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Strongly Starlike Functions and Related Classes

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Abstract. We consider univalent functions, analytic in the unit disc $|z|<1$ in the complex plane $\mathbb{C}$ which map $|z|<1$ onto a domain with some nice property. The purpose of this paper is to find some new conditions for strong starlikeness and some related results.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D}=\{z: \ |z|<1\}$ in the complex plane $\mathbb{C}$. For $a\in\mathbb{C}$ and $n\in\mathbb{N}$ we denote by

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}: f(z)=a+a_nz^n+\cdots, a_n\neq 0\}$$

and

$$\mathcal{A}_n=\left\{f\in\mathcal{H}: f(z)=z+a_{n+1}z^{n+1}+\cdots\right\},$$

so $\mathcal{A}=\mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$.

A set $E\subset\mathbb{C}$ is said to be starlike with respect to the point $0\in E$ if and only if the linear segment joining the origin to every other point $w\in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points. Let $\mathcal{S}^*$ denote the class of all functions $f\in\mathcal{S}$ such that the set $f(\mathbb{D})$ is starlike with respect 0. A natural extension of the notion starlike is to be starlike of order α. The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$\mathcal{S}^*_\alpha=\left\{f\in\mathcal{A}: \ \mathfrak{Re}\frac{zf'(z)}{f(z)}>\alpha, \ z\in\mathbb{D}\right\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha<1$

$$\mathcal{K}_\alpha:=\left\{f\in\mathcal{A}: \ \mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, \ z\in\mathbb{D}\right\}$$

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$$= \{f \in \mathcal{A} : \ zf' \in \mathcal{S}^*_\alpha\}$$

introduced Robertson in [14], see also [5]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*_0 = \mathcal{S}^*, \mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively. Furthermore, note that if $f \in \mathcal{K}_\alpha$ then $f \in \mathcal{S}^*_{\delta(\alpha)}$, see [18], where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for} \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{for} \quad \alpha = \frac{1}{2}. \end{cases} \tag{1.1}$$

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta < 2$,

$$\mathcal{SS}^*(\beta) := \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, \ \ z \in \mathbb{D} \right\}, \tag{1.2}$$

which was introduced in [15] and [1], see also [9]. Furthermore, $\mathcal{SK}(\beta) = \{f \in \mathcal{A} : \ zf' \in \mathcal{SS}^*(\beta)\}$ denote the class of strongly convex functions of order β. Analogously to (1.1), in the work [8] it was proved that if $\beta \in (0,1)$ and $f \in \mathcal{SK}(\alpha(\beta))$, then $f \in \mathcal{SS}(\beta)$, where

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left(\frac{\beta n(\beta) \sin(\pi(1-\beta)/2)}{m(\beta) + \beta n(\beta) \cos(\pi(1-\beta)/2)} \right), \tag{1.3}$$

and where

$$m(\beta) = (1+\beta)^{(1+\beta)/2} \quad n(\beta) = (1-\beta)^{(\beta-1)/2}.$$

The class $\mathcal{G}(\alpha, \gamma)$, $\gamma > 0$, $0 < \alpha \leq 1$ of γ-strongly starlike functions of order α consists of functions $f \in \mathcal{A}$ satisfying

$$\left| \arg \left\{ \left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right)^{\gamma} \right\} \right| < \frac{\alpha\pi}{2}, \ \ z \in \mathbb{D}, \tag{1.4}$$

and such that

$$f(z)f'(z) \left(1 + \frac{zf''(z)}{f'(z)} \right) \neq 0, \ \ z \in \mathbb{D} \setminus \{0\}. \tag{1.5}$$

Note that Lewandowski, S. Miller and Złotkiewicz 1974 [6] have introduced the class of γ-starlike functions, denoted here by $\mathcal{G}(1, \gamma)$, which satisfy (1.5) and such that

$$\mathfrak{Re} \left\{ \left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right)^{\gamma} \right\} > 0, \ \ z \in \mathbb{D}. \tag{1.6}$$

Lemma 1.1. (Nunokawa [8]) Let $q(z)$ be in $\mathcal{H}[1, m]$ with $q(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that $|\arg\{q(z)\}| < \pi\gamma/2$ for $|z| < |z_0|$ and $|\arg\{q(z_0)\}| = \pi\gamma/2$ for some $\gamma \in (0, 2)$, then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{2ik \arg\{q(z_0)\}}{\pi}, \tag{1.7}$$

for some $k \geq m(a + a^{-1})/2 \geq (a + a^{-1})/2$, where $\{q(z_0)\}^{1/\gamma} = \pm ia$, and $a > 0$.

