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A Method for Solving the Variational Inequality Problem and Fixed Point Problems in Banach Spaces

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Abstract. The purpose of this research is to modify Halpern iteration's process for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

1 Introduction

For the last decades, fixed point theory is a very importance tool for solving the problems in economic, computer science, physics, etc. Throughout this paper, let E be a Banach space with dual space of E^* and let C be a nonempty closed convex subset of E. We use the norm of E and E^* by the same symbol $\|\cdot\|$. We denote weak and strong convergence by notations " $\rightharpoonup$ " and " $\to$ ", respectively. Let q be a given real number with $q > 1$. The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\},$$

for all $x \in E$. If $q = 2$, then $J_2 = J$ is called *normalized duality mapping.*

Remark 1. If J_q is generalized duality mapping of E into 2^{E^*}. Then the following properties are holds:

1. $J_q(tx) = t^{q-1}J_q(x)$, for all $x \in E$ and $t \in [0, \infty)$;

2. $J_q(-x) = -J_q(x)$, for all $x \in E$.

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Definition 1. Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. Then

1. T is called a nonexpansive mapping if
$$\|Tx - Ty\| \leq \|x - y\|,$$
for all $x, y \in C$.

2. T is called an $\eta-$strictly pseudo-contractive mapping if there exists a constant $\eta \in (0,1)$ such that
$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^2 - \eta\|(I-T)x - (I-T)y\|^2, \tag{1.1}$$
for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$. It is clear that (1.1) is equivalent to the following
$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq \eta\|(I-T)x - (I-T)y\|^2, \tag{1.2}$$
for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$.

Definition 2. Let $C \subseteq E$ be closed convex and Q_C be a mapping of E onto C. The mapping Q_C is said to be sunny if $Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx$, for all $x \in E$ and $t \geq 0$. A mapping Q_C is called retraction if $Q_C^2 = Q_C$. A subset C of E is called a sunny nonexpansive retraction of E if there exists a sunny nonexpansive retraction from E onto C.

For more information about (sunny) nonexpansive retraction can be found in [13].

The modulas of smootheness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by
$$\rho_E(\tau) = \{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}. \tag{1.3}$$
A Banach space E is uniformly smooth if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is well known that E is q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. In a Hilbert space, $L_p(l_p)$ with $1 < p < \infty$ are q-uniformly smooth. Clearly every q-uniformly smooth Banach space is uniformly smooth. If E is smooth, then J_q is a single valued which is denoted by j_q.

An operator A of C into E is said to be *accretive* if there exists $j_q(x-y) \in J_q(x-y)$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \ \forall x, y \in C.$$
A mapping $A : C \to E$ is said to be *α-inverse strongly accretive* if there exists $j_q(x-y) \in J_q(x-y)$ and $\alpha > 0$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.4}$$

Remark 2. From (1.2) and (1.4), if T is an η-strictly pseudo-contractive mapping, then $I-T$ is η-inverse strongly accretive.

Let C be a nonemty subset of q-uniformly smooth Banach space E and $A : C \to E$ be a nonlinear operator. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, J_q(y - x^*)\rangle \geq 0, \;\; \forall y \in C, \tag{1.5}$$

where J_q is generalized duality mapping from E into 2^{E^*}. The set of solutions of the variational inequality in Banach space is denoted by $S_q(C, A)$. If $q = 2$, then $S_q(C, A)$ is reduced to $S(C, A)$, where $S(C, A)$ is the set of solutions of the generalized variational inequality in Banach spaces proposed by Aoyama et. al,. [1] in 2005. Many research papers have increasingly investigated variational inequality problems in Banach spaces, see, for instance, [2], [3], and the references therein.

In 1967, Halpern [4] introduced the *Halpern's iterative method* as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \forall n \geq 1,$$

where $\alpha_n \in (0, 1)$ satisfying suitable conditions, for all $n \geq 1$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of mapping T in a real Hilbert space, where T is a nonexpansive mapping. In the last decade, many authors have studied and modified Halpern's iterative method for various nonlinear mappings, see, for instance, [5], [6], [7], [8] and the references therein.

In a uniformly convex and 2-smooth Banach space, Aoyama *et al.* [1] introduced the iterative method for finding a solution of generalized variational inequality problem for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$, Q_C is a sunny nonexpansive retraction from E onto C, A is an α-inverse strongly accretive operator. Under suitable conditions, They also proved that the sequence generated by the proposed algorithm weakly converges to a solution of $S(C, A)$.

