Fixed Point of Contractive Mappings of Integral Type over an S^{JS}- Metric Space

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Abstract. We obtain sufficient conditions for existence of fixed points of integral type contractive mappings on S^{JS}- metric spaces. We also study common fixed point and couple fixed point of integral type mappings and construct examples to support our results.

1   Introduction

Nowadays fixed point theory is one of the most important and recent trends of research area in mathematics for its numerous applications. Fixed point theory has various applications in different branches of mathematics viz. boundary value problems, nonlinear differential and integral equations, nonlinear matrix equations, homotopy theory etc. The main purpose of fixed point theory is to deal with several mappings either of contractive type or expansive type in nature over various generalized spaces and to investigate the existence of their fixed points therein.

In 2002, Branciari [2] introduced integral type contractive mappings and proved some fixed point theorems. Following this, researchers have considered various types of contractive mappings of integral type in several topological spaces and proved fixed point theorems therein [13]. In addition to fixed point, researchers are also interested in investigating the existence of common fixed points of two or more mappings, coincidence points of mappings and coupled fixed points of mappings etc. (See [4], [5], [6], [7], [10] and [17]) to make further enrichment of the area of fixed point theory.

Sedghi et al. [14] introduced the concept of S-metric space by modifying D-metric and G-metric spaces. Following this article Souayan and Mlaiki [16] introduced the concept of S_b- metric space as a generalization of S-metric space and established some fixed point theorems on it. Rohen et al. [12] have given the definition of S_b-metric space in a more generalized way and they renamed the usual S_b-metric space as symmetric S_b-metric space.

Jleli and Samet [9] introduced a generalized metric space commonly known as JS-metric space, which is one of the interesting generalizations of usual metric spaces. They showed that any standard metric space, b-metric space [3], dislocated metric space [8] and Modular metric space with the Fatou property [11] are JS-metric space also. Moreover they have considered some generalized contractive type mappings in this newly introduced space and proved some fixed point theorems on it. Following this literature Senapati and Dey [15] proved some coupled fixed point theorems in the setting of partially ordered JS-metric spaces. Recently Beg et al. [1] introduced the notion of S^{JS}- metric space and proved several interesting classical results in these spaces. They also gave examples to that S- metric spaces and S_b- metric spaces are S^{JS}- metric spaces.

The aim of this paper is to prove some fixed point theorems together with common fixed point and coupled fixed point theorems for a class of integral type contractive mappings in the setting of S^{JS}- metric space. Section 2 deals with S^{JS}- metric space. In Section 3, we prove existence of fixed points and common fixed points of contractive mappings of integral type on an S^{JS}- metric space. Necessary conditions for existence of coupled fixed points are obtain in Section 4. Section 5, is conclusion and plan for future work.

2 S^{JS}- **metric spaces**

In this section we briefly recall some basic definition about S^{JS}- metric space from Beg et al. [1] for subsequent use. Let X be a nonempty set and $J : X^3 \to [0, \infty]$ be a function. Let us define the set

$$S(J, X, x) = \{\{x_n\} \subset X : \lim_{n \to \infty} J(x, x, x_n) = 0\}$$

for all $x \in X$.

Definition 1. [1] Let X be a nonempty set and $J : X^3 \to [0, \infty]$ satisfies the following conditions:

(J_1) $J(x, y, z) = 0$ implies $x = y = z$ for any $x, y, z \in X$;

(J_2) there exists some $b > 0$ such that for any $(x, y, z) \in X^3$ and $\{z_n\} \in S(J, X, z)$, we have

$$J(x, y, z) \leq b \limsup_{n \to \infty}(J(x, x, z_n) + J(y, y, z_n)).$$

Then the pair (X, J) is called an S^{JS}- metric space. Additionally if J also satisfies

(J_3) $J(x, x, y) = J(y, y, x)$ for all $x, y \in X$,

then we call it a symmetric S^{JS}-metric space.

Example 1. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + |z|$ for all $x, y, z \in X$, then clearly (J_1) is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$ then for $\{z_n\} \in S(J, X, 0)$, we have

$$J(x, y, 0) \leq \frac{1}{2} \limsup_{n \to \infty} (J(x, x, z_n) + J(y, y, z_n))$$

for all $x, y \in X$. Then (J_2) is also satisfied. So (X, J) is an $S^{JS}-$metric space. Clearly it is not symmetric.

