TAMKANG JOURNAL OF MATHEMATICS
Volume 40, Number 1, 19-29, Spring 2009

FUZZY VOLTERRA INTEGRAL EQUATIONS WITH INFINITE DELAY

P. PRAKASH AND V. KALAISELVI

Abstract. In this paper, we study the existence and uniqueness of solutions for a class of fuzzy Volterra integral equations with infinite delay by using the method of successive approximations.

1. Introduction

Kandel and Byatt [7] introduced the concept of fuzzy differential equations. Later it was applied in fuzzy processes and fuzzy dynamical systems. Since 1987, the Cauchy problem for first order fuzzy differential equations has been extensively investigated by Kaleva [6]. Song et al. [10] discussed about the existence and comparison theorems to Volterra fuzzy integral equations in (E^n, D). Diamond [4] discussed the theory of Volterra integral equations in a fuzzy context by using the interpretation equations. Balachandran and Prakash [3] studied the existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations by using the Successive approximation method. Agarwal et al. [1] have given a very general formulation of the stacking theorem approach for fuzzy Volterra integral equations. Recently, Xiaoping and Yongiang [11] establish the existence and uniqueness of solutions for set differential equations, the continuous dependence of solutions on initial values, and the structural stability of solutions by using the Banach fixed point theorem. They also discussed the relationship between small solutions and large solutions of fuzzy differential equations.

In this paper, we consider the fuzzy Volterra integral equation with infinite delay of the form

$$x'(t) = h(t, x(t)) + \int_{-\infty}^{t} q(t, s, x(s))ds, \quad t \in T = (-\infty, \infty) \tag{1}$$

where $h : T \times E^n \to E^n$ and $q : T \times T \times E^n \to E^n$ are levelwise continuous and satisfy the generalized Lipschitz conditions.

Basic Assumption: For each $t_0 \in T$, there exists a nonempty convex subset $B(t_0)$ of the space of continuous functions $\phi : T_1 = (-\infty, t_0] \to E^n$ such that $\phi \in B(t_0)$ implies

$$\int_{-\infty}^{t_0} q(t, s, \phi(s))ds := Q(t, t_0, \phi)$$

Received January 15, 2008.
2000 *Mathematics Subject Classification*. 45G10, 45M10.
Key words and phrases. Fuzzy set-valued mapping, levelwise continuous, fuzzy Volterra integral equations, fuzzy solution.

is continuous on $T_2 = [t_0, \infty)$. For a given $t_0 \in T$ and a continuous initial function $\phi : T_1 \to E^n$, we seek a continuous solution $x(t, t_0, \phi)$ satisfying (1) for $t \in [t_0, t_0 + \beta)$ for some $\beta > 0$ with $x(t, t_0, \phi) = \phi(t)$ for $t \le t_0$.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n. Addition and scalar multiplication in $P_K(R^n)$ are defined as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a-b||,\ \sup_{b\in B}\inf_{a\in A} ||a-b||\right\},$$

where $||\cdot||$ denotes the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n), d)$ becomes a complete metric space [9].

Let $I = [0,1] \subseteq R$ be a compact interval and let E^n denote the set of all $u : R^n \to I$ such that u satisfies the following conditions.

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote $[u]^\alpha = \{x \in R^n : u(x) \ge \alpha\}$. Then from (i)-(iv) it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle,we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[\tilde{g}(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n$, $0 \le \alpha \le 1$ and any continuous function g. Furthermore, we have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha and [ku]^\alpha = k[u]^\alpha$, where $k \in R$.

Theorem 2.1.([8]) *If $u \in E^n$, then*

(i) $[u]^\alpha \in P_K(R^n)$ *for* $0 \le \alpha \le 1$,

(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ *for* $0 \le \alpha_1 \le \alpha_2 \le 1$, *and*

(iii) *If* $\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then*

$$[u]^\alpha = \bigcap_{k\ge 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^\alpha : 0 \le \alpha \le 1\}$ is a family of subset A of R^n satisfying (i)-(iii), then there exists a $u \in E^n$ such that

$$[u]^\alpha = A^\alpha \text{ for } 0 < \alpha \le 1$$

and

$$[u]^0 = \overline{\bigcup_{0<\alpha\le 1} A^\alpha} \subset A^0.$$

Define the metric $D : E^n \times E^n \to R^+ \cup \{0\}$ by $D(u,v) = \sup_{0\le\alpha\le 1} d([u]^\alpha, [v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(R^n)$.

The following definitions are given in [5].