In 2013, Kangtunyakarn [9] introduced an iterative scheme for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems in a uniformly convex and 2-smooth Banach space as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \forall n \geq 1,$$

where A, B are α and β-inverse strongly accretive mappings, respectively, Q_C is a sunny nonexpansive retraction, S^A is the S^A-mapping generated by a finite family of nonexpansive mappings and a finite family of strictly pseudo-contractive mappings and finite real numbers. He also proved a strong convergence theorem of sequence $\{x_n\}$ under suitable conditions of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$, and $\{\eta_n\}$.

Motivated by the results of Aoyama *et al.* [1], Kangtunyakarn [9] and by the ongoing research in this direction, we have the following question.

Question Can we prove a strong convergence theorem of two nonlinear mapping in q-uniformly smooth Banach space ?

The purpose of this manuscript is to modify Halpern iteration's process in order to answer the question above and prove a strong convergence theorem for finding a common element of the set of solutions of (1.5) and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

2 Preliminaries

The following lemmas are important tool to prove our main results in the next section.

Lemma 2.1. *Let E be a Banach space and let $J_q : E \to 2^{E^*}$, $1 < q < \infty$ be the generalized duality mapping. Then for any $x, y \in E$, there exists $j_q(x+y) \in J_q(x+y)$ such that $\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle$.*

Lemma 2.2. *[10] Let C be a closed and convex subset of a real uniformly smooth Banach space E and $T : C \to C$ a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansie retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty} \langle u - Q_{F(T)}u, J_q(x_n - Q_{F(T)}u)\rangle \leq 0,$$

for any given $u \in C$.

Lemma 2.3. *[11] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *[12] Let $q > 1$ be a given real number and E be a real Banach space. Then the following statements are equivalent.*

(i) *E is q-uniformly smooth.*

(ii) *There is a constant $C_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q.$$

(iii) *There exists a constant d_q such that for all $x, y \in E$ and $t \in [0, 1]$,*

$$\|(1-t)x + ty\|^q \geq (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x - y\|^q,$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Lemma 2.5. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a λ-strictly pseudo contractive mapping with $F(T)\bigcap F(S) \neq \emptyset$. For every $a \in (0, 1)$, defined the mapping $H : C \to C$ by $Hx = T((1-a)I + aS)x$, for all $x \in C$ and $a \in (0, \mu)$ where $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, C_q is the best q-uniformly smooth constant of E. Then $F(H) = F(T)\bigcap F(S)$.*

Proof. It is obvious that $F(T)\bigcap F(S) \subseteq F(H)$. Let $x_0 \in F(H)$ and $x^* \in F(T)\bigcap F(S)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^q =& \|T((1-a)I + aS)x_0 - x^*\|^q \\
\leq& \|x_0 - x^* + a(Sx_0 - x_0)\|^q \\
\leq& \|x_0 - x^*\|^q + aq\langle Sx_0 - x_0, j_q(x_0 - x^*)\rangle + C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q + aq\langle Sx_0 - x^*, j_q(x_0 - x^*)\rangle + aq\langle x^* - x_0, j_q(x_0 - x^*)\rangle \\
&+ C_q a^q\|Sx_0 - x_0\|^q \\
\leq& \|x_0 - x^*\|^q + aq(\|x_0 - x^*\|^q - \lambda\|x_0 - Sx_0\|^q) - aq\|x^* - x_0\|^q \\
&+ C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q - a(q\lambda - C_q a^{q-1})\|x_0 - Sx_0\|^q.
\end{aligned} \tag{2.1}$$

From above it implies that $x_0 \in F(S)$. From the definition of H, we have

$$x_0 = Hx_0 = T((1-a)I + aS)x_0 = Tx_0.$$

Then $x_0 \in F(T)$. We can conclude that $x_0 \in F(S)\bigcap F(T)$. Hence $F(H) \subseteq F(S)\bigcap F(T)$. Applying (2.1), we have H is a nonexpansive mapping. $\square$

Example 1. Let $S : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Sx = \frac{x^2}{x+1}$, for all $x \in \mathbb{R}^+$ and let $T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Tx = \frac{3x}{4}$, for all $x \in [0, 5]$. Define the mapping $H : \mathbb{R}^+ \to \mathbb{R}^+$ by $Hx = T(\frac{9}{10}I + \frac{1}{10}S)x$ for all $x \in \mathbb{R}^+$. From Lemma 2.5, we have $F(H) = F(S) \bigcap F(T) = \{0\}$

Lemma 2.6. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $j, j_q : E \to E^*$ be a normalized duality mapping and generalized duality mapping, respectively. Let Q_C be a retraction from E onto C. Then the following are equivalent.*

(i) *Q_C is both sunny and nonexpansive,*

(ii) $\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$, *for all $x \in E$ and $y \in C$,*

(iii) $\langle x - Q_C x, J_q(y - Q_C x)\rangle \leq 0$, *for all $x \in E$ and $y \in C$.*

Proof. From [13], we have $(i) \Leftrightarrow (ii)$. Then we only show that (ii) equivalent to (iii). Since $J_q(x) = \|x\|^{q-1} J(x)$, for all $x \in E$. For every $x \in E$ and $y \in C$.
If $y - Q_C x \neq 0$, we have

$$\langle x - Q_C x, J_q(y - Q_C x)\rangle \leq 0 \Leftrightarrow \langle x - Q_C x, J(y - Q_C x)\rangle \leq 0.$$

If $y - Q_C x = 0$, we have

$$\langle x - Q_C x, J_q(y - Q_C x)\rangle = \langle x - Q_C x, J(y - Q_C x)\rangle = 0.$$

From above we can conclude the desire result. □

Remark 3. Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $x \in E$, $x_0 \in C$. From Lemma 2.6, we have

$$x_0 = Q_C x \Leftrightarrow \langle x - x_0, J_q(y - x_0)\rangle \leq 0, \forall y \in C.$$

Lemma 2.7. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $A : C \to E$ be a mapping. Then $S_q(C, A) = F(Q_C(I - \lambda A))$, for all $\lambda > 0$, where $S_q(C, A) = \{u \in C : \langle Au, J_q(y - u)\rangle \geq 0, \forall y \in C\}$.*

Proof. Let $x^* \in F(Q_C(I - \lambda A))$, for all $\lambda > 0$. Then $x^* = Q_C(I - \lambda A)x^*$. From 2.6, we have

$$\langle (I - \lambda A)x^* - x^*, J_q(y - x^*)\rangle \leq 0, \forall y \in C.$$

It follows that

$$\langle Ax^*, J_q(y - x^*)\rangle \geq 0, \forall y \in C.$$

Then $x^* \in S_q(C, A)$. Hence $F(Q_C(I - \lambda A)) \subseteq S_q(C, A)$. Similarly, we can conclude that $S_q(C, A) \subseteq F(Q_C(I - \lambda A))$.

□

3 Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S : C \to C$ be λ-strictly pseudo contractive mapping and $A : C \to E$ be a α-inverse strongly accretive operator with $\mathcal{F} = F(S) \bigcap S_q(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q_C (I - \rho A)(aI + (1 - a)S)x_n, \forall n \in \mathbb{N}, \tag{3.1}$$

where $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $a \in (0, \mu)$*, where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$*, where C_q is the q-uniformly smooth constant of E;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$*;*

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is a unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we show that $Q_C(I - \rho A)$ is a nonexpansive mapping. Let $x, y \in C$, we have

$$\begin{aligned}
\|Q_C(I - \rho A)x - Q_C(I - \rho A)y\|^q \leq& \|x - y - \rho(Ax - Ay)\|^q \\
\leq& \|x - y\|^q - \rho q \langle Ax - Ay, j_q(x - y)\rangle + C_q \rho^q \|Ax - Ay\|^q \\
\leq& \|x - y\|^q - \rho q \alpha \|Ax - Ay\|^q + C_q \rho^q \|Ax - Ay\|^q \\
\leq& \|x - y\|^q - \rho(q\alpha - C_q \rho^{q-1})\|Ax - Ay\|^q \\
\leq& \|x - y\|^q.
\end{aligned}$$

Then $Q_C(I - \rho A)$ is a nonexpansive mapping. Next we show that the sequence $\{x_n\}$ is bounded. Put $Wx = Q_C(I - \rho A)(aI + (1 - a)S)x$, for all $x \in C$. From Lemma 2.5 and 2.7, we have

$$F(W) = F(Q_C(I - \rho A)) \bigcap F(S) = S_q(C, A) \bigcap F(S)$$

and W is a nonexpansive mapping. From (3.1), we can rewrite that

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) W x_n, \forall n \in \mathbb{N}. \tag{3.2}$$

Let $x^* \in \mathcal{F}$ and the definition of x_n, we have

$$\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|W x_n - x^*\|$$

$$\begin{aligned}&\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\&\leq \max\{\|u-x^*\|,\|x_1-x^*\|\}.\end{aligned}$$

Applying induction, we have $\{x_n\}$ is bounded. From the definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leq&|\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|Wx_n-Wx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Wx_{n-1}\|\\ \leq&|\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Wx_{n-1}\|.\end{aligned}$$

Since $\{x_n\}$ is bounded sequnce, the condition (iv) and Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

From (3.2), we have

$$x_{n+1}-x_n=\alpha_n(u-x_n)+(1-\alpha_n)(Wx_n-x_n). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\lim_{n\to\infty}\|Wx_n-x_n\|=0. \tag{3.5}$$

