A Method for Solving the Variational Inequality Problem and Fixed Point Problems in Banach Spaces

Wongvisarut Khuangsatung and Atid Kangtunyakarn

Abstract. The purpose of this research is to modify Halpern iteration's process for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a strictly pseudo contractive mapping in \( q \)-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in \( q \)-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

1 Introduction

For the last decades, fixed point theory is a very importance tool for solving the problems in economic, computer science, physics, etc. Throughout this paper, let \( E \) be a Banach space with dual space of \( E^* \) and let \( C \) be a nonempty closed convex subset of \( E \). We use the norm of \( E \) and \( E^* \) by the same symbol \( \| \cdot \| \). We denote weak and strong convergence by notations “\( \rightharpoonup \)” and “\( \rightarrow \)”, respectively. Let \( q \) be a given real number with \( q > 1 \). The generalized duality mapping \( J_q : E \rightarrow 2^{E^*} \) is defined by

\[
J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\},
\]

for all \( x \in E \). If \( q = 2 \), then \( J_2 = J \) is called normalized duality mapping.

Remark 1. If \( J_q \) is generalized duality mapping of \( E \) into \( 2^{E^*} \). Then the following properties are holds:

1. \( J_q(tx) = t^{q-1}J_q(x) \), for all \( x \in E \) and \( t \in [0, \infty) \);
2. \( J_q(-x) = -J_q(x) \), for all \( x \in E \).

2010 Mathematics Subject Classification. 46B25, 47H05, 47H06, 47H10.

Key words and phrases. Strictly pseudo contractive mapping, Inverse strongly monotone accretive operator, Variational inequality problem, \( q \)-uniformly smooth Banach space.

Corresponding author: Atid Kangtunyakarn.
Definition 1. Let $C$ be a nonempty subset of a Banach space $E$ and $T : C \to C$ be a self-mapping. Then

1. $T$ is called a nonexpansive mapping if
   \[\|Tx - Ty\| \leq \|x - y\|,\]
   for all $x, y \in C$.

2. $T$ is called an $\eta$–strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that
   \[
   \langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^2 - \eta \|(I - T)x - (I - T)y\|^2,
   \]
   for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$. It is clear that (1.1) is equivalent to the following
   \[
   \langle (I - T)x - (I - T)y, j_q(x - y) \rangle \geq \eta \|(I - T)x - (I - T)y\|^2,
   \]
   for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$.

Definition 2. Let $C \subseteq E$ be closed convex and $Q_C$ be a mapping of $E$ onto $C$. The mapping $Q_C$ is said to be sunny if
   \[Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx,\]
   for all $x \in E$ and $t \geq 0$. A mapping $Q_C$ is called retraction if $Q_C^2 = Q_C$. A subset $C$ of $E$ is called a sunny nonexpansive retraction of $E$ if there exists a sunny nonexpansive retraction from $E$ onto $C$.

For more information about (sunny) nonexpansive retraction can be found in [13].

The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by
\[
\rho_E(\tau) = \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau \right\}. \tag{1.3}
\]

A Banach space $E$ is uniformly smooth if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known that $E$ is $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. In a Hilbert space, $L_p(l_p)$ with $1 < p < \infty$ are $q$-uniformly smooth. Clearly every $q$-uniformly smooth Banach space is uniformly smooth. If $E$ is smooth, then $J_q$ is a single valued which is denoted by $j_q$.

An operator $A$ of $C$ into $E$ is said to be accretive if there exists $j_q(x - y) \in J_q(x - y)$ such that
\[
\langle Ax - Ay, j_q(x - y) \rangle \geq 0, \quad \forall x, y \in C.
\]

A mapping $A : C \to E$ is said to be $\alpha$-inverse strongly accretive if there exists $j_q(x - y) \in J_q(x - y)$ and $\alpha > 0$ such that
\[
\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}
\]
Remark 2. From (1.2) and (1.4), if $T$ is an $\eta$-strictly pseudo-contractive mapping, then $I - T$ is $\eta$-inverse strongly accretive.