Definition 2.1. A mapping $F : I \to E^n$ is strongly measurable, if for all $\alpha \in [0,1]$ the set-valued mapping $F_\alpha : I \to P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(R^n)$ has the topology induced by the Hausdorff metric d.

Definition 2.2. A mapping $F : I \to E^n$ is called levelwise continuous at $t_0 \in I$ if the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0,1]$.

Definition 2.3. A mapping $F : I \to E^n$ is called integrably bounded if there exists an integrable function h such that $\|x\| \le h(t)$ for every $x \in F_0(t)$.

Definition 2.4. The integral of a fuzzy mapping $F : I \to E^n$ is defined levelwise by $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt =$ The set of all $\int_I f(t)dt$ such that $f : I \to R^n$ is a measurable selection for F_α for all $\alpha \in [0,1]$.

Theorem 2.2. ([2]) If $F : I \to E^n$ is strongly measurable and integrably bounded, then F is integrable.

It is known that $\left[\int_I F(t)dt\right]^0 = \int_I F_0(t)dt$.

Theorem 2.3. *Let* $F, G : I \to E^n$ *be integrable and* $\lambda \in R$. *Then*

(i) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt,$

(ii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt,$

(iii) $D(F,G)$ *is integrable,*

(iv) $D\left(\int_I F(t)dt, \int_I G(t)dt\right) \le \int_I D(F(t), G(t))dt.$

Definition 2.5. A mapping $F : I \to E^n$ is called differentiable at $t_0 \in I$ if, for any $\alpha \in [0,1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) | \alpha \in [0,1]\}$ define a fuzzy number $F(t_0) \in E^n$.

If $F : I \to E^n$ is differentiable at $t_0 \in I$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.4. *Let $F : I \to E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.*

Theorem 2.5. *Let $F : I \to E^n$ be differentiable and assume that the derivative F' is integrable over I. Then, for each $s \in I$, we have*

$$F(s) = F(a) + \int_a^s F'(t)\,dt.$$

Definition 2.6. A mapping $f : I \times E^n \to E^n$ is called levelwise continuous at point $(t_0, x_0) \in I \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in I, x \in E^n$.

3. Main results

Assume that $h : T_0 \times E^n \to E^n$ and $q : T_0 \times T_0 \times E^n \to E^n$ are levelwise continuous, where $T_0 = \{t \in T : t_0 \le t < t_0 + \beta\}$. Consider the fuzzy Volterra integral equation (1) where $\phi(t_0) \in E^n$. We denote $J = T_0 \times B(\phi(t_0), b)$ and $J_0 = T_0 \times T_0 \times B(\phi(t_0), b)$ where $a > 0,\ b > 0,\ \phi(t_0) \in E^n$, and

$$B(\phi(t_0), b) = \{x \in E^n : D(x, \phi(t_0)) \le b\}.$$

Definition 3.1. A mapping $x : T_0 \to E^n$ is a solution to the problem (1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = \phi(t_0) + \int_{t_0}^t h(s, x(s))\,ds + \int_{t_0}^t \int_{t_0}^u q(u, s, x(s))\,ds\,du + \int_{t_0}^t Q(u, t_0, \phi)\,du, \ \text{ for all } t \in T_0.$$

Assume that the following conditions hold.

(A) $h : J \to E^n$ is levelwise continuous and for any pair $(t,x), (t,y) \in J$ and $\alpha \in [0,1]$, we have

$$d([h(t,x)]^\alpha, [h(t,y)]^\alpha) \le k_h d([x]^\alpha, [y]^\alpha),$$

where k_h is a given constant.

(B) $q : J_0 \to E^n$ is levelwise continuous and for any pair $(t, s, x_1), (t, s, x_2) \in J_0$, $-b \le s \le t \le b$ and $\alpha \in [0,1]$, we have

$$d([q(t,s,x_1)]^\alpha, [q(t,s,x_2)]^\alpha) \le k_q \left[d([x_1]^\alpha, [x_2]^\alpha) \right],$$

where k_q is a given constant.

(C) Let $K = \max\{k_h, k_q\}$ be such that $0 < K < 1$.

Theorem 3.1. *If the conditions* (A)−(C) *hold, then there exists a unique solution* $x = x(t)$ *of* (1) *defined on the interval* $t_0 \le t < t_0 + \beta$.

Proof. Let $0 < L < \beta$ be given. Therefore $t_0 \le t \le t_0 + L$.