It should be remarked that for interesting related results associated with Lemma 2.1, the authors may refer to [7]. In this paper we consider some problems related to Lemma 2.1, which may be called as Nunokawa's lemma.

2 Main result

Lemma 2.1. *Let β be in $(0,1)$. Then we have $\forall\beta\in(0,1)$:*

$$\arg\left\{e^{i\pi(1-\beta)/2}\left[\left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2}+\left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}\right]-1\right\}\leq\pi-\frac{\pi\beta}{2}. \tag{2.1}$$

Proof. Observe that

$$\forall\beta\in(0,1):\ 1<\left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2}+\left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}<2.$$

Therefore, a geometrical observation shows that for all $\beta\in(0,1)$, we have

$$\begin{aligned}&\arg\left\{e^{i\pi(1-\beta)/2}\left[\left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2}+\left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}\right]-1\right\}\\ \leq\ &\arg\left\{e^{i\pi(1-\beta)/2}-1\right\}\\ =\ &\frac{\pi}{2}+\frac{(1-\beta)\pi}{4}\\ <\ &\pi-\frac{\pi\beta}{2}.\end{aligned}$$

□

Theorem 2.1. *Let β be in $(0,1)$, $p(z)\in\mathcal{H}[1,1]$, and suppose that*

$$\left|\arg\left\{\frac{zp'(z)}{p(z)}-p(z)\right\}\right|>\frac{\pi\alpha(\beta)}{2},\quad z\in\mathbb{D}, \tag{2.2}$$

then we have

$$|\arg\{p(z)\}|<\frac{\pi\beta}{2},\quad z\in\mathbb{D}, \tag{2.3}$$

where

$$\tan\frac{\pi\alpha(\beta)}{2}=\tan\frac{\pi\beta}{2}+\frac{\beta}{(1-\beta)\cos(\pi\beta/2)}\left(\frac{1-\beta}{1+\beta}\right)^{(1+\beta)/2}. \tag{2.4}$$

Proof. If there exists a point z_0, $|z_0|<1$, such that $|\arg\{p(z)\}|<\pi\beta/2$ for $|z|<|z_0|$ and $|\arg\{p(z_0)\}|=\pi\beta/2$ for some $0<\beta<1$, then by Lemma 1.1 we have

$$\frac{z_0p'(z_0)}{p(z_0)}=\frac{2ik\arg\{p(z_0)\}}{\pi}, \tag{2.5}$$

for some

$$k \geq (a + a^{-1})/2,$$

where

$$\{p(z_0)\}^{1/\beta} = \pm ia \tag{2.6}$$

and $a > 0$. Then, for the case $p(z_0) = a^{\beta}(\cos(\pi\beta/2) + i\sin(\pi\beta/2))$, we have from (2.5)

$$\frac{z_0 p'(z_0)}{p^2(z_0)} - 1 = e^{i\pi(1-\beta)/2}\beta k \frac{1}{a^{\beta}} - 1. \tag{2.7}$$

Note that

$$\beta k \frac{1}{a^{\beta}} \geq \frac{\beta}{2}(a^{1-\beta} + a^{-1-\beta}).$$

Let us put

$$g(a) = \frac{1}{2}(a^{1-\beta} + a^{-1-\beta}), \quad a > 0.$$

Then we have

$$g'(a) = \frac{1}{2}((1-\beta)a^{-\beta} - (1+\beta)a^{-2-\beta}),$$

and so, $g(a)$ takes its minimum value at

$$a = \sqrt{(1+\beta)/(1-\beta)}. \tag{2.8}$$

Furthermore, for the function

$$H(\beta) = \left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2} + \left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}, \quad \beta \in (0,1),$$

the $H'(\beta)$ decreases from 0, when $\beta \in (0,1)$. Therefore, $H(\beta)$ is a decreasing function, from 2 to 1:

$$\lim_{\beta \to 1^-} = \left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2} + \left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2} = 1.$$