Let $C$ be a nonempty subset of $q$-uniformly smooth Banach space $E$ and $A : C \to E$ be a nonlinear operator. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, J_q(y - x^*) \rangle \geq 0, \quad \forall y \in C,$$

where $J_q$ is generalized duality mapping from $E$ into $2E^*$. The set of solutions of the variational inequality in Banach space is denoted by $S_q(C, A)$. If $q = 2$, then $S_q(C, A)$ is reduced to $S(C, A)$, where $S(C, A)$ is the set of solutions of the generalized variational inequality in Banach spaces proposed by Aoyama et. al. [1] in 2005. Many research papers have increasingly investigated variational inequality problems in Banach spaces, see, for instance, [2], [3], and the references therein.

In 1967, Halpern [4] introduced the Halpern’s iterative method as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \forall n \geq 1,$$

where $\alpha_n \in (0, 1)$ satisfying suitable conditions, for all $n \geq 1$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of mapping $T$ in a real Hilbert space, where $T$ is a nonexpansive mapping. In the last decade, many authors have studied and modified Halpern’s iterative method for various nonlinear mappings, see, for instance, [5], [6], [7], [8] and the references therein.

In a uniformly convex and 2-smooth Banach space, Aoyama et al. [1] introduced the iterative method for finding a solution of generalized variational inequality problem for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$, $Q_C$ is a sunny nonexpansive retraction from $E$ onto $C$, $A$ is an $\alpha$-inverse strongly accretive operator. Under suitable conditions, They also proved that the sequence generated by the proposed algorithm weakly converges to a solution of $S(C, A)$.

In 2013, Kangtunyakarn [9] introduced an iterative scheme for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems in a uniformly convex and 2-smooth Banach space as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \forall n \geq 1,$$
where $A$, $B$ are $\alpha$ and $\beta$-inverse strongly accretive mappings, respectively, $Q_C$ is a sunny nonexpansive retraction, $S^A$ is the $S^A$-mapping generated by a finite family of nonexpansive mappings and a finite family of strictly pseudo-contractive mappings and finite real numbers. He also proved a strong convergence theorem of sequence $\{x_n\}$ under suitable conditions of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \text{ and } \{\eta_n\}$.

Motivated by the results of Aoyama et al. [1], Kangtunyakarn [9] and by the ongoing research in this direction, we have the following question.

**Question** Can we prove a strong convergence theorem of two nonlinear mapping in q-uniformly smooth Banach space?

The purpose of this manuscript is to modify Halpern iteration’s process in order to answer the question above and prove a strong convergence theorem for finding a common element of the set of solutions of (1.5) and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

## 2 Preliminaries

The following lemmas are important tool to prove our main results in the next section.

**Lemma 2.1.** Let $E$ be a Banach space and let $J_q : E \rightarrow 2^{E^*}$, $1 < q < \infty$ be the generalized duality mapping. Then for any $x, y \in E$, there exists $j_q(x + y) \in J_q(x + y)$ such that $\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y) \rangle$.

**Lemma 2.2.** [10] Let $C$ be a closed and convex subset of a real uniformly smooth Banach space $E$ and $T : C \rightarrow C$ a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \rightarrow F(T)$ such that

$$\limsup_{n \rightarrow \infty} \langle u - Q_{F(T)}u, J_q(x_n - Q_{F(T)}u) \rangle \leq 0,$$

for any given $u \in C$.

**Lemma 2.3.** [11] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty;$$
(2) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then, \( \lim_{n \to \infty} s_n = 0 \).

**Lemma 2.4.** [12] Let \( q > 1 \) be a given real number and \( E \) be a real Banach space. Then the following statements are equivalent.

(i) \( E \) is \( q \)-uniformly smooth.

(ii) There is a constant \( C_q > 0 \) such that for all \( x, y \in E \),

\[
\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + C_q\|y\|^q.
\]

(iii) There exists a constant \( d_q \) such that for all \( x, y \in E \) and \( t \in [0, 1] \),

\[
\|(1 - t)x + ty\|^q \geq (1 - t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x - y\|^q,
\]
where \( \omega_q(t) = t^q(1 - t) + t(1 - t)^{q-1} \).