Let

$$\delta = \min\left\{L, \sqrt{\left(\frac{M+M_2}{M_1}\right)^2 + \frac{2b}{M_1}} - \left(\frac{M+M_2}{M_1}\right)\right\},$$

where $M = D(h(t,\phi(t_0)),\hat{0})$, $\hat{0} \in E^n$, such that $\hat{0}(t) = 1$ for $t = 0$ and 0 otherwise, and for any $(t,x) \in J$ and $M_1 = D(q(u,s,\phi(t_0)),\hat{0})$ for any $(u,s,\phi(t_0)) \in J_0$, and $M_2 = D(Q(u,t_0,\phi(t_0)),\hat{0})$ for any $(u,t_0,\phi(t_0)) \in J_0$.

We will show that the sequence of functions defined inductively on $[t_0, t_0 + L]$ by

$$\begin{aligned} x_0(t) &\equiv \phi(t_0), \quad t \in T_0, \\ x_n(t) &= \phi(t_0) + \int_{t_0}^t h(s, x_{n-1}(s))\,ds + \int_{t_0}^t \int_{t_0}^u q(u,s,x_{n-1}(s))\,ds\,du \\ &\quad + \int_{t_0}^t Q(u,t_0,\phi)\,du, \qquad n = 1,2,3,\ldots, \end{aligned} \tag{2}$$

From (2), it follows that, for $n = 1$,

$$x_1(t) = \phi(t_0) + \int_{t_0}^t h(s,\phi(t_0))\,ds + \int_{t_0}^t \int_{t_0}^u q(u,s,\phi(t_0))\,ds\,du + \int_{t_0}^t Q(u,t_0,\phi(t_0))\,du, \tag{3}$$

which proves that $x_1(t)$ is levelwise continuous on $|t - t_0| \le L$ and hence on $|t - t_0| \le \delta$. Moreover, for any $\alpha \in [0,1]$, we have

$$\begin{aligned} d([x_1(t)]^\alpha, [x_0(t)]^\alpha) &= d\left(\left[\phi(t_0) + \int_{t_0}^t h(s,\phi(t_0))\,ds + \int_{t_0}^t \int_{t_0}^u q(u,s,\phi(t_0))\,ds\,du \right.\right. \\ &\qquad \left.\left. + \int_{t_0}^t Q(u,t_0,\phi(t_0))\,du\right]^\alpha, [\phi(t_0)]^\alpha\right) \\ &\le \int_{t_0}^t d([h(s,\phi(t_0))]^\alpha,\hat{0})\,ds + \int_{t_0}^t \int_{t_0}^u d([q(u,s,\phi(t_0))]^\alpha,\hat{0})\,ds\,du \\ &\quad + \int_{t_0}^t d([Q(u,t_0,\phi(t_0))]^\alpha,\hat{0})\,du, \end{aligned}$$

and by the definition of D, we get

$$
\begin{aligned}
D(x_1(t),x_0(t)) &\le \int_{t_0}^{t} D(h(s,\phi(t_0)),\hat{0})ds + \int_{t_0}^{t}\int_{t_0}^{u} D(q(u,s,\phi(t_0)),\hat{0})dsdu \\
&\quad + \int_{t_0}^{t} D(Q(u,t_0,\phi(t_0)),\hat{0})du \\
&\le (M+M_2)|t-t_0| + M_1\frac{|t-t_0|^2}{2!} \\
&\le (M+M_2)\delta + M_1\frac{\delta^2}{2!} \\
&\le b. \qquad (4)
\end{aligned}
$$

Now, assume that $x_{n-1}(t)$ is levelwise continuous on $|t-t_0|\le\delta$ and that

$$D(x_{n-1}(t),x_0(t)) \le b.$$

From (2), we deduce that $x_n(t)$ is levelwise continuous on $|t-t_0|\le\delta$ and that

$$D(x_n(t),x_0(t)) \le b.$$

Consequently, we conclude that $x_n(t)$ consists of levelwise continuous mappings on $|t-t_0|\le\delta$ and that

$$(t,x_n(t))\in J \text{ and } (t,s,x_n(t))\in J_0, \quad |t-t_0|\le\delta, n{=}1,2,\dots,$$

Let us prove that there exists a fuzzy set-valued mapping $x:[t_0,t_0{+}L]\to E^n$ such that $D(x_n(t),x(t))\to 0$ uniformly on $|t-t_0|\le\delta$ as $n\to\infty$. For $n=2$, from (2),

$$x_2(t) = \phi(t_0) + \int_{t_0}^{t} h(s,x_1(s))ds + \int_{t_0}^{t}\int_{t_0}^{u} q(u,s,x_1(s))dsdu + \int_{t_0}^{t} Q(u,t_0,\phi(t_0))du. \qquad (5)$$