Therefore, from (2.7) and from (2.8), we have

$$\begin{aligned} \arg\left\{\frac{z_0 p'(z_0)}{p^2(z_0)} - 1\right\} &= \arg\left\{e^{i\pi(1-\beta)/2}\beta k \frac{1}{a^{\beta}} - 1\right\} \\ &\leq \arg\left\{e^{i\pi(1-\beta)/2}\left(\left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2} + \left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}\right) - 1\right\} \\ &\leq \pi - \frac{\pi\beta}{2} \end{aligned} \tag{2.9}$$

because of Lemma 2.1. We also have

$$\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) = p(z_0)\left\{\frac{z_0 p'(z_0)}{p^2(z_0)} - 1\right\}$$

$$\begin{aligned} &= (ia)^{\beta}\left(i\beta k \frac{1}{(ia)^{\beta}} - 1\right) \\ &= a^{\beta} e^{i\pi\beta/2}\left(e^{i\pi(1-\beta)/2}\beta k \frac{1}{a^{\beta}} - 1\right). \end{aligned}$$

Hence we may write

$$\begin{aligned} \arg\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} &= \arg\left\{p(z_0)\left\{\frac{z_0 p'(z_0)}{p^2(z_0)} - 1\right\}\right\} \\ &= \arg\{p(z_0)\} + \arg\left\{\frac{z_0 p'(z_0)}{p^2(z_0)} - 1\right\} \\ &= \frac{\pi\beta}{2} + \arg\left\{e^{i\pi(1-\beta)/2}\frac{\beta k}{a^{\beta}} - 1\right\} \\ &\le \frac{\pi\beta}{2} + \arg\left\{e^{i\pi(1-\beta)/2}\left(\left(\frac{1+\beta}{1-\beta}\right)^{(1-\beta)/2} + \left(\frac{1+\beta}{1-\beta}\right)^{-(1+\beta)/2}\right) - 1\right\} \\ &= \frac{\pi\beta}{2} + \tan^{-1}\frac{\frac{\beta}{1-\beta}\left(\frac{1-\beta}{1+\beta}\right)^{(1+\beta)/2}\sin\frac{\pi(1-\beta)}{2}}{-1 + \frac{\beta}{1-\beta}\left(\frac{1-\beta}{1+\beta}\right)^{(1+\beta)/2}\cos\frac{\pi(1-\beta)}{2}} \\ &= \frac{\pi\beta}{2} + \tan^{-1}\frac{\beta n(\beta)\sin\frac{\pi(1-\beta)}{2}}{m(\beta) + \beta n(\beta)\cos\frac{\pi(1-\beta)}{2}}, \end{aligned}$$

where

$$m(\beta) = (1+\beta)^{(1+\beta)/2}, \quad n(\beta) = (1-\beta)^{(\beta-1)/2}.$$

This contradicts the hypothesis (2.2) and for the case $p(z_0) = -\pi\beta/2$, applying the same method as the above and Lemma 2.1, we can have

$$\arg\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} \ge -\frac{\pi\beta}{2} - \tan^{-1}\frac{\beta n(\beta)\sin\frac{\pi(1-\beta)}{2}}{m(\beta) + \beta n(\beta)\cos\frac{\pi(1-\beta)}{2}}.$$

This contradicts the hypothesis (2.2) and therefore, it completes the proof of Theorem 2.1. □

For $\beta = 1/2$ Theorem 2.1 becomes the following corollary.

Corollary 2.2. *Let $p(z)$ be in $\mathcal{H}[1,1]$, and suppose that*

$$\left|\arg\left\{\frac{zp'(z)}{p(z)} - p(z)\right\}\right| > \frac{\pi}{2}\frac{2\tan^{-1}(1+\sqrt[4]{108})}{\pi} = \frac{\pi}{2}(0.85\ldots) = 1.3383\ldots, \quad z \in \mathbb{D}$$

then we have

$$|\arg\{p(z)\}| < \frac{\pi}{4}, \quad z \in \mathbb{D}.$$

For $p(z) = zf'(z)/f(z)$, $f(z) \in \mathcal{A}$, Theorem 2.1 becomes the following corollary.

Corollary 2.3. *Let β be in $(0,1)$. Assume that $f(z)\in\mathcal{A}$ and $zf'(z)/f(z)\in\mathcal{H}$ in $\mathbb{D}$, and that*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)}{f(z)}\right\}\right|>\frac{\pi\alpha(\beta)}{2},\quad z\in\mathbb{D},$$

then we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi\beta}{2},\quad z\in\mathbb{D},$$

where $\alpha(\beta)$ is defined in (2.4). This means that $f(z)\in\mathcal{SS}^*(\beta)$ or $f(z)$ is a strongly starlike function of order β, see (1.2).

Corollary 2.2 and 2.3 give together the following result.