**Lemma 2.5.** Let \( C \) be a nonempty closed convex subset of \( q \)-uniformly smooth Banach space \( E \). Let \( T : C \to C \) be a nonexpansive mapping and \( S : C \to C \) be a \( \lambda \)-strictly pseudo contractive mapping with \( F(T) \cap F(S) \neq \emptyset \). For every \( a \in (0, 1) \), defined the mapping \( H : C \to C \) by \( Hx = T((1 - a)I + aS)x \), for all \( x \in C \) and \( a \in (0, \mu) \) where \( \mu = \min\{1, \left(\frac{2\lambda}{C_qq^{\frac{1}{q-1}}}\right)^{\frac{1}{q-1}}\} \), \( C_q \) is the best \( q \)-uniformly smooth constant of \( E \). Then \( F(H) = F(T) \cap F(S) \).

**Proof.** It is obvious that \( F(T) \cap F(S) \subseteq F(H) \). Let \( x_0 \in F(H) \) and \( x^* \in F(T) \cap F(S) \), we have

\[
\|x_0 - x^*\|^q = \|T((1 - a)I + aS)x_0 - x^*\|^q \\
\leq \|x_0 - x^* + a(Sx_0 - x_0)\|^q \\
\leq \|x_0 - x^*\|^q + aq\langle Sx_0 - x_0, j_q(x_0 - x^*) \rangle + C_qa^q\|Sx_0 - x_0\|^q \\
= \|x_0 - x^*\|^q + aq\langle Sx_0 - x^*, j_q(x_0 - x^*) \rangle + aq\langle x^* - x_0, j_q(x_0 - x^*) \rangle \\
+ C_qa^q\|Sx_0 - x_0\|^q \\
\leq \|x_0 - x^*\|^q + aq(\|x_0 - x^*\|^q - \lambda\|x_0 - Sx_0\|^q) - aq\|x^* - x_0\|^q \\
+ C_qa^q\|Sx_0 - x_0\|^q \\
= \|x_0 - x^*\|^q - a(q\lambda - C_qa^{q-1})\|x_0 - Sx_0\|^q.
\]

From above it implies that \( x_0 \in F(S) \). From the definition of \( H \), we have

\[
x_0 = Hx_0 = T((1 - a)I + aS)x_0 = Tx_0.
\]

Then \( x_0 \in F(T) \). We can conclude that \( x_0 \in F(S) \cap F(T) \). Hence \( F(H) \subseteq F(S) \cap F(T) \). Applying (2.1), we have \( H \) is a nonexpansive mapping.
Example 1. Let $S : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Sx = \frac{x^2}{x+1}$, for all $x \in \mathbb{R}^+$ and let $T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Tx = \frac{3x}{4}$, for all $x \in [0, 5]$. Define the mapping $H : \mathbb{R}^+ \to \mathbb{R}^+$ by $Hx = T\left(\frac{9}{10}I + \frac{1}{10}S\right)x$ for all $x \in \mathbb{R}^+$. From Lemma 2.5, we have $F(H) = F(S) \cap F(T) = \{0\}$.

Lemma 2.6. Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$. Let $j, j_q : E \to E^*$ be a normalized duality mapping and generalized duality mapping, respectively. Let $Q_C$ be a retraction from $E$ onto $C$. Then the following are equivalent.

(i) $Q_C$ is both sunny and nonexpansive,

(ii) $\langle x - Q_Cx, J(y - Q_Cx) \rangle \leq 0$, for all $x \in E$ and $y \in C$,

(iii) $\langle x - Q_Cx, j_q(y - Q_Cx) \rangle \leq 0$, for all $x \in E$ and $y \in C$.

Proof. From [13], we have $(i) \iff (ii)$. Then we only show that $(ii)$ equivalent to $(iii)$. Since $j_q(x) = \|x\|^{q-1}J(x)$, for all $x \in E$. For every $x \in E$ and $y \in C$.

