TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 3, Autumn 2002

AUTOMORPHISM GROUPS OF COMPLEX ANALYTIC GROUPS

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Abstract. Let G be a faithfully representable complex analytic group and $Aut\,G$ the group of all complex analytic automorphisms of G. We prove that the identity component of $Aut\,G$ can be identified as an algebraic group.

Introduction

If H is a connected complex affine algebraic group, in [4], G. Hochschild and G. D. Mostow prove that the connected component of the neutral element in the group of all rational automorphisms of H is algebraic. It is our main purpose here to seek a reasonable analogous result in the category of complex analytic groups. More precisely, if G is a faithfully representable complex analytic group, we show that the connected component $(Aut\,G)_\circ$ of the neutral element in the group of all complex analytic automorphisms of G can be identified as an algebraic group. Furthermore, as an application, we show that the fixer of an element of G in $(Aut\,G)_\circ$ can also be identified as an algebraic group.

Notations. Let G be a group. If f is a function on G and H is a subset of G, we denote by $f|H$ the restriction of f to H. If, in addition, G is a Lie group and K is a subgroup of G, we denote by $L(G)$ the Lie algebra of G and by $K_\circ$ the connected component of K that contains the identity element. We denote by $\mathbf{C}^*$ the multiplicative group of nonzero complex numbers.

1. Preliminaries.

Let G be a faithfully representable complex analytic group. If f is a function on G and $x \in G$, we define the left (resp. right) translate $x \cdot f$ (resp. $f \cdot x$) of f by x to be $(x \cdot f)(y) = f(yx)$ (resp. $(f \cdot x)(y) = f(xy)$). We denote by $R(G)$ the Hopf algebra of all complex analytic functions f of G into $\mathbf{C}$ for which the complex vector space spanned by all left translates of f is finite-dimensional. Let $S \subset R(G)$. We say that S is left (resp. right) stable if S is stable under all left (resp. right) translations. We say that S is bistable if S is both left stable and right stable. An element f of $R(G)$ is

Received June 14, 2001
2000 *Mathematics Subject Classification.* Primary 22D45, 22E55.
Key words and phrases. Faithfully representable complex analytic groups, automorphism groups, algebraic groups, representative function.

called semisimple if the representation of G by left translations on the finite-dimensional space spanned by all left translates of f is semisimple. If B is a subset of $R(G)$, we use B_s to denote the collection of all semisimple functions in B. Let $\mathrm{Hom}(G, \mathbf{C})$ be the collection of all complex analytic homomorphisms of G into the additive group $\mathbf{C}$. Let $Q = \{\exp \circ f : f \in \mathrm{Hom}(G, \mathbf{C})\}$, where exp:$\mathbf{C} \to \mathbf{C}^*$ is the exponential map. A subalgebra B of $R(G)$ is called a basic subalgebra if elements of Q is free over B and $R(G) = B[Q]$. A normal basic subalgebra B of $R(G)$ is a left stable basic subalgebra such that B_s is bistable. A subalgebra of $R(G)$ is called fully stable if it is bistable and is stable under the antipode $\eta : R(G) \to R(G)$ sending every element f of $R(G)$ onto the function defined by $\eta(f)(x) = f(x^{-1})$. A subalgebra of $R(G)$ is called a regular subalgebra if it is fully stable and finitely generated as a $\mathbf{C}$-algebra, and contains a normal basic subalgebra of $R(G)$.

Now, let K be a nucleus of G. Then there exists a normal basic subalgebra B of $R(G)$ such that K is the kernel of the representation of G by left translations on B_S ([2, Theorem 1.1]), and there is a unique regular subalgebra $R(G, K)$ of $R(G)$ such that, if B is any normal baic subalgebra of $R(G)$ associated with K as above, then $R(G, K)$ coincides with the smallest fully stable subalgebra of $R(G)$ that contains B. We call $R(G, K)$ the nuclear subalgebra of $R(G)$ that is associated with K ([3, Section 5]). Put $S = R(G, K)$. Let A_S be the group of all proper automorphisms of S (that is, algebra automorphisms of S that commute with right translations) and G_S the group of all left translations on S. Since every normal basic subalgebra separates points of G, we may identify G with G_S via the map sending every element x of G onto the restriction of the left translation by x to S. By [3, Theorem 3.1], A_S is an algebraic group as well as a complex analytic group, G is closed and algebraically dense in A_S, and A_S and G share the same commutative subgroup. What follows are the statements of two results from [1] that will be used later.

Lemma 1.1.([1, Lemma 1.1(1)]) *Let G be a faithfully representable complex analytic group. Let K be a nucleus of G and S the associated nuclear subalgebra of $R(G)$. If U is the unipotent radical of A_S and A is the maximal reductive complex Lie subgroup of the center of A_S, then $U \cdot A$ is the maximal nilpotent normal subgroup of A_S.*

Theorem 1.2.([1, Theorem 2.5]) *Let G be a faithfully representable complex analytic group and A the maximal reductive complex Lie subgroup of the center of G. Then there is a nucleus K of G such that $\theta(K) \subseteq KA_\circ$ for all $\theta \in Aut\,G$.*

2. Automorphism Groups.

From now on, all notations will be fixed. Let G be a faithfully representable complex analytic group and $Aut\,G$ the gruop of all complex analytic automorphisms of G. Identifying with a subgroup of $GL(L(G))$, $Aut\,G$ has a natural structure of a complex Lie group.

Let K be a fixed nucleus of G. Let N be the representation radical of G (that is, the intersection of the kernels of all semisimple representations of G) and A the maximal reductive complex Lie subgroup of $Z(G)$. Let $\mathrm{Aut}(G,K)$ be the closed complex Lie subgroup of $Aut\,G$ that consists of all those $\theta \in Aut\,G$ for which $\theta(K) \subset KA_\circ$. Let $W(G,K)$ be the closed comple Lie subgroup of $Aut\,G$ that consists of all those elements in $Aut\,G$ that leave K invariant, and let $\mathrm{Hom}_N(K,A_\circ)$ be the group of all complex analytic homomorphisms of K into $A_\circ$ that are trivial on N.

Let $\tau \in \mathrm{Hom}_N(K,A_\circ)$. Fix a maximal reductive complex Lie subgroup H of G. Define $\tau^+ : G \to G$ by $\tau^+(kh) = kh\tau(k)$ for all $k \in K$ and $h \in H$. From the facts that τ is trivial on N and $[G,K] \subset N$, one checks readily that $\tau^+ \in Aut(G,K)$. Because any two maximal reductive complex Lie subgroups of G are conjugate by elements of N, all maximal reductive complex Lie subgroups of G are contained in HN. Since τ^+ is trivial on HN, the definition of τ^+ above is independent of the choice of H.

Let $V(G,K) = \{\tau^+ \in Aut(G,K) : \tau \in \mathrm{Hom}_N(K,A_\circ)\}$. If $\gamma, \sigma \in \mathrm{Hom}_N(K,A_\circ)$ and $\rho(k) = \gamma(k)^{-1}$ for all $k \in K$, one checks directly that $\gamma^+ \circ \sigma^+ = (\gamma\sigma)^+, \rho \in \mathrm{Hom}_N(K,A_\circ)$ and $\rho^+ = (\gamma^+)^{-1}$. It follows that $V(G,K)$ is a closed complex Lie subgroup of $\mathrm{Aut}(G,K)$. Now, we are ready to describe a semidirect product decomposition of $\mathrm{Aut}(G,K)$.

Theorem 2.1. *Let G be a faithfully representable complex analytic group and K a nucleus of G. Then $Aut(G,K)$ is a semidirect product $V(G,K)\cdot W(G,K)$ with $V(G,K)$ being normal in $Aut(G,K)$.*

Proof. Let $\alpha \in Aut(G,K)$. By definition, α determines uniquely two maps $\beta : K \to K$ and $\tau : K \to A_\circ$ such that $\alpha(k) = \beta(k)\tau(k)$ for every $k \in K$. From the fact that N and $A_\circ$ are characteristic subgroups of G, one checks readily that $\beta \in Aut\,K$ and $\tau \in \mathrm{Hom}_N(K,A_\circ)$. Define $\delta : K \to A_\circ$ by $\delta(k) = (\tau \circ \beta^{-1})(k^{-1})$. It follows from $\beta(N) = N$ that $\delta \in \mathrm{Hom}_N(K,A_\circ)$; and hence, $\delta^+ \in Aut(G,K)$. Put $\epsilon = \delta^+ \circ \alpha$. Since $\epsilon(k) = \beta(k)\tau(k)\delta(\beta(k)) = \beta(k)$ for every $k \in K$, we have $\epsilon \in W(G,K)$. This proves $\mathrm{Aut}(G,K) = V(G,K)W(G,K)$. Moveover, suppose $\gamma \in \mathrm{Hom}_N(K,A_\circ)$. Then $\alpha^{-1} \circ \gamma \circ \beta \in \mathrm{Hom}_N(K,A_\circ)$. Because $A_\circ$ is contained in any maximal reductive complex Lie subgroup of G and the difinition of γ^+ is independent of any maximal reductive complex Lie subgroup of G, we have $(\alpha^{-1} \circ \gamma^+ \circ \alpha)(kh) = (\alpha^{-1} \circ \gamma^+)(\beta(k)\tau(k)\alpha(h)) = \alpha^{-1}(\alpha(kh)\gamma(\beta(k))) = kh(\alpha^{-1} \circ \gamma \circ \beta)(k)$ for every $k \in K$ and $h \in H$; that is $\alpha^{-1} \circ \gamma^+ \circ \alpha = (\alpha^{-1} \circ \gamma \circ \beta)^+$. This proves that $V(G,K)$ is normal in $\mathrm{Aut}(G,K)$. Since $A_\circ$ is a characteristic subgroup of G, we see that $V(G,K)$ meets $W(G,K)$ trivially. This completes the proof.

Let $S = R(G,K)$ be the nuclear subalgebra of $R(G)$ that is associated with K and A_S the group of all proper automorphisms of S. Let $A_S = U \cdot M$, where U is the unipotent radical of A_S and M is a maximal fully reducible subgroup of A_S. We shall identify G as a closed complex analytic subgroup of A_S. Let $Aut\,A_S$ be the group of all complex analytic automorphisms of A_S. We are going to embed $\mathrm{Aut}(G,K)$ into $\mathrm{Aut}\ A_S$. To do this, let $V_1(A_S)$ be the collection of all those $\alpha \in Aut\,A_S$ which leave $N \cdot M$ pointwise fixed and satisfy $\alpha(x)x^{-1} \in A_\circ$ for every $x \in U \cdot A_\circ$. It follows from Lemma 1.1 that $U \cdot A_\circ$ is the maximal nilpotent normal complex analytic subgroup of A_S. Thus, by [4, Theorem 1], $V_1(A_S)$ is a closed normal complex analytic subgroup of $\mathrm{Aut}A_S$ and its canonical image in $GL(L(A_S))$ is an algebraic subgroup of $GL(L(A_S))$.

Let $\mathrm{Hom}_N(U, A_\circ)$ be the collection of all complex analytic homomorphisms of U into $A_\circ$ that are trivial on N. Let $\epsilon \in \mathrm{Hom}_N(U, A_\circ)$. Define $\epsilon^* : A_S \to A_S$ by $\epsilon^*(um) = um\epsilon(u)$. Just like the case for $\mathrm{Hom}_N(K, A_\circ)$, one checks readily that $\epsilon^* \in Aut\, A_S$ and the definition of ϵ^* is independent of the choice of M. Let $V(A_S)$ be the collection of all ϵ^* as ϵ ranges over all elements of $\mathrm{Hom}_N(U, A_\circ)$. Since $U \cdot A$ is the maximal nilpotent normal subgroup of A_S by Lemma 1.1, one concludes readily that $V(A_S) = V_1(A_S)$.

Lemma 2.2. *$V(A_S)$ and $V(G, K)$ are isomorphic as complex analytic groups.*

Proof. By [3, Theorem 3.1 and Theorem 4.1], there is a complex toroid X such that $A_S = X \cdot G$, $M = X \cdot (M \cap G)$ is a direct product, $(X \cdot U) \cap G = K$ and $H = M \cap G$ is a maximal reductive complex Lie subgroup of G. (A complex Lie group is said to be a complex toroid if it is isomorphic with the complex analytic group $(\mathbf{C}^*)^d$ for some positive integer d.)

Let $\epsilon \in \mathrm{Hom}_N(U, A_\circ)$. If $k \in K$, we write $k = k_U k_X$ with $k_U \in U$ and $k_X \in X$, then $\epsilon^*(k) = k\epsilon(k_U) \in KA_\circ$. Together with the fact that ϵ^* is identity on H, we see that $\epsilon^*|G \in Aut\, G$. Let $\gamma : K \to A_\circ$ be defined by $\gamma(k) = \epsilon(k_U)$. From the fact that $[X, U] \subset N$, one checks readily that $\gamma \in \mathrm{Hom}_N(K, A_\circ)$; and hence, $\gamma^+ \in V(G, K)$. Since ϵ^* and γ^+ coincide on K and are both identity on H, we see that $\epsilon^*|G = \gamma^+$. It follows that the map $\psi : V(A_S) \to V(G, K)$ defined by $\psi(\epsilon^*) = \epsilon^*|G$, where $\epsilon \in \mathrm{Hom}_N(U, A_\circ)$, is a complex analytic homomorphism.

Suppose $\epsilon \in \mathrm{Hom}_N(U, A_\circ)$ such that ϵ^* is identity on G. Because $X \subset M$, ϵ^* is identity on X. Consequently, ϵ^* is identity on A_S. This shows that ψ is injective. To see that ψ is surjective, let $\gamma \in \mathrm{Hom}_N(K, A_\circ)$. Define $\epsilon_1 : A_S \to A_S$ by $\epsilon_1(xkh) = x\gamma^+(kh)$ for $x \in X$, $k \in K$ and $h \in H$. From the facts that X centralizes H and $[M, K] \subset N$, one checks readily that $\epsilon_1 \in Aut\, A_S$. Since $\epsilon_1(U) \subset UA_\circ$ (by Lemma 1.1), by the definition of ϵ_1, we see that there is an $\epsilon \in \mathrm{Hom}_N(U, A_\circ)$ such that $\epsilon_1(u) = u\epsilon(u)$ for all $u \in U$. Since ϵ_1 and ϵ^* coincide on U and both of them are identity on M, $\epsilon_1 = \epsilon^*$ on A_S. Because γ^+ is identity on H and $\epsilon_1 = \gamma^+$ on K, we may conclude that $\psi(\epsilon^*) = \epsilon_1|G = \gamma^+$. This proves that ψ is surjective and the proof of the lemma is complete.

Let $W(A_S)$ be the collection of all rational automorphisms of A_S and $W(A_S, G, K) = \{\alpha \in W(A_S) : \alpha(G) = G, \alpha(K) = K\}$. Clearly, $W(A_S)$ and $W(A_S, G, K)$ are closed complex Lie subgroups of $\mathrm{Aut}A_S$.

Lemma 2.3. *$W(G, K)$ is isomorphic with $W(A_S, G, K)$ as complex Lie groups.*

Proof. Let $\theta \in W(G, K)$. Clearly, θ induces an algebra automorphism ${}^t\theta$ of $R(G)$ defined by ${}^t\theta(f) = f \circ \theta$. Suppose that B is a normal basic subalgebra of $R(G)$ such that K is the kernel of the representation of G by left translations on B_s. Since $\theta(K) = K$, one checks readily that ${}^t\theta(B)$ is again a normal basic subalgebra of $R(G)$ that is associated with K. By [2, Theorem 4.1], ${}^t\theta(B) = B \cdot x$ for some $x \in N$. Since ${}^t\theta(S)$ is again a fully stable subalgebra of $R(G)$, it follows from the uniqueness of S that ${}^t\theta(S) = S$. Hence ${}^t\theta$ induces an automorphism $\tilde{\theta}$ of the collection of all algebra homomorphisms of S into $\mathbf{C}$. In turn, $\tilde{\theta}$ induces an automorphism $\theta^\#$ of A_S because S is the algebra of polynomial functions on A_S. More precisely, if $\alpha \in A_S$, $f \in S$ and $x \in G$, then $\theta^{\#}(\alpha)(f)(x) = \alpha((f \cdot x) \circ \theta)(e)$, where e is the identity element of G. Clearly, $\theta^{\#}$ leaves G and K invariant. Together with the result ${}^t\theta(S) = S$, we see that $\theta^{\#} \in W(A_S, G, K)$. It is then straightforward to check that the map $\theta \mapsto \theta^{\#}$ is a complex analytic homomorphism of $W(G, K)$ into $W(A_S, G, K)$. Since S separates points of G, one sees readily that the map $\theta \mapsto \theta^{\#}$ is injective. Let $\phi \in W(A_S, G, K)$. Clearly, $\phi|G \in W(G, K)$. Then it is straightforward to check that $(\phi|G)^{\#} = \phi$ on G. Since $(\phi|G)^{\#}$ and ϕ are rational automorphisms on A_S and G is algebraically dense in A_S, we may conclude that $(\phi|G)^{\#} = \phi$ on A_S. This proves that the map $\theta \mapsto \theta^{\#}$ is surjective and the proof of the lemma is complete.

Theorem 2.4. *Let G be a faithfully representable complex analytic group, K a nucleus of $G, S = R(G, K)$ and A_S the group of all proper automorphisms of S. Then $Aut(G, K)_\circ$ can be identified as an algebraic subgroup of $(Aut\, A_S)_\circ$.*

Proof. By [4, Theorem 1 and Theorem 3], $\mathrm{Aut}A_S$ is a semidirect product $V(A_S)$. $W(A_S)$, $V(A_S)$ is a connected algebraic group and $(\mathrm{Aut}A_S)_\circ = V(A_S) \cdot W(A_S)_\circ$ is an algebraic group. Clearly, $E = \{\alpha \in W(A_S)_\circ : \alpha(G) = G \text{ and } \alpha(K) = K\}$ is an algebraic subgroup of $W(A_S)_\circ$ and $E_\circ = W(A_S, G, K)_\circ$; and hence, $W(A_S, G, K)_\circ$ is an algebraic group. Together with Theorem 2.1, Lemma 2.2 and Lemma 2.3, we may conclude that $\mathrm{Aut}(G, K)_\circ = V(G, K) \cdot W(G, K)_\circ$ is isomorphic with the algebraic subgroup $V(A_S) \cdot W(A_S, G, K)_\circ$ of $W(A_S)_\circ$ as complex analytic groups. This completes the proof.

Let J be the maximal nilpotent normal subgroup of the radical R of G. By Theorem 1.2, there is a nucleus K of G that is contained in R such that $J = (J \cap K) \cdot (J \cap A)$ and $\theta(K) \subseteq KA_\circ$ for all $\theta \in AntG$. Together with Theorem 2.4, we have our main result.

Theorem 2.5. *Let G be a faithfully representable complex analytic group. Then $(Aut\, G)_\circ$ can be identified as an algebraic subgroup of $(AutA_S)_\circ$, where K is a nucleus of G such that $\theta(K) \subseteq KA_\circ$ for all $\theta \in AntG$, and S is the nuclear subalgebra of $R(G)$ that is associated with K.*

As a corollary, we have the following.

Corollary 2.6. *Let G be a faithfully representable complex analytic group, $x \in G$ and $Aut_\circ(G, \{x\}) = \{\alpha \in (Aut\, G)_\circ : \alpha(x) = x\}$. Then $Aut_\circ(G, \{x\})$ can be identified as an algebraic subgroup of $(AutA_S)_\circ$, where K is a nucleus of G such that $\theta(K) \subseteq KA_\circ$ for all $\theta \in AntG$, and S is the nuclear subalgebra of $R(G)$ that is associated with K.*

Proof. Fix a maximal reductive complex Lie subgroup H of G and write $x = kh$ with $k \in K$ and $h \in H$. Let $\alpha \in Aut_\circ(G, \{x\})$. By Theorem 2.1 and Theorem 2.4, $\alpha = \gamma\alpha^+ \circ \beta_\alpha$ with $\gamma_\alpha \in \mathrm{Hom}_N(K, A_\circ)$ and $\beta_\alpha \in W(G, K)_\circ$. Since $\alpha(x) = x$, we have $kh = \gamma\alpha^+(\beta_\alpha(kh)) = \beta_\alpha(kh)\gamma_\alpha(\beta_\alpha(k))$, which implies $\beta_\alpha(x)x^{-1} = \gamma_\alpha(\beta_\alpha(k^{-1})) = \beta_\alpha(k)\gamma_\alpha^+(\beta_\alpha(k)^{-1})$. This proves that $\mathrm{Aut}_\circ(G, \{x\})$ consists of all those elements $\alpha \in (Aut\, G)_\circ$ for which $\beta_\alpha(x)x^{-1} = \beta_\alpha(k)\gamma_\alpha^+(\beta_\alpha(k)^{-1})$. By Theorem 2.5, $(Aut\, G)_\circ$ can be identified as an algebraic subgroup of $(\mathrm{Aut}A_S)_\circ$. It follows that $\mathrm{Aut}_\circ(G, \{x\})$ can be identified as an algebraic subgroup of $(\mathrm{Aut}A_S)_\circ$. This completes the proof.

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