Corollary 2.4. *Assume that $f(z)\in\mathcal{A}$ and $zf'(z)/f(z)\in\mathcal{H}$ in $\mathbb{D}$, and that*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)}{f(z)}\right\}\right|>\frac{\pi}{2}\frac{2\tan^{-1}(1+\sqrt[4]{108})}{\pi}=\frac{\pi}{2}(0.85\ldots)=1.3383\ldots,$$

$z\in\mathbb{D}$, then we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{4},\quad z\in\mathbb{D},$$

or $f(z)$ is a strongly starlike function of order $1/2$, see (1.2).

Corollary 2.5. *Assume that $f(z)\in\mathcal{A}$ and $zf'(z)/f(z)\in\mathcal{H}$ in $\mathbb{D}$, and that*

$$\left|\arg\left\{\frac{z(2f''(z)+zf'''(z))}{2(f'(z)+zf''(z))}-\frac{zf'(z)}{2f(z)}-\sqrt{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)}\right\}\right|>\frac{\pi\alpha(\beta)}{2},\quad z\in\mathbb{D},$$

then we have

$$\left|\arg\left\{\sqrt{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)}\right\}\right|<\frac{\pi\beta}{2},\quad z\in\mathbb{D},$$

where $\alpha(\beta)$ is defined in (2.4), or $f(z)$ is a $1/2$-strongly starlike functions of order β, see (1.6).

Theorem 2.6. *[12] If $f(z)$ is analytic, $h(z)$ is convex univalent in $\mathbb{D}$, and that*

$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right|\le\frac{\alpha\pi}{2},\quad z\in\mathbb{D}\tag{2.10}$$

for some α, $0\le\alpha\le1$, then

$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right\}\right|\le\frac{\alpha\pi}{2}\tag{2.11}$$

for all $z_1\in\mathbb{D}$, and $z_2\in\mathbb{D}$.

Notice that (2.10) implies that $f(z)$ satisfies Ozaki's univalence condition [11], or that $f(z)$ strongly close-to-convex of order α with respect to g. Recall [13], that $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha)$ of strongly close-to-convex functions of order α, $0 < \alpha \leq 1$, if and only if there exist $g \in \mathcal{K}$, $\varphi \in \mathbb{R}$, such that

$$\left|\arg\left\{\frac{f'(z)}{e^{i\varphi}g'(z)}\right\}\right| < \frac{\pi\alpha}{2}, \quad z \in \mathbb{U}. \tag{2.12}$$

If $\alpha = 1$, then $\mathcal{C}(\alpha)$ becomes the well known class of close-to-convex functions, Kaplan [3]. Functions defined by (2.12) with $\varphi = 0$, $\alpha = 1$ where considered earlier by Ozaki [11], see also Umezawa [16, 17]. Moreover, Lewandowski [4, 5] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{U})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathcal{C}(1)$. It is worthy to note that $f \in \mathcal{A}$ satisfies the condition (2.11) with some convex univalent g and $\alpha = 1$ if and only if f is close-to-convex function, see [2, p. 31].

Theorem 2.6 was improved to the following.

Theorem 2.7. *[10] If $f(z)$ is analytic, $g(z)$ is convex univalent in $|z| < 1$, $f'(0) = g'(0)$ and*

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad \text{in } |z| < 1, \tag{2.13}$$

for some α, $0 \leq \alpha \leq 1$, then

$$\left|\arg\left\{\frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right\}\right| < \alpha\left(\frac{\pi}{2} - \log 2\right), \quad |z_1| < 1, \quad |z_2| < 1, \tag{2.14}$$

for all $z_1 \in \mathbb{D}$, and $z_2 \in \mathbb{D}$, where $\pi/2 - \log 2 = 0.877649147\ldots$.

In this paper, we will improve Theorem 2.6 by adding some conditions.

Theorem 2.8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$, and suppose that*

$$\frac{g(z)}{zg'(z)} \prec \frac{1+\delta z}{1-\delta z}, \quad z \in \mathbb{D} \tag{2.15}$$

for some δ, $0 < \delta \leq 1$, and

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\left\{\alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha(1-\delta)^2}{1+2\alpha\delta-\delta^2}\right\}, \quad z \in \mathbb{D}, \tag{2.16}$$

and

$$\tan\frac{\alpha\pi}{2} \leq \frac{(1-\delta)^2}{2\delta} \tag{2.17}$$

for some α, $0 < \alpha < 1$. Then we have

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. \tag{2.18}$$

Proof. Let us put

$$p(z)=\frac{f(z)}{g(z)},\ p(0)=1,\quad z\in\mathbb{D}. \tag{2.19}$$

Then it follows that

$$f(z)=p(z)g(z),\quad f'(z)=g'(z)p(z)+g(z)p'(z),$$

and

$$\frac{f'(z)}{g'(z)}=p(z)+\frac{g(z)}{g'(z)}p'(z)=p(z)\left(1+\frac{g(z)}{zg'(z)}\frac{zp'(z)}{p(z)}\right). \tag{2.20}$$