If $y - Q_Cx \neq 0$, we have

$$\langle x - Q_Cx, j_q(y - Q_Cx) \rangle \leq 0 \iff \langle x - Q_Cx, J(y - Q_Cx) \rangle \leq 0.$$

If $y - Q_Cx = 0$, we have

$$\langle x - Q_Cx, j_q(y - Q_Cx) \rangle = \langle x - Q_Cx, J(y - Q_Cx) \rangle = 0.$$

From above we can conclude the desire result. □

Remark 3. Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$ and let $x \in E$, $x_0 \in C$. From Lemma 2.6, we have

$$x_0 = Q_Cx \iff \langle x - x_0, j_q(y - x_0) \rangle \leq 0, \forall y \in C.$$

Lemma 2.7. Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$ and let $A : C \to E$ be a mapping. Then $S_q(C, A) = F(Q_C(I - \lambda A))$, for all $\lambda > 0$, where $S_q(C, A) = \{u \in C : \langle Au, j_q(y - u) \rangle \geq 0, \forall y \in C\}$.

Proof. Let $x^* \in F(Q_C(I - \lambda A))$, for all $\lambda > 0$. Then $x^* = Q_C(I - \lambda A)x^*$. From 2.6, we have

$$\langle (I - \lambda A)x^* - x^*, j_q(y - x^*) \rangle \leq 0, \forall y \in C.$$

It follows that

$$\langle Ax^*, j_q(y - x^*) \rangle \geq 0, \forall y \in C.$$

Then $x^* \in S_q(C, A)$. Hence $F(Q_C(I - \lambda A)) \subseteq S_q(C, A)$. Similarly, we can conclude that $S_q(C, A) \subseteq F(Q_C(I - \lambda A))$. □
3 Main results

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$ and let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let $S : C \to C$ be $\lambda$-strictly pseudo contractive mapping and $A : C \to E$ be a $\alpha$-inverse strongly accretive operator with $F = F(S) \cap S_q(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho A)(aI + (1 - a)S)x_n, \forall n \in \mathbb{N},$$

where $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $a \in (0, \mu)$, where $\mu = \min\{1, (\frac{A}{C_q})^{\frac{1}{q-1}}\}$, where $C_q$ is the $q$-uniformly smooth constant of $E$;

(iii) $0 < \rho < (\frac{C_q}{\mu})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{F}u$, where $Q_{F}$ is a unique sunny nonexpansive retraction of $C$ onto $F$.

**Proof.** First, we show that $Q_C(I - \rho A)$ is a nonexpansive mapping. Let $x, y \in C$, we have

$$\|Q_C(I - \rho A)x - Q_C(I - \rho A)y\|^q \leq \|x - y - \rho(Ax - Ay)\|^q$$

$$\leq \|x - y\|^q - \rho q\langle Ax - Ay, j_q(x - y)\rangle + C_q\rho^q\|Ax - Ay\|^q$$

$$\leq \|x - y\|^q - \rho q \alpha \|Ax - Ay\|^q + C_q\rho^q\|Ax - Ay\|^q$$

$$\leq \|x - y\|^q - \rho (q\alpha - C_q\rho^{q-1})\|Ax - Ay\|^q$$

$$\leq \|x - y\|^q.$$

Then $Q_C(I - \rho A)$ is a nonexpansive mapping. Next we show that the sequence $\{x_n\}$ is bounded. Put $Wx = Q_C(I - \rho A)(aI + (1 - a)S)x$, for all $x \in C$. From Lemma 2.5 and 2.7, we have

$$F(W) = F(Q_C(I - \rho A)) \cap F(S) = S_q(C, A) \cap F(S)$$

and $W$ is a nonexpansive mapping. From (3.1), we can rewrite that

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Wx_n, \forall n \in \mathbb{N}.$$  

(3.2)

Let $x^* \in F$ and the definition of $x_n$, we have

$$\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\| + (1 - \alpha_n)\|Wx_n - x^*\|.$$
\[ \leq \alpha_n \|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \]
\[ \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}. \]

Applying induction, we have \(\{x_n\}\) is bounded. From the definition of \(\{x_n\}\), we have
\[ \|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_{n-1}|\|u\| + (1 - \alpha_n)\|Wx_n - Wx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Wx_{n-1}\|. \]

Since \(\{x_n\}\) is bounded sequence, the condition \((iv)\) and Lemma 2.3, we have
\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad (3.3) \]

From (3.2), we have
\[ x_{n+1} - x_n = \alpha_n (u - x_n) + (1 - \alpha_n)(Wx_n - x_n). \quad (3.4) \]

From (3.3) and (3.4), we have
\[ \lim_{n \to \infty} \|Wx_n - x_n\| = 0. \quad (3.5) \]

From Lemma 2.2 and (3.5), we have
\[ \limsup_{n \to \infty} \langle u - z_0, j_q(x_n - z_0) \rangle \leq 0, \quad (3.6) \]
where \(z_0 = Q_Fu\). Finally, we show that the sequence \(\{x_n\}\) converges strongly to \(z_0 = Q_Fu\).

By using the method of proof in Theorem 3.1, we have the following theorems.

**Theorem 3.2.** Let \(C\) be a nonempty closed convex subset of \(q\)-uniformly smooth Banach space \(E\) and let \(Q_C\) be a sunny nonexpansive retraction from \(E\) onto \(C\). Let \(S : C \to C\) be be \(\lambda\)-strictly pseudo contractive mapping and \(T : C \to E\) be a nonexpansive mapping with \(F = F(S) \cap F(T) \neq \emptyset\). Let \(\{x_n\}\) be the sequence generated by \(x_1, u \in C\) and
\[ x_{n+1} = \alpha_n u + (1 - \alpha_n)T(aI + (1 - a)S)x_n, \forall n \in \mathbb{N}, \]
(3.7)

where \(\alpha_n \in [0, 1], a \in (0, 1)\) and \(\rho > 0\) satisfy the following conditions:

(i) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty;\)
(ii) $\alpha \in (0, \mu)$, where $\mu = \min\{1, (\frac{A}{C_q})^{\frac{1}{q-1}}\}$, where $C_q$ is the $q$-uniformly smooth constant of $E$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence \( \{x_n\} \) converges strongly to $z_0 = Q_F u$, where $Q_F$ is a unique sunny nonexpansive retraction of $C$ onto $F$.

Proof. Applying the method of Theorem 3.1 and Lemma 2.5, we can conclude the desired result. \( \square \)

4 Application

In this section, we use the main results to obtain fixed points theorems for a finite family of strictly pseudo contractive mappings in $q$-uniformly smooth Banach space. Before prove this theorems, we need the following results.

Lemma 4.1. [14] Let $E$ be a smooth Banach space and $C$ be a nonempty convex subset of $E$. Given an integer $N \geq 1$, assume that for each $i \in \Lambda$, $T_i : C \to C$ is a $\lambda_i$-strict pseudocontraction for some $0 \leq \lambda_i < 1$. Assume that \( \{\eta_i\}_{i=1}^{N} \) is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$, then $\sum_{i=1}^{N} \eta_i T_i : C \to C$ is a $\lambda_i$-strict pseudocontraction with $\lambda = \min\{\lambda_i : 1 \leq i \leq N\}$.

Lemma 4.2. [14] Let $E$ be a smooth Banach space and $C$ be a nonempty convex subset of $E$. Given an integer $N \geq 1$, assume that for each $i \in \Lambda$, \( \{T_i\}_{i=1}^{N} : C \to C \) is a finite family of $\lambda_i$-strict pseudocontraction for some $0 \leq \lambda_i < 1$ such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume that \( \{\eta_i\}_{i=1}^{N} \) is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$. Then $F(\sum_{i=1}^{N} \eta_i T_i) = F$.

Theorem 4.1. Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$ and let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let \( T_i : C \to C \) is a $\lambda_i$-strict pseudocontraction for some $0 \leq \lambda_i < 1$ and $A : C \to E$ be a $\alpha$-inverse strongly accretive operator with $F = \bigcap_{i=1}^{N} F(T_i) \cap S_q(C, A) \neq \emptyset$. Let \( \{x_n\} \) be the sequence generated by $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho A)(aI + (1-a) \sum_{i=1}^{N} \eta_i T_i)x_n, \forall n \in \mathbb{N}, \quad (4.1)$$

where \( \{\eta_i\}_{i=1}^{N} \) is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$, $\alpha_n \in [0, 1]$, $\alpha \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\alpha \in (0, \mu)$, where $\mu = \min\{1, (\frac{A}{C_q})^{\frac{1}{q-1}}\}$, where $C_q$ is the $q$-uniformly smooth constant of $E$;
(iii) \( 0 < \rho < \left( \frac{a}{C_q} \right)^{\frac{1}{q-1}} \);

(iv) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \)

Then the sequence \( \{x_n\} \) converges strongly to \( z_0 = Q_F u \), where \( Q_F \) is the unique sunny nonexpansive retraction of \( C \) onto \( F \).

**Proof.** From Theorem 3.1, Lemma 4.1 and 4.2, we can conclude the desired result. \( \square \)

**Lemma 4.3.** Let \( C \) be a nonempty closed convex subset of \( q \)-uniformly smooth Banach space \( E \) and let \( S : C \to C \) be \( \kappa \)-strictly pseudo contractive mapping with \( F(S) \neq \emptyset \). Then \( F(S) = S_q(C, I - S) \).

**Proof.** Obvious that \( F(S) \subseteq S_q(C, I - S) \). Let \( x_0 \in S_q(C, I - S) \) and \( x^* \in F(S) \). Then

\[
\langle (I - S)x_0, j_q(y - x_0) \rangle \geq 0, \forall y \in C.
\]

Put \( A = I - S \). Since \( S : C \to C \) is \( \kappa \)-strictly pseudo contractive mapping, then there exists \( j_q(x_0 - x^*) \) such that

\[
\langle Sx_0 - Sx^*, j_q(x_0 - x^*) \rangle = \langle (I - A)x_0 - (I - A)x^*, j_q(x_0 - x^*) \rangle
\]

\[
= \langle x_0 - x^*, j_q(x_0 - x^*) \rangle - \langle Ax_0 - Ax^*, j_q(x_0 - x^*) \rangle
\]

\[
= \|x_0 - x^*\|^q - \langle (I - S)x_0, j_q(x_0 - x^*) \rangle
\]

\[
\leq \|x_0 - x^*\|^q - \kappa \| (I - S)x_0 \|^q.
\]

It implies that

\[
\kappa \| (I - S)x_0 \|^q \leq \langle (I - S)x_0, j_q(x_0 - x^*) \rangle \leq 0.
\]

Then \( x_0 \in F(S) \). Hence \( S_q(C, I - S) \subseteq F(S) \). \( \square \)

**Corollary 4.2.** Let \( C \) be a nonempty closed convex subset of \( q \)-uniformly smooth Banach space \( E \) and let \( Q_C \) be a sunny nonexpansive retraction from \( E \) onto \( C \). Let \( T_i : C \to C \) is a \( \lambda_i \)-strictly pseudo contractive mapping for some \( 0 \leq \lambda_i < 1 \) and \( S : C \to E \) be a \( \alpha \)-strictly pseudo contractive mapping with \( F = \bigcap_{i=1}^{N} F(T_i) \bigcap F(S) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated by \( x_1, u \in C \) and

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho(I - S))(aI + (1 - a)\sum_{i=1}^{N} \eta_i T_i)x_n, \forall n \in \mathbb{N}, \quad (4.2)
\]

where \( \{\eta_i\}_{i=1}^{N} \) is a positive sequence such that \( \sum_{i=1}^{N} \eta_i = 1, \alpha_n \in [0, 1], a \in (0, 1) \) and \( \rho > 0 \) satisfy the following conditions:
(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(ii) \( a \in (0, \mu) \), where \( \mu = \min\{1, (\frac{q}{C_q})^{\frac{1}{q-1}}\} \), where \( C_q \) is the \( q \)-uniformly smooth constant of \( E \);

(iii) \( 0 < \rho < (\frac{a \mu}{C_q})^{\frac{1}{q-1}} \);

(iv) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \).

Then \( \{x_n\} \) converges strongly to \( z_0 = Q_F u \), where \( Q_F \) is a unique sunny nonexpansive retraction of \( C \) onto \( F \).

**Proof.** From Theorem 4.1 and Lemma 4.3, we can conclude the desired result. \( \square \)

## 5 Example and Numerical results

In this section, we give numerical results to illustrate the main theorem.

**Example 2.** Let \( \mathbb{R} \) be a set of real number. Let \( S : [0, 10] \to [0, 1] \) be a mapping defined by \( Sx = \frac{2x^2}{x+2x} \), for all \( x \in [0, 1] \) and let \( A : [0, 10] \to \mathbb{R} \) defined by \( Ax = 3x^2 \) for all \( x \in [0, 10] \). Suppose the sequence \( \{x_n\} \) generated by (3.1), where \( \alpha_n = \frac{1}{60n} \), \( \rho = \frac{1}{100} \), and \( a = \frac{1}{80} \). Then the sequence \( \{x_n\} \) converges strongly to 0.

**Solution.** It is obvious that \( S \) is \( \frac{1}{50} \)-strictly pseudo contractive mapping and \( A \) is \( \frac{1}{60} \)-inverse strongly accretive operator with \( F(S) \cap S_2(C, A) = \{0\} \). Since \( \{x_n\} \) generated by (3.1), we have

\[
x_{n+1} = \frac{1}{60n} u + \left(1 - \frac{1}{60n}\right) Q_{[0,10]} \left(I - \frac{1}{100}A\right) \left(\frac{1}{80}I + (1 - \frac{1}{80})S\right)x_n,
\]

where \( u, x_1 \in [0, 10] \). It is easy to see that \( \alpha_n \), for all \( n \geq 1 \), \( a, \rho \) satisfied all condition in Theorem 3.1. From Theorem 3.1, we have the sequence \( \{x_n\} \) coonvergence strongly to 0.

Putting \( u = 0.55 \) and \( x_1 = 0.99 \) in (5.1), we have the numerical results as shown in the following Figure 1 and Table 1.
Table 1: The values of the sequences \( \{x_n\} \) with initial values \( u = 0.55, x_1 = 0.99 \) and \( n = N = 50 \).

<table>
<thead>
<tr>
<th>n</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.990000</td>
</tr>
<tr>
<td>2</td>
<td>0.649212</td>
</tr>
<tr>
<td>3</td>
<td>0.430824</td>
</tr>
<tr>
<td>4</td>
<td>0.287969</td>
</tr>
<tr>
<td>5</td>
<td>0.193551</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>46</td>
<td>0.000650</td>
</tr>
<tr>
<td>47</td>
<td>0.000635</td>
</tr>
<tr>
<td>48</td>
<td>0.000621</td>
</tr>
<tr>
<td>49</td>
<td>0.000607</td>
</tr>
<tr>
<td>50</td>
<td>0.000594</td>
</tr>
</tbody>
</table>

Figure 1: The behavior of the sequences \( \{x_n\} \) with initial values \( u = 0.55, x_1 = 0.99 \) and \( n = N = 50 \).

Acknowledgments

The authors would like to thank the referees for valuable comments and suggestions for improving this work. We also would like to thank Rajamangala University of Technology Thanyaburi and King Mongkut's Institute of Technology Ladkrabang for the financial support.

References


Wongvisarut Khuangsatung  Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathumthani, 12110, Thailand.
E-mail: wongvisarut_k@rmutt.ac.th

Atid Kangtunyakarn  Department of Mathematics, Faculty of Science, King Mongkut’s Institute of Technology Ladkrabang, Bangkok 10520, Thailand.
E-mail: beawrock@hotmail.com