From (3) and (5), we have

$$
\begin{aligned}
&d([x_2(t)]^\alpha,[x_1(t)]^\alpha) \\
&\quad = d\Bigg(\Bigg[\int_{t_0}^{t} h(s,x_1(s))ds + \int_{t_0}^{t}\int_{t_0}^{u} q(u,s,x_1(s))dsdu + \int_{t_0}^{t} Q(u,t_0,\phi(t_0))du\Bigg]^\alpha, \\
&\qquad \Bigg[\int_{t_0}^{t} h(s,\phi(t_0))ds + \int_{t_0}^{t}\int_{t_0}^{u} q(u,s,\phi(t_0))dsdu + \int_{t_0}^{t} Q(u,t_0,\phi(t_0))du\Bigg]^\alpha\Bigg) \\
&\quad \le k_h\int_{t_0}^{t} d([x_1(s)]^\alpha,[\phi(t_0)]^\alpha)ds + k_q\int_{t_0}^{t}\int_{t_0}^{u} d([x_1(s)]^\alpha,[\phi(t_0)]^\alpha)dsdu,
\end{aligned}
$$

So by the definition of D, we have

$$D(x_2(t),x_1(t)) \le k_h\int_{t_0}^{t} D(x_1(s),\phi(t_0))ds + k_q\int_{t_0}^{t}\int_{t_0}^{u} D(x_1(s),\phi(t_0))dsdu. \qquad (6)$$

Now, we can apply the first inequality (4) in the right-hand side of (6) to get

$$\begin{aligned} &D(x_2(t),x_1(t)) \\ &\quad\le (M+M_2)K\frac{|t-t_0|^2}{2!}+M_1K\frac{|t-t_0|^3}{3!}+(M+M_2)K\frac{|t-t_0|^3}{3!}+M_1K\frac{|t-t_0|^4}{4!} \\ &\quad\le K\left[(M+M_2)\frac{\delta^2}{2!}+(M+M_1+M_2)\frac{\delta^3}{3!}+M_1\frac{\delta^4}{4!}\right]. \end{aligned} \tag{7}$$

Starting from (4) and (7), assume that

$$\begin{aligned} &D(x_n(t),x_{n-1}(t)) \\ &\quad\le K^{n-1}\left[{}^{(n-1)}C_0(M+M_2)\frac{\delta^n}{n!}+\left[{}^{(n-1)}C_1(M+M_2)+{}^{(n-1)}C_0M_1\right]\frac{\delta^{n+1}}{(n+1)!}+\cdots\right. \\ &\qquad\left.+\left[{}^{(n-1)}C_{n-1}(M+M_2)+{}^{(n-1)}C_{n-2}M_1\right]\frac{\delta^{2n-1}}{(2n-1)!}+M_1\frac{\delta^{2n}}{2n!}\right], \end{aligned} \tag{8}$$

and we prove that such an inequality holds for $D(x_{n+1}(t),x_n(t))$. Indeed, from (2) and the assumptions, it follows that

$$\begin{aligned} &d([x_{n+1}(t)]^\alpha,[x_n(t)]^\alpha)= \\ &\quad\le k_h\int_{t_0}^t d([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha)ds+k_q\int_{t_0}^t\int_{t_0}^u d([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha)dsdu, \end{aligned}$$

for any $\alpha\in[0,1]$ and from the condition on D, we have

$$D(x_{n+1}(t),x_n(t))\le k_h\int_{t_0}^t D(x_n(s),x_{n-1}(s))ds+k_q\int_{t_0}^t\int_{t_0}^u D(x_n(s),x_{n-1}(s))dsdu.$$

According to (8), we get

$$\begin{aligned} &D(x_{n+1}(t),x_n(t)) \\ &\quad\le K^n\left[{}^nC_0(M+M_2)\frac{\delta^{n+1}}{(n+1)!}+\left[{}^nC_1(M+M_2)+{}^nC_0M_1\right]\frac{\delta^{n+2}}{(n+2)!}\right. \\ &\qquad\left.+\cdots+\left[{}^nC_n(M+M_2)+{}^nC_{n-1}M_1\right]\frac{\delta^{2n+1}}{(2n+1)!}+M_1\frac{\delta^{2n+2}}{(2n+2)!}\right]. \end{aligned}$$

