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ON HUA'S INEQUALITY IN REAL INNER PRODUCT SPACES

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Abstract. Generalization of results in [1] concerning the Hua's inequality in real inner product spaces are given.

1. Introduction

The following generalizations of the Hua's inequality in real inner product spaces was given in [1]:

Theorem 1.1. *Let $(X;(,))$ be a real inner product space and $\alpha>0,\beta>0$. For all $x,y\in X$ one has the inequality*

$$(\delta-(x,y))^2+\alpha\|x\|^2\geq\alpha\delta^2/(\alpha+\|y\|^2). \tag{1.1}$$

The equality holds in (1.1) *iff*

$$x=(\delta/(\alpha+\|y\|^2))y.$$

Theorem 1.2. *Let $(X;(,))$ be an inner product space and $x_i\in X(i=1,2,\dots,n),y\in X$ and $\alpha>0$. Then one has the inequality:*

$$\left\|y-\sum_{i=1}^n x_i\right\|^2+\alpha\sum_{i=1}^n\|x_i\|^2\geq(\alpha/(n+\alpha))\|y\|^2. \tag{1.2}$$

The equality holds in (1.2) *iff*

$$x_i=(1/(n+\alpha))y,\quad \textit{for all } i=1,2,\dots,n.$$

In this paper we shall give some new generalization of these results.

2. Results

First we shall prove the following result:

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Theorem 2.1. *Let the conditions of Theorem* 1.1. *be satisfied and let* $(x, y) < \delta, p, q > 1, 1/p + 1/q = 1$. *Then one has the inequality*

$$(\delta - (x, y))^p + \alpha^{p-1}\|x\|^p \geq (\alpha/(\alpha + \|y\|^q))^{p-1}\delta^p \tag{2.1}$$

The equality holds in (2.1) *iff*

$$x = (\delta\|y\|^{q-2}/\alpha + \|y\|^q))y. \tag{2.2}$$

Proof. Applying Hölder's and Schwarz's inequality we have

$$\begin{aligned}(1 + \alpha^{-1}\|y\|^q)^{1/q}[(\delta - (x, y))^p + \alpha^{-1}\alpha^p\|x\|^p]^{1/p} \\ \geq \delta - (x, y) + \alpha^{-1}\|y\|\alpha\|x\| = \delta - (x, y) + \|x\|\ \|y\| \geq \delta\end{aligned}$$

This inequality is equivalent to (2.1). From the conditions for equality in Holder's inequality we have

$$\alpha^p\|x\|^p = C\|y\|^q,\ (\delta - (x, y))^p = C \tag{2.3}$$

while the equality in Schwarz's inequality gives

$$x = \lambda y, \quad (\lambda > 0). \tag{2.4}$$

From (2.4) and (2.3) we have

$$\alpha^p\lambda^p\|y\|^p = C\|y\|^q,$$

that is

$$C = \alpha^p\lambda^p\|y\|^{p-q}. \tag{2.5}$$

From (2.4), (2.3) and (2.5) we have

$$(\delta - (x, y))^p = \alpha^p\lambda^p\|y\|^{p-q}$$

i.e.

$$\begin{aligned}\delta - \lambda\|y\|^2 &= \alpha\lambda\|y\|^{1-q/p} \\ &= \alpha\lambda\|y\|^{2-q}\end{aligned}$$

where from we have

$$\lambda = \delta\|y\|^{q-2}/(\alpha + \|y\|^q). \tag{2.6}$$

Now, (2.4) and (2.6) gives (2.2).

Remark. For $p = 2$, we have Theorem 1.1.

The following theorem is also valid:

Theorem 2.2. *Let* x, y, δ, α *satisfy the conditions of Theorem* 2.1 *and let* f *be a convex nondecreasing function on* $[0, \infty)$. *Then*

$$f(\delta - (x, y)) + \alpha^{-1}\|y\|f(\alpha\|x\|) \geq ((\alpha + \|y\|)/\alpha)f(\alpha\delta/(\alpha + \|y\|)) \tag{2.7}$$

If f is strictly convex, then the equality holds in (2.7) *iff*

$$x = (\delta/(\|y\|(\alpha + \|y\|)))y. \tag{2.8}$$

Proof. By applying Jensen's and Schwarz's inequlity and the nondecreasing property of f, we have

$$\begin{aligned} & f(\delta - (x, y)) + \alpha^{-1}\|y\| f(\alpha\|x\|) \\ & \geq (1 + \|y\|/\alpha) f((\delta - (x, y) + \|x\|\ \|y\|)/(1 + \|y\|/\alpha)) \\ & \geq ((\alpha + \|y\|)/\alpha) f(\alpha\delta/(\alpha + \|y\|)) \end{aligned}$$

The case of equality for Jensen's and Schwarz's inequality gives respectivity

$$\delta - (x, y) = \alpha\|x\| \quad \text{and } x = \lambda y (\lambda > 0),$$

where from we have

$$\lambda = \delta/(\|y\|(\alpha + \|y\|)),$$

and so we have (2.8).