Example 2. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + 2|z|$ for all $x, y, z \in X$. Clearly the conditions (J_1) and (J_3) are satisfied. Also one can check that for any $x, y, z \in X$

$$J(x, y, z) \leq \limsup_{n \to \infty} (J(x, x, z_n) + J(y, y, z_n))$$

for any sequence $\{z_n\} \in S(J, X, z)$. Therefore (J_2) is also satisfied and hence X is a symmetric S^{JS}-metric space.

Remark 1. (1) Let (X, S) be an S-metric space (See [14]). Clearly S- satisfies condition (J_1). Now let $(x, y, z) \in X^3$ and $\{z_n\}$ converges to z in (X, S), then $S(z, z, z_n) \to 0$ as $n \to \infty$ and from the condition (ii) we have

$$S(x, y, z) \leq \limsup_{n \to \infty} (S(x, x, z_n) + S(y, y, z_n))$$

Therefore S satisfies (J_2) also. Hence X is an S^{JS}-metric space. It is also symmetric.

(2) Let (X, S) be an S_b-metric space with coefficient $s \geq 1$ (See [12]). Then clearly S satisfies (J_1) and it also satisfies (J_2) for $b = s$. So an S_b-metric space is an S^{JS}-metric space.

Definition 2. [1] Let (X, J) be an S^{JS}-metric space, then a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J, X, x)$.

Definition 3. [1] Let (X, J) be an S^{JS}-metric space. A sequence $\{x_n\} \subset X$ is said to be Cauchy if $\lim_{n,m \to \infty} J(x_n, x_n, x_m) = 0$.

Definition 4. [1] An S^{JS}-metric space is said to be complete if every Cauchy sequence in X is convergent.

Definition 5. [1] Let (X, J) be an S^{JS}-metric space and $T : X \to X$ be a self mapping. Then T is called continuous at $a \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$, $J(Ta, Ta, Tx) < \epsilon$ whenever $J(a, a, x) < \delta$.

Theorem 2.1. *[1] In an S^{JS}-metric space (X, J) : (i) A convergent sequence $\{x_n\}$ has unique limit point x and $J(x, x, x) = 0$.(ii) If a Cauchy sequence $\{x_n\}$ has a convergent subsequence then $\{x_n\}$ is also convergent in X.(iii) If T is continuous at $a \in X$ then for any sequence $\{x_n\} \in S(J, X, a)$, $\{Tx_n\} \in S(J, X, Ta)$.*

3   Common fixed point

Let us consider the following set:

$$\Phi = \{\varphi : [0,\infty) \to [0,\infty) : \varphi \text{ is bounded, Lebesgue-integrable, summable and}$$

$$\text{for each } \epsilon > 0, \int_0^\epsilon \varphi(t)dt > 0\}.$$

In an S^{JS}-metric space (X, J), $d_J : X^2 \to [0, \infty]$ stands for the function defined as $d_J(x,y) = J(x, x, y)$ for any $x, y \in X$.

Lemma 3.1. *Let* $\varphi \in \Phi$ *and* $\{a_\lambda : \lambda \in \Lambda\} \subset \mathbb{R}^+$ *be a nonempty set. If for some* $M > 0$, $\int_0^{a_\lambda} \varphi(t)dt \leq M$ *for all* $\lambda \in \Lambda$ *then* $\int_0^a \varphi(t)dt \leq M$, *where* $a = \sup\{a_\lambda : \lambda \in \Lambda\} < \infty$.

Proof. For any $n \in \mathbb{N}$ there exists $\alpha \in \Lambda$ such that $a_\alpha + \frac{1}{n} > a$. Therefore

$$\begin{aligned}
\int_0^a \varphi(t)dt &\leq \int_0^{a_\alpha + \frac{1}{n}} \varphi(t)dt \\
&= \int_0^{a_\alpha} \varphi(t)dt + \int_{a_\alpha}^{a_\alpha + \frac{1}{n}} \varphi(t)dt \\
&\leq M + \int_{a_\alpha}^{a_\alpha + \frac{1}{n}} \varphi(t)dt
\end{aligned} \tag{3.1}$$

Since φ is bounded, there exists $L > 0$ such that $\varphi(t) \leq L$ for all $t \in [0, \infty)$. Therefore from (3.1) for any $n \in \mathbb{N}$ we have $\int_0^a \varphi(t)dt \leq M + L\frac{1}{n}$. Hence $\int_0^a \varphi(t)dt \leq M$. □

Theorem 3.1. *Let* (X, J) *be a complete* S^{JS}-*metric space and* $T : X \to X$ *be a self mapping. Also let* T *satisfies*

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leq k \int_0^{d_J(x,y)} \varphi(t)dt \tag{3.2}$$

for some $\varphi \in \Phi$, $k \in [0, 1)$ *and for all* $x, y \in X$. *If there exists* $x_0 \in X$ *such that* $\delta(J, T, x_0) = \sup\{d_J(T^i x_0, T^j x_0) : i, j \geq 1\} < \infty$ *then* T *has at least one fixed point in* X.