From the hypothesis, we have

$$\frac{1-\delta}{1+\delta}<\mathfrak{Re}\left\{\frac{g(z)}{zg'(z)}\right\}<\frac{1+\delta}{1-\delta},\quad z\in\mathbb{D} \tag{2.21}$$

and

$$\frac{-2\delta}{1-\delta^2}<\mathfrak{Im}\left\{\frac{g(z)}{zg'(z)}\right\}<\frac{2\delta}{1-\delta^2},\quad z\in\mathbb{D}. \tag{2.22}$$

If there exists a point z_0 with $|z_0|<1$, such that $|\arg\{p(z)\}|<\pi\alpha/2$ for $|z|<|z_0|$ and $|\arg\{p(z_0)\}|=\pi\alpha/2$ for some $0<\alpha<1$, then by Lemma 2.1 we have

$$\frac{z_0p'(z_0)}{p(z_0)}=\frac{2ik\arg\{p(z_0)\}}{\pi}, \tag{2.23}$$

for some

$$k\geq(a+a^{-1})/2,$$

where

$$\{p(z_0)\}^{1/\alpha}=\pm ia$$

and $a>0$. Then, for the case $p(z_0)=a^\alpha(\cos(\pi\alpha/2)+i\sin(\pi\alpha/2))$, we have from (2.21)

$$\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\}\ =\ \arg\left\{p(z_0)\left(1+\frac{g(z_0)}{z_0g'(z_0)}\frac{z_0p'(z_0)}{p(z_0)}\right)\right\}$$

Notice that if we assume $\arg\{w\}\in[0,\pi]$, then under the hypothesis of Theorem 2.8, we may apply the formula

$$\arg\left\{p(z_0)\left(1+\frac{g(z_0)}{z_0g'(z_0)}\frac{z_0p'(z_0)}{p(z_0)}\right)\right\}=\arg\{p(z_0)\}+\arg\left\{1+\frac{g(z_0)}{z_0g'(z_0)}\frac{z_0p'(z_0)}{p(z_0)}\right\}.$$

It is because of that

$$\begin{aligned}\arg\{p(z_0)\}&\in[0,\pi],\\ \arg\left\{1+\frac{g(z_0)}{z_0g'(z_0)}\frac{z_0p'(z_0)}{p(z_0)}\right\}&\in[0,\pi],\end{aligned}$$

$$\arg\{p(z_0)\} + \arg\left\{1 + \frac{g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right\} \quad \in \quad [0,\pi], \tag{2.24}$$

where (2.24) holds because of (2.17), namely

$$\begin{aligned}
&\arg\{p(z_0)\} + \arg\left\{1 + \frac{g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right\} \in [0,\pi] \\
\Leftrightarrow\quad & \frac{\alpha\pi}{2} + \pi - \tan^{-1}\frac{\alpha\frac{1-\delta}{1+\delta}}{\alpha\frac{2\delta}{1-\delta^2}} \in [0,\pi] \\
\Leftrightarrow\quad & \frac{\alpha\pi}{2} + \pi - \tan^{-1}\frac{\frac{1-\delta}{1+\delta}}{\frac{2\delta}{1-\delta^2}} \in [0,\pi) \\
\Leftrightarrow\quad & 0 \le \frac{\alpha\pi}{2} + \pi - \tan^{-1}\frac{\frac{1-\delta}{1+\delta}}{\frac{2\delta}{1-\delta^2}} \le \pi \\
\Leftrightarrow\quad & 0 \le \tan^{-1}\frac{\frac{1-\delta}{1+\delta}}{\frac{2\delta}{1-\delta^2}} - \frac{\alpha\pi}{2} \le \pi \\
\Leftrightarrow\quad & \frac{\alpha\pi}{2} \le \tan^{-1}\frac{\frac{1-\delta}{1+\delta}}{\frac{2\delta}{1-\delta^2}} \\
\Leftrightarrow\quad & \frac{\alpha\pi}{2} \le \tan^{-1}\frac{(1-\delta)^2}{2\delta} \\
\Leftrightarrow\quad & \tan\frac{\alpha\pi}{2} \le \frac{(1-\delta)^2}{2\delta}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\} &= \arg\{p(z_0)\} + \arg\left\{1 + \frac{g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right\} \\
&= \frac{\alpha\pi}{2} + \arg\left\{1 + i\alpha k\frac{g(z_0)}{z_0 g'(z_0)}\right\}.
\end{aligned} \tag{2.25}$$

We know that

$$\frac{g(z_0)}{z_0 g'(z_0)}$$

lies in a rectangle described by (2.21), (2.22) in the right half-plane. Therefore,

$$\left\{i\alpha\frac{g(z_0)}{z_0 g'(z_0)}\right\}$$

lies in a rectangle in the upper half-plane and

$$-\alpha\frac{2\delta}{1-\delta^2} < \mathfrak{Re}\left\{i\alpha\frac{g(z)}{z g'(z)}\right\} < -\alpha\frac{-2\delta}{1-\delta^2}, \quad z \in \mathbb{D} \tag{2.26}$$

and

$$\alpha\frac{1-\delta}{1+\delta} < \mathfrak{Im}\left\{i\alpha\frac{g(z)}{z g'(z)}\right\} < \alpha\frac{1+\delta}{1-\delta}, \quad z \in \mathbb{D}. \tag{2.27}$$

This gives

$$1-\alpha\frac{2\delta}{1-\delta^2}<\mathfrak{Re}\left\{1+i\alpha\frac{g(z)}{zg'(z)}\right\}<1+\alpha\frac{2\delta}{1-\delta^2},\quad z\in\mathbb{D}$$

and

$$\alpha\frac{1-\delta}{1+\delta}<\mathfrak{Im}\left\{1+i\alpha\frac{g(z)}{zg'(z)}\right\}<\alpha\frac{1+\delta}{1-\delta},\quad z\in\mathbb{D},$$

and so

$$\tan^{-1}\frac{\alpha\frac{1-\delta}{1+\delta}}{1+\alpha\frac{2\delta}{1-\delta^2}}\leq\arg\left\{1+i\alpha k\frac{g(z_0)}{z_0g'(z_0)}\right\}. \tag{2.28}$$

Inequality (2.28) together with (2.25) contradicts condition (2.16). For the case $\arg\{p(z_0)\}=-\pi\alpha/2$, applying the same method as above we can get

$$\arg\left\{\frac{f(z_0)}{g(z_0)}\right\}\leq\frac{\pi}{2}\left\{\alpha+\frac{2}{\pi}\tan^{-1}\frac{\alpha(1-\delta)^2}{1+2\alpha\delta-\delta^2}\right\}.$$

This is also contradicts hypothesis and therefore it completes the proof of Theorem 2.8. □

Observe that condition (2.15) satisfies the function

$$g(z)=\frac{z}{(1+\delta z)^2}$$

so Theorem 2.8 becomes the following corollary.

Corollary 2.9. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in $\mathbb{D}$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be analytic in $\mathbb{D}$, and suppose that*

$$\left|\arg\left\{(1-\delta z)\frac{f'(z)}{(1+\delta z)^3}\right\}\right|<\frac{\pi}{2}\left\{\alpha+\frac{2}{\pi}\tan^{-1}\frac{\alpha(1-\delta)^2}{1+2\alpha\delta-\delta^2}\right\},\quad z\in\mathbb{D}, \tag{2.29}$$

and

$$\tan\frac{\alpha\pi}{2}\leq\frac{(1-\delta)^2}{2\delta} \tag{2.30}$$

for some α, $0<\alpha<1$. Then we have

$$\left|\arg\left\{(1+\delta z)^2\frac{f(z)}{z}\right\}\right|\leq\frac{\alpha\pi}{2},\quad z\in\mathbb{D}. \tag{2.31}$$

For $\alpha=1/2$ and $\delta=2-\sqrt{3}$ Theorem 2.8 becomes the following corollary.

Corollary 2.10. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in $\mathbb{D}$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be analytic in $\mathbb{D}$, and suppose that*

$$\frac{g(z)}{zg'(z)}\prec\frac{1+(2-\sqrt{3})z}{1-(2-\sqrt{3})z},\quad z\in\mathbb{D}$$

and

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}\frac{2\sqrt{3}-1}{11}\right\} = \frac{\pi}{2}\cdot 0.64\ldots, \quad z\in\mathbb{D},$$

then we have

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \le \frac{\pi}{4}, \quad z\in\mathbb{D}.$$

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