Consequently, inequality (8) holds for $n=1,2,\cdots$ We can also write

$$\begin{aligned} &D(x_n(t),x_{n-1}(t)) \\ &\quad\le\frac{K^n}{K}\left[{}^{n-1}C_0(M+M_2)\frac{\delta^n}{n!}\left[{}^{(n-1)}C_1(M+M_2)+{}^{(n-1)}C_0M_1\right]\frac{\delta^{n+1}}{(n+1)!}+\cdots\right. \\ &\qquad\left.+\left[{}^{(n-1)}C_{(n-1)}(M+M_2)+{}^{(n-1)}C_{n-2}M_1\right]\frac{\delta^{2n-1}}{(2n-1)!}+M_1\frac{\delta^{2n}}{(2n)!}\right], \end{aligned} \tag{9}$$

for $n=1,2,\ldots,$ and $|t-t_0|\le\delta$.

Let us mention that

$$x_n(t) = x_0(t) + [x_1(t) - x_0(t)] + \cdots + [x_n(t) - x_{n-1}(t)],$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$x_0(t) + \sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)]$$

have the same convergence properties. From (9), it follows that $D(x_n(t), x_{n-1}(t)) \to 0$ uniformly on $|t - t_0| \le \delta$ as $n \to \infty$. Hence, there exists a fuzzy set-valued mapping $x : [t_0, t_0 + L] \to E^n$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t - t_0| \le \delta$ as $n \to \infty$. From the assumptions, we get

$$d([h(t, x_n(t))]^\alpha, [h(t, x(t))]^\alpha) \le k_h d([x_n(t)]^\alpha, [x(t)]^\alpha)$$

for any $\alpha \in [0,1]$, and so

$$D(h(t, x_n(t)), h(t, x(t))) \le k_h D(x_n(t), x(t)) \to 0 \tag{10}$$

uniformly on $|t - t_0| \le \delta$ as $n \to \infty$. Furthermore,

$$d([q(t, s, x_n(s))]^\alpha, [q(t, s, x(s))]^\alpha), \le k_q d([x_n(s)]^\alpha, [x(s)]^\alpha)$$

for any $\alpha \in [0,1]$, and

$$D(q(t, s, x_n(s)), q(t, s, x(s))) \le k_q D(x_n(s), x(s)) \to 0 \tag{11}$$

uniformly on $|t - t_0| \le \delta$ as $n \to \infty$.

Taking (10) and (11) into account, from (2), we obtain,

$$x(t) = \phi(t_0) + \int_{t_0}^{t} h(s, x(s))ds + \int_{t_0}^{t}\int_{t_0}^{u} q(u, s, x(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du$$

for $n \to \infty$,

Consequently, there is at least one levelwise continuous solution of (1).

We want to prove now that this solution is unique, that is, from

$$y(t) = \phi(t_0) + \int_{t_0}^{t} h(s, y(s))ds + \int_{t_0}^{t}\int_{t_0}^{u} q(u, s, y(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du \tag{12}$$

on $|t - t_0| \le \delta$, we want to show that $D(x(t), y(t)) \equiv 0$. Indeed, from (2) and (12), we have

$$d([y(t)]^\alpha, [x_n(t)]^\alpha) \le k_h \int_{t_0}^{t} d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha)ds + k_q \int_{t_0}^{t}\int_{t_0}^{u} d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha)dsdu$$

for any $\alpha \in [0,1]$, $n = 1, 2, \ldots$.

By the definition of D, we have

$$D(y(t),x_n(t)) \leq K\int_{t_0}^{t} D(y(s),x_{n-1}(s))ds + K\int_{t_0}^{t}\int_{t_0}^{u} D(y(s),x_{n-1}(s))dsdu \tag{13}$$

But $D(y(t),x_0(t)) \leq b$ on $|t-t_0| \leq \delta$, $y(t)$ being a solution of (12). It follows from (13) that

$$\begin{aligned} D(y(t),x_1(t)) &\leq K\int_{t_0}^{t} D(y(s),x_0(s))ds + K\int_{t_0}^{t}\int_{t_0}^{u} D(y(s),x_0(s))dsdu \\ &\leq Kb\left[|t-t_0| + \frac{|t-t_0|^2}{2!}\right] \end{aligned}$$