Proof. Now since T satisfies (3.2), for any $n \in \mathbb{N}$ we have

$$\int_0^{d_J(T^{n+i}x_0, T^{n+j}x_0)} \varphi(t)dt \leq k \int_0^{d_J(T^{n-1+i}x_0, T^{n-1+j}x_0)} \varphi(t)dt \tag{3.3}$$

for all $i, j \geq 1$. Let us take $\delta(J, T^{p+1}, x_0) = \sup\{d_J(T^{p+i}x_0, T^{p+j}x_0) : i, j \in \mathbb{N}\}$ for any non-negative integer p and for any $x_0 \in X$. Then for all $i, j \geq 1$

$$\int_0^{d_J(T^{n+i}x_0, T^{n+j}x_0)} \varphi(t)dt \leq k \int_0^{\delta(J, T^n, x_0)} \varphi(t)dt. \tag{3.4}$$

Since $\delta(J, T^{p+1}, x_0) \leq \delta(J, T, x_0) < \infty$ for any $p \geq 1$, from Lemma 3.1 it implies that

$$\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \leq k \int_0^{\delta(J,T^n,x_0)} \varphi(t)dt \qquad (3.5)$$

for any $n \in \mathbb{N}$. It further implies

$$\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \leq k^n \int_0^{\delta(J,T,x_0)} \varphi(t)dt \qquad (3.6)$$

for all $n \geq 1$. Taking $n \to \infty$ we have $\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \to 0$. Since $\varphi \in \Phi$, we get $\lim_{n\to\infty} \delta(J, T^{n+1}, x_0) = 0$. Now for any $1 \leq n < m$ it follows that $d_J(T^n x_0, T^m x_0) \leq \delta(J, T^n, x_0)$ which tends to 0 as n tends to ∞. Thus $\{T^n x_0\}$ is a Cauchy sequence in X. By the completeness of X there exists some $z \in X$ such that $\{T^n x_0\} \in S(J, X, z)$. Now for any $n \in \mathbb{N}$ we have

$$\int_0^{d_J(Tz,T^{n+1}x_0)} \varphi(t)dt \leq k \int_0^{d_J(z,T^n x_0)} \varphi(t)dt \qquad (3.7)$$

Taking $n \to \infty$ we get $\int_0^{d_J(Tz,T^{n+1}x_0)} \varphi(t)dt \to 0$ and therefore $\lim_{n\to\infty} d_J(Tz, T^{n+1}x_0) = 0$. From Theorem 2.1 it follows that $Tz = z$. Hence T has a fixed point in X. $\qquad\square$

Theorem 3.2. *If z and z' are two fixed points of T in Theorem 3.1 such that $d_J(z, z') < \infty$ then $z = z'$.*

Proof. Since z and z' are fixed points of T satisfying condition (3.2) then we obtain

$$\int_0^{d_J(z,z')} \varphi(t)dt = \int_0^{d_J(Tz,Tz')} \varphi(t)dt \leq k \int_0^{d_J(z,z')} \varphi(t)dt.$$

It implies that $\int_0^{d_J(z,z')} \varphi(t)dt = 0$ as $0 \leq k < 1$. Since $\varphi \in \Phi$ we get $d_J(z, z') = 0$. Hence $z = z'$. $\qquad\square$

Corollary 3.3. *Let (X, S) be a complete S-metric space and $T : X \to X$ satisfies*

$$S(Tx, Tx, Ty) \leq LS(x, x, y)$$

for all $x, y \in X$, where $0 \leq L < 1$. Then T has a unique fixed point in X.