Similarly we can prove the following result.

Theorem 2.3. *Let the conditions of Theorem* 1.2. *be satisfied. If f is a convex nondecreasing function on* $[0, \infty)$, *then*

$$f(\|y - \sum_{i=1}^{n} x_i\|) + \alpha^{-1} \sum_{i=1}^{n} f(\alpha\|x_i\|) \geq ((\alpha + n)/\alpha) f(\alpha\|y\|/(\alpha + n)) \tag{2.9}$$

If f is a strictly convex, then equality holds iff

$$x_i = (1/(n + \alpha))y \quad \textit{for all } i = 1, 2, \ldots, n. \tag{2.10}$$

Proof. By the triangle inequality we have

$$\Big\|y - \sum_{i=1}^{n} x_i\Big\| \geq \Big|\|y\| - \Big\|\sum_{i=1}^{n} x_i\Big\|\Big| \tag{2.11}$$

Hence since f is nondecreasing on $[0, \infty)$ we have

$$f\Big(\Big\|y - \sum_{i=1}^{n} x_i\Big\|\Big) \geq f\Big(\Big|\ \|y\| - \Big\|\sum_{i=1}^{n} x_i\Big\|\ \Big|\Big) \tag{2.12}$$

Now Jensen's inequality for convex functions gives

$$\begin{aligned} & \alpha^{-1} \sum_{i=1}^{n} f(\alpha\|x_i\|) = (n/\alpha)(1/n) \sum_{i=1}^{n} f(\alpha\|x_i\|) \\ & \geq (n/\alpha) f((\alpha/n) \sum_{i=1}^{n} \|x_i\|) \geq (n/\alpha) f((\alpha/n)\| \sum_{i=1}^{n} x_i\|) \end{aligned} \tag{2.13}$$

Note that $f(|t|)$ is also a convex function. So by (2.12) and (2.13) we have

$$f\Big(\Big\|y-\sum_{i=1}^{n}x_i\Big\|\Big)+\alpha^{-1}\sum_{i=1}^{n}f(\alpha\|x_i\|)$$
$$\geq f\Big(\Big|\ \|y\|-\Big\|\sum_{i=1}^{n}x_i\Big\|\ \Big|\Big)+(n/\alpha)f\Big((\alpha/n)\Big\|\sum_{i=1}^{n}x_i\Big\|\Big)$$
$$\geq ((\alpha+n)/\alpha)f(|\ \alpha\|y\|/(\alpha+n)\ |)=((\alpha+n)/\alpha)f(\alpha\ \|y\|/(\alpha+n)). \tag{2.14}$$

Now we address the condition for equality in the event that f is strictly convex. Since it is nondecreasing it must be strictly increasing on $[0,\infty)$. The first inequality in (2.13) becomes equality iff

$$\|x_1\|=\|x_2\|=\cdots=\|x_n\| \tag{2.15}$$

While the second inequality in (2.14) becomes an equality iff

$$\Big\|\sum_{i=1}^{n}x_i\Big\|=n\|y\|/(\alpha+n) \tag{2.16}$$

The second inequality in (2.13) becomes an equality iff

$$\Big\|\sum_{i=1}^{n}x_i\Big\|=\sum_{i=1}^{n}\|x_i\| \tag{2.17}$$

i.e. there exists $\lambda_{ij}>0$ so that $x_i=\lambda_{ij}x_j$ for all $i,j\varepsilon\{1,2,\dots,n\}$ with $j\neq j$. With respect to (2.15) it is clear that $\lambda_{ij}=1$. From (2.16) and (2.17) we have

$$n\|x_i\|=n\|y\|/(\alpha+n)\quad i=1,2,\dots,n \tag{2.18}$$

Moreover equality in (2.11) is valid if

$$\sum_{i=1}^{n}x_i=\lambda y\quad \text{i.e. } nx_i=\lambda y$$

which together with (2.18) gives (2.10).

Remark. For $f(x)=x^2$, we have Theorem 1.2, while for $f(x)=x^p,\ p>1$ we obtain

$$\Big\|y-\sum_{i=1}^{n}x_i\Big\|^p+\alpha^{p-1}\sum_{i=1}^{n}\|x_i\|^p\geq(\alpha/(\alpha+n))^{p-1}\|y\|^p \tag{2.19}$$

References

[1] S. S. Dragomir, G. S. Yang, *On Hua's inequality in real inner product spaces,* Tamkang J. Math. **27** (1996), 227-232.

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