on $|t-t_0| \leq \delta$. Also,

$$\begin{aligned} D(y(t),x_2(t)) &\leq K\int_{t_0}^{t} D(y(s),x_1(s))ds + K\int_{t_0}^{t}\int_{t_0}^{u} D(y(s),x_1(s))dsdu \\ &\leq K^2 b\left[\frac{|t-t_0|^2}{2!} + 2\frac{|t-t_0|^3}{3!} + \frac{|t-t_0|^4}{4!}\right] \end{aligned}$$

on $|t-t_0| \leq \delta$. Now assume that

$$\begin{aligned} D(y(t),x_n(t)) \leq K^n b\Bigg[&{}^{(n-1)}C_0\frac{|t-t_0|^n}{n!} + {}^{(n-1)}C_1\frac{|t-t_0|^{n+1}}{(n+1)!} \\ &+ \cdots + {}^{(n-1)}C_n\frac{|t-t_0|^{2n}}{(2n)!}\Bigg] \end{aligned} \tag{14}$$

on the interval $|t-t_0| \leq \delta$. From

$$D(y(t),x_{n+1}(t)) \leq K\int_{t_0}^{t} D(y(s),x_n(s))ds + K\int_{t_0}^{t}\int_{t_0}^{u} D(y(s),x_n(s))dsdu$$

and (14), one obtains

$$D(y(t),x_{n+1}(t)) \leq K^{n+1}b\left[{}^{n}C_0\frac{|t-t_0|^{n+1}}{(n+1)!} + {}^{n}C_1\frac{|t-t_0|^{n+2}}{(n+2)!} + \cdots + {}^{n}C_n\frac{|t-t_0|^{2n+1}}{(2n+1)!}\right] \tag{15}$$

Consequently, (14) holds for any n, which leads to the conclusion

$$D(y(t),x_n(t)) = D(x(t),x_n(t)) \to 0$$

on the interval $|t-t_0| \leq \delta$ as $n \to \infty$. Thus, there exists an unique solution on $[t_0, t_0+L]$. Since L is arbitrary in $(0,\beta)$.Therefore, there exists an unique solution on $[t_0, t_0+\beta)$.

Example. Consider the fuzzy Volterra integral equation with infinite delay

$$\begin{aligned} x'(t) &= \left(\frac{5}{6} - \frac{6}{11e}\right)x(t) + \int_{-\infty}^{t} e^{(s-t-1)}x(s)ds, \\ [x(t)]^\alpha &= [\phi(t)]^\alpha = e^{2t}[(1+\alpha),(3-\alpha)], \qquad t \in (-\infty,0], \alpha \in [0,1]. \end{aligned}$$

Since the conditions (A)-(C) hold, from Theorem 3.1 the above equation has a unique solution.

Acknowledgement

The work is supported by Department of Science and Technology, New Delhi, India.

References

[1] R. P. Agarwal, D. O. Regan and V. Laksmikantham, Fuzzy Volterra integral equations: a Stacking theorem approach, *Applicable Analysis* **83** 521–532 (2004).

[2] R. J. Aumann, Integrals of set-valued functions, *Journal of Mathematical Analysis and Applications* **12** (1965), 1–12.

[3] K. Balachandran and P. Prakash, Existence of solutions of nonlinear fuzzy volterra-Fredholm integral equations, *Indian Journal of Pure Applied Mathematical* **33** (2002), 329–343.

[4] P. Diamond, Theory and applications of fuzzy Volterra integral equations, *IEEE Transactions on Fuzzy Systems* **10** (2002), 97–101.

[5] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems* **24** (1987), 301-317.

[6] O. Kaleva, The Cauchy problem for fuzzy differential equations, *Fuzzy Sets and Systems* **35** (1990), 389–396.

[7] A. Kandel and W. J. Byatt, Fuzzy differential equations, Proceedings International Conference Cybernetics and Society, Tokyo, (1978), 1213–1216.

[8] C. V. Negoita and D. A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis*, Wiley, New York, Toronto, 1975.

[9] M. L. Puri and D. A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications* **114** (1986), 409–422.

[10] Shiji Song, Qin-Yu Liu, and Qi-chun Xu, Existence and comparison theorems to Volterra fuzzy integral equation in (E^n, D), *Fuzzy Sets and Systems* **104** (1999), 315–321.

[11] X. Xiaoping and F. Yongqiang, On the structure of solutions for fuzzy initial value problem, *Fuzzy Sets and Systems* **157** (2006), 212–229.

Department of Mathematics, Periyar University, Salem - 636 011, India.

E-mail:kp-prakash@lycos.com

Department of Mathematics, Periyar University, Salem - 636 011, India.

E-mail: