π-periodic and Lebesgue integrable function, then for $0 \le \delta < \lambda < 2$, the best approximation of g in $B_q^\lambda(L^r)$, $r \ge 1, 1 < q \le \infty$ space using $\Delta_H N^{p,q}$ means, is given by*

$$\|t_m^{\Delta_H N^{p,q}}(z) - g(z)\| = \begin{cases} O\left(m+1\right)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\ O\left(m+1\right)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty. \end{cases}$$

4 Lemmas

Lemma 4.1. *Let $K_m^{\Delta_H N^{p,q}}(\eta) := \int_0^1 M_m(\eta) d\gamma(y)$*
where

$$M_m(\eta) := \left[\sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{\frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_\nu \frac{\sin\left(\nu + \frac{1}{2}\right)\eta}{2 \sin\frac{\eta}{2}}\right\}\right],$$

then

$$K_m^{\Delta_H N^{p,q}}(\eta) = \begin{cases} O(m+1), & 0 \le \eta \le \frac{1}{(m+1)}; \\ O\left(\frac{1}{\eta}\right), & \frac{1}{(m+1)} \le \eta \le \pi. \end{cases}$$

Proof. For $0 \le \eta \le \frac{1}{m+1}$, we have $\sin m\eta \le m \sin \eta$, then

$$
\begin{aligned}
&M_m(\eta)\\
&= \left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu \frac{\sin(\nu+\frac{1}{2})\eta}{2\sin\frac{\eta}{2}}\right\}\right]\\
&\le \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu \frac{(2\nu+1)\sin\frac{\eta}{2}}{\sin\frac{\eta}{2}}\right\}\right]\\
&\le \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu (2\nu+1)\right\}\right]\\
&= \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\frac{1}{R_j}\left\{p_j q_0(2\cdot 0+1) + p_{j-1}q_1(2\cdot 1+1)+\cdots+p_0 q_j(2\cdot j+1)\right\}\right]\\
&\le \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\frac{1}{R_j}\left\{p_j q_0(2j+1) + p_{j-1}q_1(2j+1)+\cdots+p_0 q_j(2j+1)\right\}\right]\\
&= \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\frac{1}{R_j}(2j+1)\sum_{\nu=0}^{j} p_{j-\nu}q_\nu\right]\\
&= \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\frac{1}{R_j}(2j+1)O(|R_j|)\right]\\
&= O\left[\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{n-j}(2j+1)\right]\\
&= O\left[2\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{n-j}\cdot j + \sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\right]. \qquad (4.1)
\end{aligned}
$$

Now, solving first term of (4.1),

$$
\begin{aligned}
\sum_{j=0}^{m}\binom{m}{j} y^j (1-y)^{m-j}\cdot j &= (1-y)^m \sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{1-y}\right)^j \cdot j\\
&= (1-y)^m \sum_{j=0}^{m}\binom{m}{j} p^j \cdot j,
\end{aligned}
$$

where $\frac{y}{1-y} = p \Rightarrow 1+p = \frac{1}{1-y}$.

$$
\sum_{j=0}^{m}\binom{m}{j} p^j \cdot j = \binom{m}{0}p^0\cdot 0 + \binom{m}{1}p^1\cdot 1 + \binom{m}{2}p^2\cdot 2 + \cdots + \binom{m}{m}p^m\cdot m
$$

$$= \binom{m}{1}p + 2\binom{m}{2}p^2 + 3\binom{m}{3}p^3 + \cdots + m\binom{m}{m}p^m. \tag{4.2}$$

We know that

$$(1+p)^m = \binom{m}{0} + \binom{m}{1} \cdot p + \binom{m}{2} \cdot p^2 + \cdots + \binom{m}{m} \cdot p^m.$$

By diffrentiating with respect to p, we have

$$m(1+p)^{m-1} = 0 + \binom{m}{1} + \binom{m}{2} \cdot 2p + \binom{m}{3} \cdot 3p^2 + \cdots + \binom{m}{m} \cdot mp^{m-1}.$$

Multiplying above equation by p on both side, we have

$$mp(1+p)^{m-1} = \binom{m}{1}p + 2\binom{m}{2}p^2 + 3\binom{m}{3}p^3 + \cdots + m\binom{m}{m}p^m. \tag{4.3}$$

Now, from (4.2) and (4.3), we have

$$\sum_{j=0}^{m} \binom{m}{j} \left(\frac{y}{1-y}\right)^j \cdot j = mp(1+p)^{m-1}$$

$$(1-y)^m \sum_{j=0}^{m} \binom{m}{j} \left(\frac{y}{1-y}\right)^j \cdot j = (1-y)^m \left\{mp(1+p)^{m-1}\right\}$$

$$\sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j = (1-y)^m \left\{m \cdot \frac{y}{1-y} \cdot \frac{1}{(1-y)^{m-1}}\right\}$$

$$\sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j = my. \tag{4.4}$$

Now, solving second term of (4.1),

$$\begin{aligned}
&\sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \\
&\quad = \binom{m}{0} y^0 (1-y)^m + \binom{m}{1} y^1 (1-y)^{m-1} + \cdots + \binom{m}{m} y^m (1-y)^{m-m} \\
&\quad = (1-y+y)^m \\
&\quad = 1.
\end{aligned} \tag{4.5}$$

Now, from (4.1), (4.4) and (4.5), we get

$$M_m(\eta) = O\left(2my + 1\right).$$

Thus,

$$\begin{aligned}K_m^{\Delta_H N^{p,q}}(\eta) &= \int_0^1 M_m(\eta) d\gamma(y)\\ &= O(1)\int_0^1 (2my+1)\,dy\\ &= O(m+1).\end{aligned}$$

For $\frac{1}{m+1} \leq \eta \leq \pi$, by Jordan's lemma we have, $\sin\frac{\eta}{2} \geq \frac{\eta}{\pi}$ and $\sin n\eta \leq 1$. Thus,

$$\begin{aligned}M_m(\eta) &= \left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu \frac{\sin(\nu+\frac{1}{2})\eta}{2\sin\frac{\eta}{2}}\right\}\right]\\ &\leq \frac{1}{2}\left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu \frac{1}{\frac{\eta}{\pi}}\right\}\right]\\ &\leq \frac{\pi}{2\eta}\left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j} p_{j-\nu}q_\nu\right\}\right]\\ &= \frac{\pi}{2\eta}\left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\left\{\frac{1}{R_j}O\left(|R_j|\right)\right\}\right]\\ &= \frac{\pi}{2\eta}\left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\{O(1)\}\right]\\ &= O\left(\frac{\pi}{2\eta}\right)\left[\sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j}\right]\\ &= O\left(\frac{1}{\eta}\right) \quad \text{since} \quad \sum_{j=0}^{m}\binom{m}{j} y^j(1-y)^{m-j} = 1.\end{aligned}$$

Thus,

$$\begin{aligned}K_m^{\Delta_H N^{p,q}}(\eta) &= \int_0^1 M_m(\eta) d\gamma(y)\\ &= \int_0^1 O\left(\frac{1}{\eta}\right) dy\\ &= O\left(\frac{1}{\eta}\right)\int_0^1 dy\\ &= O\left(\frac{1}{\eta}\right).\end{aligned}$$

□

Lemma 4.2. *([12]) Let $1 \le r \le \infty$ and $0 < \lambda < 2$. If $g \in L^r$ then for $0 < l, \eta \le \pi$:*

(i) $\|\Phi(\cdot, l, \eta)\|_r \le 4w_j(g, l)_r$,

(ii) $\|\Phi(\cdot, l, \eta)\|_r \le 4w_j(g, \eta)_r$,

(iii) $\|\Phi(\eta)\|_r \le 2w_j(g, \eta)_r$,

where $j = [\lambda] + 1$.

Lemma 4.3. *Let $0 \le \delta < \lambda < 2$. If $g \in B_q^\lambda(L^r), r \ge 1, 1 < q < \infty$, then*

$$\text{(i)} \int_0^\pi |K_m^{\Delta_H N^{p,q}}(\eta)| \left(\int_0^\eta \frac{\|\Phi(\cdot, l, \eta)\|_r^q}{l^{\delta q}} \frac{dl}{l}\right)^{\frac{1}{q}} d\eta = O(1) \left\{\int_0^\pi (\eta^{\lambda-\delta} |K_m^{\Delta_H N^{p,q}}(\eta)|)^{\frac{q}{q-1}} d\eta\right\}^{1-(1/q)},$$

$$\text{(ii)} \int_0^\pi |K_m^{\Delta_H N^{p,q}}(\eta)| \left(\int_\eta^\pi \frac{\|\Phi(\cdot, l, \eta)\|_r^q}{l^{\delta q}} \frac{dl}{l}\right)^{\frac{1}{q}} d\eta = O(1) \left\{\int_0^\pi (\eta^{\lambda-\delta+(1/q)} |K_m^{\Delta_H N^{p,q}}(\eta)|)^{\frac{q}{q-1}} d\eta\right\}^{1-(1/q)}.$$

Proof. This Lemma can be proved along the same lines of the proof of Lemma 1 of [12]. □

Lemma 4.4. *([12]) Let $0 \le \delta < \lambda < 2$. If $g \in B_q^\lambda(L^r), r \ge 1, q = \infty$, then*

$$\sup_{0<l,\eta\le\pi} (l^{-\delta} \|\Phi(\cdot, l, \eta)\|_r) = O(\eta^{\lambda-\delta}). \tag{4.6}$$

5 Proof of the Main theorem

Proof. Following [5], $s_m(g, z)$ of Fourier series is given by

$$s_m(g; z) - g(z) = \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \frac{\sin(m + \frac{1}{2})\eta}{\sin \frac{\eta}{2}} d\eta.$$

Denoting the $N^{p,q}$ summability transform of $s_m(g; z)$ by $t_m^{p,q}(z)$, we get

$$t_m^{p,q}(z) - g(z) = \sum_{j=0}^m t_j^{p,q} \{s_j(g; z) - g(z)\}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\phi_z(\eta)\sum_{j=0}^{m} t_j^{p,q}\frac{\sin\left(m+\frac{1}{2}\right)\eta}{2\sin\frac{\eta}{2}}\right)d\eta$$

$$= \left\{\frac{1}{2\pi}\int_0^{\pi}\phi_z(\eta)\sum_{j=0}^{m}\left(\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{\sin\frac{\eta}{2}}\right)\right\}d\eta.$$

The Hausdorff transform of $t_m^{p,q}(z)$ i.e., $\Delta_H N^{p,q}$ transform of $s_m(g;z)$ denoted by $t_m^{\Delta_H N^{p,q}}$, is given by

$$\begin{aligned}
&t_m{}^{\Delta_H N^{p,q}}(z) - g(z)\\
&\quad= \sum_{j=0}^{m} h_{m,j}\left\{t_m^{p,q}(z) - g(z)\right\}\\
&\quad= \sum_{j=0}^{m}\binom{m}{j}\Delta^{m-j}\mu_j\left\{t_m^{p,q}(z) - g(z)\right\}\\
&\quad= \sum_{j=0}^{m}\binom{m}{j}\Delta^{m-j}\mu_j\left\{\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu s_\nu\right\}\\
&\quad= \frac{1}{\pi}\int_0^{\pi}\phi_z(\eta)\sum_{j=0}^{m}\binom{m}{j}\Delta^{m-j}\mu_j\left\{\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{2\sin\frac{\eta}{2}}\right\}d\eta\\
&\quad= \frac{1}{2\pi}\int_0^{\pi}\phi_z(\eta)\sum_{j=0}^{m}\binom{m}{j}\int_0^1 y^j(1-y)^{m-j}d\gamma(y)\left\{\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{\sin\frac{\eta}{2}}\right\}d\eta\\
&\quad= \frac{1}{2\pi}\int_0^{\pi}\phi_z(\eta)\left\{\int_0^1\sum_{j=0}^{m}\binom{m}{j}y^j(1-y)^{m-j}\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{\sin\frac{\eta}{2}}d\gamma(y)\right\}d\eta\\
&\quad= \int_0^{\pi}\phi_z(\eta)K_m^{\Delta_H N^{p,q}}(\eta)d\eta.
\end{aligned}$$

Let

$$l_m(z) := t_m^{\Delta_H N^{p,q}}(z) - g(z) = \frac{1}{\pi}\int_0^{\pi}\phi_z(\eta)K_m^{\Delta_H N^{p,q}}(\eta)d\eta, \tag{5.1}$$

where

$$\begin{aligned}
K_m^{\Delta_H N^{p,q}}(\eta) &= \int_0^1\sum_{j=0}^{m}\binom{m}{j}y^j(1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{\sin\frac{\eta}{2}}\right\}d\gamma(y)\\
&= \int_0^1 M_m(\eta)d\gamma(y),
\end{aligned}$$

where

$$M_m(\eta) = \sum_{j=0}^{m}\binom{m}{j}y^j(1-y)^{m-j}\left\{\frac{1}{R_j}\sum_{\nu=0}^{j}p_{j-\nu}q_\nu\frac{\sin\left(m+\frac{1}{2}\right)\eta}{\sin\frac{\eta}{2}}\right\}.$$

We write,

$$\Phi(z,l,\eta)=\begin{cases}\phi_{z+l}(\eta)-\phi_z(\eta), & 0<\lambda<1,\\ \phi_{z+l}(\eta)+\phi_{z-l}(\eta)-2\phi_z(\eta), & 1\le\lambda<2.\end{cases}$$

and

$$\mathcal{L}_m(z,l)=\begin{cases}l_m(z+l)-l_m(z), & 0<\lambda<1,\\ l_m(z+l)+l_m(z-l)-2l_m(z), & 1\le\lambda<2.\end{cases}$$

Now, we have

$$\mathcal{L}_m(z,l)=\frac{1}{\pi}\int_0^\pi K_m^{\Delta_H N^{p,q}}(\eta)\Phi(z,l,\eta)d\eta \quad \text{and} \quad \omega_j(l_m,l)_r=\|\mathcal{L}_m(\cdot,l)\|_r.$$

Case I : For $1<q<\infty, r\ge 1, 0\le\delta<\lambda<2.$

By definition, we have

$$\|l_m(\cdot)\|_{B_q^\delta(L^r)}=\|l_m(\cdot)\|_r+\|w_j(l_m,\cdot)\|_{\delta,q}. \tag{5.2}$$

Using generalized Minkowski's inequality [3], Lemma 4.2 (iii) and (5.1), we have

$$\begin{aligned}\|l_m(\cdot)\|_r&\le\frac{1}{\pi}\int_0^\pi\|\phi(\eta)\|_r|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta\\ &\le\frac{2}{\pi}\int_0^\pi w_j(g,\eta)_r|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta.\end{aligned}$$

Using Hölder's inequality and definition of Besov space, we get

$$\begin{aligned}\|l_m(\cdot)\|_r&\le 2\left\{\int_0^\pi\left(|K_m^{\Delta_H N^{p,q}}(\eta)|\eta^{\lambda+q^{-1}}\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}\times\left\{\int_0^\pi\left(\frac{w_j(g,\eta)_r}{\eta^{\lambda+q^{-1}}}\right)^q d\eta\right\}^{q^{-1}}\\ &=O(1)\left\{\int_0^\pi\left(|K_m^{\Delta_H N^{p,q}}(\eta)|\eta^{\lambda+q^{-1}}\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}\\ &=O(1)\left\{\left(\int_0^{\frac{1}{m+1}}+\int_{\frac{1}{m+1}}^\pi\right)\left(|K_m^{\Delta_H N^{p,q}}(\eta)|\eta^{\lambda+q^{-1}}\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}\\ &:=O(1)\left[I_1+I_2\right].\end{aligned} \tag{5.3}$$

Using Lemma 4.1 for $0\le\eta\le\frac{1}{m+1}$, we get

$$\begin{aligned}I_1&=\left\{\int_0^{\frac{1}{m+1}}\left(|K_m^{\Delta_H N^{p,q}}(\eta)|\eta^{\lambda+q^{-1}}\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}\\ &=O(m+1)\left\{\int_0^{\frac{1}{m+1}}\left(\eta^{\lambda+q^{-1}}\right)^{\frac{q}{q-1}}d\eta\right\}^{1-q^{-1}}\end{aligned}$$

$$
\begin{aligned}
&= O(m+1)\left\{\int_0^{\frac{1}{m+1}} \eta^{\frac{q}{q-1}(\lambda+q^{-1})} d\eta\right\}^{1-q^{-1}} \\
&= O((m+1)^{-\lambda}).
\end{aligned} \tag{5.4}
$$

By using Lemma 4.1 for $\frac{1}{m+1} \le \eta \le \pi$, we get

$$
\begin{aligned}
I_2 &= \left\{\int_{\frac{1}{m+1}}^{\pi} \left(|K_m^{\Delta_H N^{p,q}}(\eta)|\eta^{\lambda+q^{-1}}\right)^{q/(q-1)} d\eta\right\}^{1-q^{-1}} \\
&= O(1)\left\{\int_{\frac{1}{m+1}}^{\pi} \left(\eta^{\lambda+q^{-1}-1}\right)^{q/(q-1)} d\eta\right\}^{1-q^{-1}} \\
&= O(1)\left\{\int_{\frac{1}{m+1}}^{\pi} \eta^{\frac{q}{q-1}(\lambda+q^{-1}-1)} d\eta\right\}^{1-q^{-1}} \\
&= O(1)\left\{\int_{\frac{1}{m+1}}^{\pi} \eta^{\frac{q}{q-1}\lambda-1} d\eta\right\}^{1-q^{-1}} \\
&= O((m+1)^{-\lambda}).
\end{aligned} \tag{5.5}
$$

From (5.3), (5.4) and (5.5), we have

$$
\|l_m(\cdot)\|_r = O((m+1)^{-\lambda}). \tag{5.6}
$$

Now, using generalized Minkowski's inequality and using Lemma 4.3, we have

$$
\begin{aligned}
&\|w_j\left(l_m,\cdot\right)\|_{\delta,q} \\
&= \left\{\int_0^{\pi} \left(\frac{w_j\left(l_m,l\right)r}{l^{\delta}}\right)^q \frac{dl}{l}\right\}^{q^{-1}} \\
&= \left\{\int_0^{\pi} \left(\frac{\|\mathcal{L}_m\left(\cdot,l\right)\|_r}{l^{\delta}}\right)^q \frac{dl}{l}\right\}^{q^{-1}} \\
&= \left\{\int_0^{\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathcal{L}_m\left(z,l\right)|^r dz\right)^{q/r} \frac{dl}{l^{\delta q+1}}\right\}^{q^{-1}} \\
&= \left\{\int_0^{\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{\pi}\int_0^{\pi} \Phi\left(z,l,\eta\right) K_m^{\Delta_H N^{p,q}}(\eta)d\eta\right|^r dz\right)^{q/r} \frac{dl}{l^{\delta q+1}}\right\}^{q^{-1}} \\
&\le \frac{1}{\pi}\left[\int_0^{\pi} \left(\frac{1}{2\pi}\right)^{q/r} \times \left\{\int_0^{\pi} \left(\int_0^{2\pi} |\Phi\left(z,l,\eta\right)|^r |K_m^{\Delta_H N^{p,q}}(\eta)|^r dz\right)^{1/r} d\eta\right\} \frac{dl}{l^{\delta q+1}}\right]^{q^{-1}} \\
&= \frac{1}{\pi}\left[\int_0^{\pi} \left\{\int_0^{\pi} \|\Phi\left(\cdot,l,\eta\right)\|_r |K_m^{\Delta_H N^{p,q}}(\eta)d\eta\right\} \frac{dl}{l^{\delta q+1}}\right]^{q^{-1}}
\end{aligned}
$$

$$\le \frac{1}{\pi}\int_0^{\pi}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta\left(\int_0^{\pi}\frac{\|\Phi(\cdot,l,\eta)\|_r^q}{l^{\delta q}}\frac{dl}{l}\right)^{q^{-1}}$$
$$=\frac{1}{\pi}\int_0^{\pi}|K_m^{\Delta_H N^{p,q}}(\eta)d\eta\left\{\left(\int_0^{\eta}+\int_{\eta}^{\pi}\right)\frac{\|\Phi(\cdot,l,\eta)\|_r^q}{l^{\delta q}}\frac{dl}{l}\right\}^{q^{-1}}$$
$$\le \frac{1}{\pi}\int_0^{\pi}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta\left\{\int_0^{\eta}\frac{\|\Phi(\cdot,l,\eta)\|_r^q}{l^{\delta q}}\frac{dl}{l}\right\}^{q^{-1}}$$
$$+\frac{1}{\pi}\int_0^{\pi}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta\left\{\int_{\eta}^{\pi}\frac{\|\Phi(\cdot,l,\eta\|_r^q)}{l^{\delta q}}\frac{dl}{l}\right\}^{q^{-1}}$$
$$=O(1)\left\{\int_0^{\pi}\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}d\eta\right\}^{1-(1/q)}$$
$$+O(1)\left\{\int_0^{\pi}\left(\eta^{\lambda-\delta+(1/q)}|K_m^{\Delta_H N^{p,q}}(\eta)\right)^{q/(q-1)}d\eta\right\}^{1-(1/q)}$$
$$:=O(1)\left(J_1+J_2\right). \tag{5.7}$$

Since $(x+y)^r\le x^r+y^r$ for positive x, y and $0<r\le 1$, then

$$J_1=\left\{\int_0^{\pi}\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}$$
$$=\left\{\left(\int_0^{1/(m+1)}+\int_{1/(m+1)}^{\pi}\right)\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}\right\}^{1-q^{-1}}$$
$$\le\left\{\int_0^{1/(m+1)}\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}$$
$$+\left\{\int_{1/(m+1)}^{\pi}\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}$$
$$=I_{11}+I_{12}. \tag{5.8}$$

Using Lemma 4.1 for $0\le\eta\le\frac{1}{m+1}$, we have

$$I_{11}=\left\{\int_0^{1/(m+1)}\left(\eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|\right)^{q/(q-1)}d\eta\right\}^{1-q^{-1}}$$
$$=O\left(m+1\right)\left\{\int_0^{1/(m+1)}\eta^{\frac{q}{q-1}(\lambda-\delta)}d\eta\right\}^{1-q^{-1}}$$
$$=O\left(m+1\right)\left\{\int_0^{1/(m+1)}\eta^{\frac{q}{q-1}(\lambda-\delta+1-(1/q))-1}d\eta\right\}^{1-q^{-1}}$$
$$=O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right). \tag{5.9}$$

Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we have

$$\begin{aligned} I_{12} &= \left\{ \int_{1/(m+1)}^{\pi} \left(\eta^{\lambda-\delta} |K_m^{\Delta_H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(1) \left\{ \int_{1/(m+1)}^{\pi} \left(\eta^{\lambda-\delta-1} \right)^{\frac{q}{q-1}} d\eta \right\}^{1-q^{-1}} \\ &= O(1) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\frac{q}{q-1}(\lambda-\delta-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(1) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\frac{q}{q-1}(\lambda-\delta-(1/q))-1} d\eta \right\}^{1-q^{-1}} \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \end{aligned} \tag{5.10}$$

From (5.8), (5.9) and (5.10), we have

$$\begin{aligned} J_1 &:= I_{11} + I_{12} \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \end{aligned} \tag{5.11}$$

Now,

$$\begin{aligned} J_2 &= \left\{ \int_0^{\pi} \left(\eta^{\lambda-\delta+1/q} |K_m^{\Delta_H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= \left[\left\{ \int_0^{1/(m+1)} + \int_{1/(m+1)}^{\pi} \right\} \left(\eta^{\lambda-\delta+(1/q)} |K_m^{\Delta_H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right]^{1-q^{-1}} \\ &:= J_{11} + J_{12}. \end{aligned} \tag{5.12}$$

Using Lemma 4.1 for $0 \leq \eta \leq \frac{1}{m+1}$, we have

$$\begin{aligned} J_{11} &= \left\{ \int_0^{1/(m+1)} \left(\eta^{\lambda-\delta+(1/q)} |K_m^{\Delta_H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(m+1) \left\{ \int_0^{1/(m+1)} \left(\eta^{\lambda-\delta+(1/q)} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(m+1) \left\{ \int_0^{1/(m+1)} \eta^{\frac{q}{q-1}(\lambda-\delta+1)-1} d\eta \right\}^{1-q^{-1}} \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}} \right). \end{aligned} \tag{5.13}$$

Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we have

$$\begin{aligned} J_{12} &= \left\{ \int_{1/(m+1)}^{\pi} \left(\eta^{\lambda-\delta+(1/q)} |K_m^{\Delta_H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(1) \left\{ \int_{1/(m+1)}^{\pi} \left(\eta^{\lambda-\delta+(1/q)-1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\ &= O(1) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\frac{q}{q-1}(\lambda-\delta)-1} d\eta \right\}^{1-q^{-1}} \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}} \right). \end{aligned} \tag{5.14}$$

From (5.12), (5.13) and (5.14), we have

$$\begin{aligned} J_2 &:= J_{11} + J_{12} \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}} \right). \end{aligned} \tag{5.15}$$

From (5.7), (5.11) and (5.15), we get

$$\begin{aligned} \|w_j(l_m, \cdot)\|_{\delta,q} &= O(1)(J_1 + J_2) \\ &= O(1) \left[O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}} \right) \right]. \end{aligned} \tag{5.16}$$

From (5.2), (5.6) and (5.16), we get

$$\begin{aligned} \|l_m(\cdot)\|_{B_q^\delta(L^r)} &= \|l_m(\cdot)\|_r + \|w_j(l_m, \cdot)\|_{\delta,q} \\ &= O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}} \right). \end{aligned}$$

This completes the proof of case I.

Case II : For $q = \infty$, $0 \leq \delta < \lambda < 2$.

We have

$$\|l_m(\cdot)\|_{B_\infty^\delta(L^r)} = \|l_m(\cdot)\|_r + \|w_j(l_m, \cdot)\|_{\delta,\infty}. \tag{5.17}$$

Using (1.6), we have

$$\begin{aligned} \|l_m(\cdot)\|_r &\leq 2 \int_0^{\pi} w_j(g, \eta)_r |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \\ &= O(1) \left\{ \int_0^{1/(m+1)} \eta^\lambda |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta + \int_{1/(m+1)}^{\pi} \eta^\lambda |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \right\} \end{aligned}$$

$$:= O\left(1\right)\left(I_2 + J_2\right). \tag{5.18}$$

Using Lemma 4.1 for $0 \le \eta \le \frac{1}{m+1}$, we get

$$\begin{aligned} I_2 &= \int_0^{1/(m+1)} \eta^{\lambda} |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \\ &= O\left(m+1\right) \int_0^{1/(m+1)} \eta^{\lambda} d\eta \\ &= O\left(m+1\right)^{-\lambda}. \end{aligned} \tag{5.19}$$

Again, using Lemma 4.1 for $\frac{1}{m+1} \le \eta \le \pi$, we get

$$\begin{aligned} J_2 &= \int_{1/(m+1)}^{\pi} \eta^{\lambda} |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \\ &= O\left(1\right) \int_{1/(m+1)}^{\pi} \eta^{\lambda - 1} d\eta \\ &= O\left(m+1\right)^{-\lambda}. \end{aligned} \tag{5.20}$$

From (5.18), (5.19) and (5.20), we get

$$\begin{aligned} \|l_m\left(\cdot\right)\|_r &= O\left(1\right)\left(I_2 + J_2\right) \\ &= O\left(m+1\right)^{-\alpha}. \end{aligned} \tag{5.21}$$

Using generalized Minkowski's inequality and Lemma 4.4, we get

$$\begin{aligned} &\|w_j\left(l_m, \cdot\right)\|_{\delta,\infty} \\ &= \sup_{l>0} \left(l^{-\delta} w_j\left(l_m, l\right)_r\right) \\ &= \sup_{l>0} \left(l^{-\delta} \|\mathcal{L}_m\left(\cdot, l\right)\|_r\right) \\ &= \sup_{l>0} \left[l^{-\delta} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{\pi} K_m^{\Delta_H N^{p,q}}(\eta) \phi\left(z, l, \eta\right) d\eta \right|^r dz \right)^{1/r} \right] \\ &\le \sup_{l>0} \left[\frac{l^{-\delta}}{\pi} \left(\frac{1}{2\pi}\right)^{1/r} \int_0^{\pi} \left\{ \int_0^{2\pi} |K_m^{\Delta_H N^{p,q}}(\eta)|^r |\phi\left(z, l, \eta\right)|^r dz \right\}^{1/r} d\eta \right] \\ &= \sup_{l>0} \left[\frac{l^{-\delta}}{\pi} \int_0^{\pi} \|\phi\left(\cdot, l, \eta\right)\|_r |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \right] \\ &= \frac{1}{\pi} \int_0^{\pi} \left(\sup_{l>0} l^{-\delta} \|\Phi\left(\cdot, l, \eta\right)\|_r \right) |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \\ &= O\left(1\right) \int_0^{\pi} \eta^{\lambda - \delta} |K_m^{\Delta_H N^{p,q}}(\eta)| d\eta \end{aligned}$$

$$= O(1)\left[\int_0^{1/(m+1)} \eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)d\eta + \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta\right]$$

$$= O(1)\left[I_3 + J_3\right]. \tag{5.22}$$

Using Lemma 4.1 for $0 \leq \eta \leq \frac{1}{m+1}$, we get

$$\begin{aligned} I_3 &= \int_0^{1/(m+1)} \eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta \\ &= O(m+1)\int_0^{1/(m+1)} \eta^{\lambda-\delta}d\eta \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \end{aligned} \tag{5.23}$$

Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we get

$$\begin{aligned} J_3 &= \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta}|K_m^{\Delta_H N^{p,q}}(\eta)|d\eta \\ &= O(1)\int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta-1}d\eta \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \end{aligned} \tag{5.24}$$

From (5.22), (5.23) and (5.24), we get

$$\begin{aligned} \|w_j(l_m,\cdot)\|_{\delta,\infty} &= O(1)\left[I_3 + J_3\right] \\ &= O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \end{aligned} \tag{5.25}$$

From (5.17), (5.21) and (5.25), we have

$$\begin{aligned} \|l_m(\cdot)\|_{B_\infty^\beta(L^r)} &= \|l_m(\cdot)\|_r + \|w_j(l_m,\cdot)\|_{\delta,\infty} \\ &= O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \end{aligned}$$

This completes the proof of case II. □

6 Corollary

The following corollary are derived from our main theorem.

Corollary 6.1. *If $q_m = 1\ \forall\ m$, then $\Delta_H N^{p,q}$ means reduces to $\Delta_H N^{p_m}$ means and the best approximation of $g \in B_q^\lambda(L^r)$ space by $\Delta_H N^{p_m}$ means of Fourier series is*

$$\|t_m^{\Delta_H N^{p_m}}(z) - g(z)\| = \begin{cases} O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; 1 < q < \infty \\ O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ; q = \infty. \end{cases}$$

Corollary 6.2. *If* $p_m = 1\,\forall\, m$, *then* $\Delta_H N^{p,q}$ *reduces to* $\Delta_H \tilde{N}^{q_m}$ *means and the best approximation of* $g \in B_q^\lambda(L^r)$ *space by* $\Delta_H \tilde{N}^{q_m}$ *means of Fourier series is*

$$\|t_m^{\Delta_H \tilde{N}^{q_m}}(z)-g(z)\| = \begin{cases} O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;1<q<\infty \\ O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;q=\infty. \end{cases}$$

Corollary 6.3. *If* $p_m = \binom{m+\alpha-1}{\alpha-1}$ $\alpha > 0$, *and* $q_m = 1\,\forall\, m$, *then* $\Delta_H N^{p,q}$ *means reduces to* $\Delta_H C_\alpha$ *means of and the best approximation of* $g \in B_q^\lambda(L^r)$ *space by* $\Delta_H C_\alpha$ *means of Fourier series is*

$$\|t_m^{\Delta_H C_\alpha}(z) - g(z)\| = \begin{cases} O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;1<q<\infty \\ O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;q=\infty. \end{cases}$$

Corollary 6.4. *If* $h_{m,j} = \begin{cases} \binom{m}{j}\frac{q^{m-j}}{(1+q)^m}, & if\ \ 0 \le j \le m, \\ 0, & if\ \ j > m. \end{cases}$, *then* $\Delta_H N^{p,q}$ *means reduces to* $E_q N^{p,q}$ *means and the best approximation of* $g \in B_q^\lambda(L^r)$ *space by* $E_q N^{p,q}$ *means of Fourier series is*

$$\|t_m^{E_q N^{p,q}}(z) - g(z)\| = \begin{cases} O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;1<q<\infty \\ O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;q=\infty. \end{cases}$$

Corollary 6.5. *If* $h_{m,j} = \begin{cases} \frac{1}{m+1}, & if\ \ 0 \le j \le m, \\ 0, & if\ \ j > m. \end{cases}$, *then* $\Delta_H N^{p,q}$ *means reduces to* $C_1 N^{p,q}$ *means and the best approximation of* $g \in B_q^\lambda(L^r)$ *space by* $C_1 N^{p,q}$ *means of Fourier series is*

$$\|t_m^{C_1 N^{p,q}}(z) - g(z)\| = \begin{cases} O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;1<q<\infty \\ O\,(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right) & ;q=\infty. \end{cases}$$

7 Particular cases

(i) In view of Remark 2 (i) and 3 (ii), our result becomes a particular case of [10].

(ii) In view of Remark 2 (i) and 3 (iv), our result becomes a particular case of [26].

(iii) In view of Remark 2 (ii) and 3 (v), our result becomes a particular case of [27].

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