Proof. If we set $d_S(x, y) = S(Tx, Tx, Ty)$ for all $x, y \in X$ then for any $x_0 \in X$ and for $1 \leq i < j$ we have

$$\begin{aligned}
d_S(T^i x_0, T^j x_0) &= S(T^i x_0, T^i x_0, T^j x_0) \\
&\leq 2d_S(T^i x_0, T^{i+1} x_0) + S(T^{i+1} x_0, T^{i+1} x_0, T^j x_0)
\end{aligned}$$

$$
\begin{aligned}
&\leq\ 2\left[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0)\right] + \\
&\quad\ S(T^{i+2} x_0, T^{i+2} x_0, T^j x_0) \\
&\quad\ ... \\
&\leq\ 2[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0) + ... + \\
&\quad\ d_S(T^{j-2} x_0, T^{j-1} x_0)] + S(T^{j-1} x_0, T^{j-1} x_0, T^j x_0) \\
&\leq\ 2[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0) + ... + \\
&\quad\ d_S(T^{j-1} x_0, T^j x_0)] \\
&\leq\ 2[L^i + L^{i+1} + ... + L^{j-1}] d_S(x_0.Tx_0) \\
&\leq\ 2L^i \frac{1 - L^{j-i}}{1 - L} d_S(x_0.Tx_0) \\
&\leq\ 2\frac{L^i}{1 - L} d_S(x_0.Tx_0).
\end{aligned}
$$

For any $i, j \geq 1$ we can arrange them as $1 \leq i < j$ and so

$$
\begin{aligned}
\sup\{d_S(T^i x_0, T^j x_0) : i, j \geq 1\} &\leq \sup\{2\frac{L^i}{1 - L} d_S(x_0.Tx_0) : i \geq 1\} \\
&= \frac{2L d_S(x_0, Tx_0)}{1 - L} < \infty.
\end{aligned}
$$

Now if we take $\phi(t) = 1$ for all $t \geq 0$ then we get

$$
\int_0^{d_S(Tx,Ty)} \varphi(t) dt \leq L \int_0^{d_S(x,y)} \varphi(t) dt
$$

for all $x, y \in X$. Also from Remark 1 it follows that any S-metric space is a symmetric S^{JS}-metric space. So all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied and hence T has a unique fixed point in X. $\square$

Remark 2. Above Corollary 3.3 is a theorem proved in Sedghi et al. [14].

Proposition 3.1. *In an S^{JS}-metric space (X, J) if two mappings $T, S : X \to X$ satisfy $T(X) \subset S(X)$ then for any $x_0 \in X$ there exists a sequence $\{y_n\}$, where $y_n = T(x_n) = S(x_{n+1})$ for all non-negative integers n.*

Proof. Let $x_0 \in X$. Then $Tx_0 \in T(X) \subset S(X)$ and therefore there exists $x_1 \in X$ such that $Tx_0 = Sx_1$. Next $Tx_1 \in T(X) \subset S(X)$, so there exists some $x_2 \in X$ such that $Tx_1 = Sx_2$. Proceeding similarly we can construct a sequence $\{y_n\}$ in such a way that $y_n = Tx_n = Sx_{n+1}$ for any $n \geq 0$. $\square$

Theorem 3.4. *Let (X, J) be a complete S^{JS}-metric space and $T, S : X \to X$ be two commutative mappings such that S is continuous and $T(X) \subset S(X)$. Also let T and S satisfy the following condition:*

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \le k \int_0^{d_J(Sx,Sy)} \varphi(t)dt \qquad (3.8)$$

for all $x, y \in X$, where $0 \le k < 1$ and $\varphi \in \Phi$. If there exists $x_0 \in X$ such that

$$\delta(J, T, S, x_0) = \sup\{d_J(y_i, y_j) : i, j \in \{0\} \cup \mathbb{N}\} < \infty$$

then T and S have at least one common fixed point in X.

Proof. Let us denote $\delta(J, T, S, p+1, x_0) = \sup\{d_J(y_{p+i}, y_{p+j}) : i, j \in \{0\} \cup \mathbb{N}\}$ for any $p \ge 0$. Since T satisfies the condition (3.8) we obtain for any $n \in \mathbb{N}$

$$
\begin{aligned}
\int_0^{d_J(y_{n+i}, y_{n+j})} \varphi(t)dt &= \int_0^{d_J(Tx_{n+i}, Tx_{n+j})} \varphi(t)dt \\
&\le k \int_0^{d_J(Sx_{n+i}, Sx_{n+j})} \varphi(t)dt \\
&= k \int_0^{d_J(y_{n-1+i}, y_{n-1+j})} \varphi(t)dt \qquad (3.9)
\end{aligned}
$$

for all $i, j \ge 0$, which in turn implies that $\int_0^{d_J(y_{n+i}, y_{n+j})} \varphi(t)dt \le k \int_0^{\delta(J,T,S,n,x_0)} \varphi(t)dt$ for all $i, j \ge 0$. Since $\delta(J, T, S, n+1, x_0) \le \delta(J, T, S, x_0) < \infty$ then by Lemma 3.1 it follows that $\int_0^{\delta(J,T,S,n+1,x_0)} \varphi(t)dt \le k \int_0^{\delta(J,T,S,n,x_0)} \varphi(t)dt$ for any $n \ge 1$. Therefore for any $n \in \mathbb{N}$ we obtain

$$\int_0^{\delta(J,T,S,n+1,x_0)} \varphi(t)dt \le k^n \int_0^{\delta(J,T,S,x_0)} \varphi(t)dt. \qquad (3.10)$$

Which implies that $\lim_{n\to\infty} \int_0^{\delta(J,T,S,n+1,x_0)} \varphi(t)dt = 0$. Since $\varphi \in \Phi$ it follows that

$$\lim_{n\to\infty} \delta(J, T, S, n+1, x_0) = 0.$$

Now for any $1 \le n < m$ we have $d_J(y_n, y_m) \le \delta(J, T, S, n, x_0)$ which tends to 0 as $n \to \infty$. So $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists $z \in X$ such that $\{y_n\} \in S(J, X, z)$. So $Tx_n = Sx_{n+1} \to z$ as $n \to \infty$. Since S is continuous, $\{Sy_n\}$ converges to Sz. Now for any $n \in \mathbb{N}$ $Sy_n = S(Tx_n) = T(Sx_n) = Ty_{n-1}$ and from (3.8) it follows that

$$\int_0^{d_J(Tz,Ty_n)} \varphi(t)dt \le k \int_0^{d_J(Sz,Sy_n)} \varphi(t)dt. \qquad (3.11)$$

Therefore $\lim_{n\to\infty} d_J(Tz, Ty_n) = 0$ since $\varphi \in \Phi$ and we get $Tz = Sz$. Also using (3.8) we get

$$\int_0^{d_J(Tz,T^2z)} \varphi(t)dt \le k \int_0^{d_J(Sz,S(Tz))} \varphi(t)dt$$

$$= k \int_0^{d_J(Sz,T(Sz))} \varphi(t)dt$$

$$= k \int_0^{d_J(Tz,T^2z)} \varphi(t)dt. \tag{3.12}$$

Since $k \in [0,1)$ we obtain $\int_0^{d_J(Tz,T^2z)} \varphi(t)dt = 0$, which implies $T^2z = Tz$. Now, $S(Tz) = T(Sz) = T(Tz) = Tz$. Hence Tz is a common fixed point of T and S. $\square$

Theorem 3.5. *If u and u' are two common fixed points of T and S in Theorem 3.4 such that $d_J(u,u') < \infty$ then $u = u'$.*

Proof. Given that u and u' are two common fixed points of T and S, so from (3.8) we obtain

$$\int_0^{d_J(u,u')} \varphi(t)dt = \int_0^{d_J(Tu,Tu')} \varphi(t)dt$$

$$\leq k \int_0^{d_J(Su,Su')} \varphi(t)dt$$

$$= k \int_0^{d_J(u,u')} \varphi(t)dt. \tag{3.13}$$

Since $0 \leq k < 1$ we have $\int_0^{d_J(u,u')} \varphi(t)dt = 0$ implying that $u = u'$. $\square$

Corollary 3.6. *Let T and S be two commuting self mappings of a complete S^{JS}-metric space (X,J). Suppose that S is continuous, $T(X) \subset S(X)$ and $\sup\{d_J(x,Tx) : x \in X\} < \infty$. Also let T and S satisfy the following condition for some positive integer p :*

$$\int_0^{d_J(T^px,T^py)} \varphi(t)dt \leq k \int_0^{d_J(Sx,Sy)} \varphi(t)dt \tag{3.14}$$

for all $x,y \in X$, for some $k \in [0,1)$ and $\varphi \in \Phi$. If there exists $x_0 \in X$ such that $\delta(J,T^p,S,x_0) < \infty$ then T and S have at least one common fixed point in X.

Proof. Clearly T^p and S commute with each other and also $T^p(X) \subset T(X) \subset S(X)$. So all the conditions of Theorem 3.4 are satisfied and thus T^p and S have a common fixed point in X, say z. Then $T^pz = Sz = z$. Since $T^p(Tz) = Tz = T(Sz) = S(Tz)$, it follows that Tz is also a common fixed point of T and S in X. By the given condition $d_J(z,Tz) < \infty$ and thus from Theorem 3.5 it follows that $Tz = z$. Hence z is a common fixed point of T and S. $\square$

Example 3. Let us take the complete S^{JS}-metric space (X,J) given in Example 1. Let $T : X \to X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$ and $\varphi(t) = \frac{t}{t+1}$ for all $t \in [0,\infty)$. Then clearly

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leq \frac{1}{2} \int_0^{d_J(x,y)} \varphi(t)dt \tag{3.15}$$

for all $x, y \in X$. Also for any $x_0 \in X \backslash \{-\infty, \infty\}$, $\delta(J, T, x_0) \leq \frac{3}{2}|x_0|$ and we see that 0 is a fixed point of T in X. Other fixed points of T are $-\infty$ and ∞.

Example 4. Let us consider $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x - 1| + |y - 1| + |z - 1|$ for all $x, y, z \in X$. Then it is clearly an S^{JS}-metric space which is not symmetric. Also let $T : X \to X$ be defined by $Tx = \frac{x+1}{2}$ for all $x \in X$. If we take $\varphi(t) = t^2$ for all $t \geq 0$ then

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leq \frac{1}{4} \int_0^{d_J(x,y)} \varphi(t)dt \tag{3.16}$$

for any $x, y \in X$. Also for any $x_0 \in X \backslash \{-\infty, \infty\}$, $\delta(J, T, x_0) \leq \frac{3}{2}|x_0 - 1|$ and we see that 1 is the only finite fixed point of T in X.

Example 5. Let us consider $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |y + z - 2x| + |x - z| + |y - z|$ for all $x, y, z \in X$. Then it is clearly a symmetric S^{JS}-metric space. Also let $T : X \to X$ be defined by $Tx = \frac{x}{4} + \frac{1}{4}$ for all $x \in X$. If we take $\varphi(t) = 1$ for all $t \geq 0$ then we see that

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leq \frac{1}{4} \int_0^{d_J(x,y)} \varphi(t)dt \tag{3.17}$$

for any $x, y \in X$. Also for any $x_0 \in X \backslash \{-\infty, \infty\}$, $\delta(J, T, x_0) \leq \frac{|3x_0 - 1|}{4}$ and we have $\frac{1}{3}$ is the only finite fixed point of T in X.

Example 6. Let us consider the complete S^{JS}-metric space (X, J) given in Example 1. Let us define $T, S : X \to X$ by $Tx = \frac{x}{6}$ and $Sx = \frac{x}{2}$ for all $x \in X$. Also let us take φ defined in Example 3. Then for any $x, y \in X$, T and S satisfy

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leq \frac{1}{3} \int_0^{d_J(Sx,Sy)} \varphi(t)dt. \tag{3.18}$$

Also for any $x_0 \in X \backslash \{-\infty, \infty\}$ the sequence $\{y_n\}_{n \geq 0}$ is given by $y_n = \frac{x_0}{2.3^{n+1}}$ for all $n \in \{0\} \cup \mathbb{N}$ and we have $\delta(J, T, S, x_0) \leq \frac{1}{2}|x_0|$. Here 0 is the unique finite common fixed point of T and S in X. Also $-\infty$ and ∞ are common fixed points of T and S in X.

4 Coupled fixed point

The notion of coupled fixed point was introduced in 1987 by Guo and Lakshmikantham (See [7]). In this section we prove a coupled fixed point theorem in the setting of S^{JS}-metric space. First we define coupled fixed point of a mapping.

Definition 6. Let X be a non-empty set and $f : X^2 \to X$ be a mapping. A point $(a, b) \in X^2$ is said to be a coupled fixed point of f if $f(a, b) = a$ and $f(b, a) = b$.

For any $(a, b) \in X^2$ we can construct two iterative sequences using f in the following way

$$f^2(a, b) = f(f(a, b), f(b, a)) \quad , \quad f^2(b, a) = f(f(b, a), f(a, b))$$
$$f^3(a, b) = f(f^2(a, b), f^2(b, a)) \quad , \quad f^3(b, a) = f(f^2(b, a), f^2(a, b))$$

Proceeding in the similar manner we can get

$$f^{n+1}(a, b) = f(f^n(a, b), f^n(b, a)) \quad , \quad f^{n+1}(b, a) = f(f^n(b, a), f^n(a, b))$$

for any $n \geq 1$. So we get two iterative sequences $\{f^n(a, b)\}$ and $\{f^n(b, a)\}$.

Theorem 4.1. *Let (X, J) be a complete S^{JS}-metric space and $f : X^2 \to X$. Also let f satisfies*

$$\int_0^{d_J(f(x,y), f(u,v))} \varphi(t)dt \leq k \int_0^{\frac{1}{2}[d_J(x,u)+d_J(y,v)]} \varphi(t)dt \tag{4.1}$$

for all $x, y, u, v \in X$, for some $k \in [0, 1)$ and $\varphi \in \Phi$. If there exists $(x_0, y_0) \in X^2$ such that

$$\delta(J, f, (x_0, y_0)) = \{d_J(f^i(x_0, y_0), f^j(x_0, y_0)) : i, j \geq 1\} < \infty$$

and

$$\delta(J, f, (y_0, x_0)) = \{d_J(f^i(y_0, x_0), f^j(y_0, x_0)) : i, j \geq 1\} < \infty$$

then f has at least one coupled fixed point in X.

Proof. For any $n \in \mathbb{N}$ from (4.1) we have

$$\int_0^{d_J(f^{n+i}(x_0,y_0), f^{n+j}(x_0,y_0))} \varphi(t)dt$$
$$= \int_0^{d_J\left(f(f^{n-1+i}(x_0,y_0), f^{n-1+i}(y_0,x_0)), f(f^{n-1+j}(x_0,y_0), f^{n-1+j}(y_0,x_0))\right)} \varphi(t)dt$$
$$\leq k \int_0^{\frac{1}{2}\left[d_J(f^{n-1+i}(x_0,y_0), f^{n-1+j}(x_0,y_0)) + d_J(f^{n-1+i}(y_0,x_0), f^{n-1+j}(y_0,x_0))\right]} \varphi(t)dt \tag{4.2}$$

for any $i, j \geq 1$. Let us take

$$\delta(J, f^{q+1}, (x_0, y_0)) = \sup\{d_J(f^{q+i}(x_0, y_0), f^{q+j}(x_0, y_0) : i, j \in \mathbb{N}\}$$

and

$$\delta(J, f^{q+1}, (y_0, x_0)) = \sup\{d_J(f^{q+i}(y_0, x_0), f^{q+j}(y_0, x_0) : i, j \in \mathbb{N}\}$$

for any $q \geq 0$. Then for all $i, j \in \mathbb{N}$

$$\int_0^{d_J(f^{n+i}(x_0,y_0), f^{n+j}(x_0,y_0))} \varphi(t)dt \leq k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt \tag{4.3}$$

for any $n \geq 1$. Since $\delta(J, f^{q+1}, (x_0, y_0)) \leq \delta(J, f, (x_0, y_0)) < \infty$ for any $q \geq 1$ then using Lemma 3.1 we get

$$\int_0^{\delta(J,f^{n+1},(x_0,y_0))} \varphi(t)dt \leq k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt \tag{4.4}$$

for any $n \in \mathbb{N}$. Similarly we can obtain for any $n \geq 1$

$$\int_0^{\delta(J,f^{n+1},(y_0,x_0))} \varphi(t)dt \leq k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt. \tag{4.5}$$

Let $M_n = \max\{\delta(J, f^n, (x_0, y_0)), \delta(J, f^n, (y_0, x_0))\}$ for all $n \in \mathbb{N}$. Then from (4.4) and (4.5) we have

$$\int_0^{M_{n+1}} \varphi(t)dt \leq k \int_0^{M_n} \varphi(t)dt \tag{4.6}$$

for all $n \geq 1$. Then we obtain $\int_0^{M_{n+1}} \varphi(t)dt \leq k^n \int_0^{M_1} \varphi(t)dt$ for all $n \in \mathbb{N}$. Since $M < \infty$ it follows that

$$\lim_{n\to\infty} \int_0^{M_{n+1}} \varphi(t)dt = 0.$$

This shows that $\lim_{n\to\infty} M_n = 0$ as $\varphi \in \Phi$. Therefore we get $\lim_{n\to\infty} \delta(J, f^n, (x_0, y_0))$ $= 0 = \lim_{n\to\infty} \delta(J, f^n, (y_0, x_0))$. Now for any $1 \leq n < m$ we get $d_J(f^n(x_0, y_0), f^m(x_0, y_0))$ $\leq \delta(J, f^n, (x_0, y_0)) \to 0$ as $n \to \infty$. So $\{f^n(x_0, y_0)\}$ is Cauchy in X and since X is complete so there exists $z_1 \in X$ such that $f^n(x_0, y_0) \to z_1$ as n tending to ∞. In a similar way we can find $z_2 \in X$ such that $f^n(y_0, x_0) \to z_2$ as n tending to ∞. Now

$$\int_0^{d_J(f(z_1,z_2),f^n(x_0,y_0))} \varphi(t)dt = \int_0^{d_J\left(f(z_1,z_2),f(f^{n-1}(x_0,y_0),f^{n-1}(y_0,x_0))\right)} \varphi(t)dt$$
$$\leq k \int_0^{\frac{1}{2}\left[d_J(z_1,f^{n-1}(x_0,y_0))+d_J(z_2,f^{n-1}(y_0,x_0))\right]} \varphi(t)dt \tag{4.7}$$

Since $f^n(x_0, y_0) \to z_1$ and $f^n(y_0, x_0) \to z_2$ as $n \to \infty$ so we obtain

$$\lim_{n\to\infty} \int_0^{d_J(f(z_1,z_2),f^n(x_0,y_0))} \varphi(t)dt = 0.$$

Therefore it follows that $\lim_{n\to\infty} f^n(x_0, y_0) = f(z_1, z_2)$. Then by Theorem 2.1 we get $f(z_1, z_2) = z_1$. In a similar manner we have $f(z_2, z_1) = z_2$. Hence (z_1, z_2) is a coupled fixed point of f in X. □

Theorem 4.2. *If (z_1, z_2) and (z'_1, z'_2) are two coupled fixed points of f in Theorem 4.1 such that $d_J(z_1, z'_1) < \infty$ and $d_J(z_2, z'_2) < \infty$ then $(z_1, z_2) = (z'_1, z'_2)$.*

Proof. Now

$$
\int_0^{d_J(z_1,z_1')} \varphi(t)dt = \int_0^{d_J\left(f(z_1,z_2),f(z_1',z_2')\right)} \varphi(t)dt
$$
$$
\leq k \int_0^{\frac{1}{2}\left[d_J(z_1,z_1')+d_J(z_2,z_2')\right]} \varphi(t)dt \tag{4.8}
$$

and also

$$
\int_0^{d_J(z_2,z_2')} \varphi(t)dt = \int_0^{d_J\left(f(z_2,z_1),f(z_2',z_1')\right)} \varphi(t)dt
$$
$$
\leq k \int_0^{\frac{1}{2}\left[d_J(z_1,z_1')+d_J(z_2,z_2')\right]} \varphi(t)dt \tag{4.9}
$$

If we set $L = \max\{d_J(z_1, z_1'), d_J(z_2, z_2')\}$ then from (4.8) and (4.9) we can get

$$
\int_0^L \varphi(t)dt \leq k \int_0^L \varphi(t)dt. \tag{4.10}
$$

Since $k \in [0, 1)$ then we obtain $\int_0^L \varphi(t)dt = 0$ and therefore $L = 0$. Then $d_J(z_1, z_1') = 0 = d_J(z_2, z_2')$ that is $z_1 = z_1'$ and $z_2 = z_2'$. Hence $(z_1, z_2) = (z_1', z_2')$. □

Example 7. Let us consider the complete S^{JS}-metric space (X, J) defined in Example 1 and φ as defined in Example 3. Let $f : X^2 \to X$ be defined by $f(x, y) = \frac{x+y}{3}$ for all $x, y \in X$. Then for any $x, y, u, v \in X$ we have

$$
\int_0^{d_J(f(x,y),f(u,v))} \varphi(t)dt \leq \frac{2}{3} \int_0^{\frac{1}{2}[d_J(x,u)+d_J(y,v)]} \varphi(t)dt. \tag{4.11}
$$

For any $a, b \in X \backslash \{-\infty, \infty\}$ we get $\delta(J, f, (a, b)) \leq \frac{4}{3}|a + b|$ and also $\delta(J, f, (b, a)) \leq \frac{4}{3}|a + b|$. Here we see that $(0, 0)$, $(-\infty, -\infty)$ and (∞, ∞) are all coupled fixed points of f.

5   Conclusion

We proved some fixed point theorems together with common fixed point and coupled fixed point theorems for a class of integral type contractive mappings in the setting of S^{JS}- metric space. Our results generalized several known resutts. In future we further plan to study fixed point of multivalued mappings and variational principle in S^{JS}- metric spaces.

Conflict of interest. The authors declare that they have no conflict of interest.

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