From Lemma 2.2 and (3.5), we have

$$\limsup_{n\to\infty}\langle u-z_0,j_q(x_n-z_0)\rangle\leq 0, \tag{3.6}$$

where $z_0=Q_{\mathcal{F}}u$. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0=Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-z_0\|^q \leq&\|\alpha_n(u-x^*)+(1-\alpha_n)(Wx_n-z_0)\|^q\\ \leq&(1-\alpha_n)\|x_n-z_0\|^q+q\alpha_n\langle u-z_0,j_q(x_{n+1}-z_0)\rangle.\end{aligned}$$

From Lemma 2.3 and (3.6), we have the sequence $\{x_n\}$ converges strongly to $z_0=Q_{\mathcal{F}}u$. □

By using the method of proof in Theorem 3.1, we have the following theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S:C\to C$ be be λ-strictly pseudo contractive mapping and $T:C\to E$ be a nonexpansive mapping with $\mathcal{F}=F(S)\bigcap F(T)\neq\emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1,u\in C$ and*

$$x_{n+1}=\alpha_nu+(1-\alpha_n)T(aI+(1-a)S)x_n,\forall n\in\mathbb{N}, \tag{3.7}$$

where $\alpha_n\in[0,1]$, $a\in(0,1)$ and $\rho>0$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = Q_{\mathcal{F}}u$*, where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$*.*

Proof. Applying the method of Theorem 3.1 and Lemma 2.5, we can conclude the desired result. □

4 Application

In this section, we use the main results to obtain fixed points theorems for a finite family of strictly pseuso contractive mappings in q-uniformly smooth Banach space. Before prove this theorems, we need the following results.

Lemma 4.1. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*, then* $\sum_{i=1}^N \eta_i T_i : C \to C$ *is a* λ_i*-strict pseudocontraction with* $\lambda = min\{\lambda_i : 1 \leq i \leq N\}$*.*

Lemma 4.2. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $\{T_i\}_{i=1}^N : C \to C$ *is a finite family of* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *such that* $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*. Then* $F(\sum_{i=1}^N \eta_i T_i) = F$

Theorem 4.1. *Let* C *be a nonempty closed convex subset of* q*-uniformly smooth Banach space* E *and let* Q_C *be a sunny nonexpansive retraction from* E *onto* C*. Let* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *and* $A : C \to E$ *be a* α*-inverse strongly accretive operator with* $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \bigcap S_q(C, A) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q_C (I - \rho A)(aI + (1 - a) \sum_{i=1}^{N} \eta_i T_i) x_n, \forall n \in \mathbb{N}, \tag{4.1}$$

where $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*,* $\alpha_n \in [0, 1]$*,* $a \in (0, 1)$ *and* $\rho > 0$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. From Theorem 3.1, Lemma 4.1 and 4.2, we can conclude the desired result. $\square$

Lemma 4.3. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $S : C \to C$ be κ-strictly pseudo contractive mapping with $F(S) \neq \emptyset$. Then $F(S) = S_q(C, I - S)$.*

Proof. Obvious that $F(S) \subseteq S_q(C, I - S)$. Let $x_0 \in S_q(C, I - S)$ and $x^* \in F(S)$. Then

$$\langle (I - S)x_0, j_q(y - x_0)\rangle \geq 0, \forall y \in C.$$

Put $A = I - S$. Since $S : C \to C$ is κ-strictly pseudo contractive mapping, then there exists $j_q(x_0 - x^*)$ such that

$$\begin{aligned}
\langle Sx_0 - Sx^*, j_q(x_0 - x^*)\rangle &= \langle (I - A)x_0 - (I - A)x^*, j_q(x_0 - x^*)\rangle \\
&= \langle x_0 - x^*, j_q(x_0 - x^*)\rangle - \langle Ax_0 - Ax^*, j_q(x_0 - x^*)\rangle \\
&= \|x_0 - x^*\|^q - \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \\
&\leq \|x_0 - x^*\|^q - \kappa\|(I - S)x_0\|^q.
\end{aligned}$$

It implies that

$$\kappa\|(I - S)x_0\|^q \leq \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \leq 0.$$

Then $x_0 \in F(S)$. Hence $S_q(C, I - S) \subseteq F(S)$. $\square$

Corollary 4.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \to C$ is a λ_i-strictly pseudo contractive mapping for some $0 \leq \lambda_i < 1$ and $S : C \to E$ be a α-strictly pseudo contractive mapping with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \bigcap F(S) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho(I - S))(aI + (1 - a)\sum_{i=1}^{N} \eta_i T_i)x_n, \forall n \in \mathbb{N}, \tag{4.2}$$

where $\{\eta_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$, $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $a \in (0,\mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E;

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$.

Then $\{x_n\}$ *converses strongly to* $z_0 = Q_{\mathcal{F}}u$, *where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. From Theorem 4.1 and Lemma 4.3, we can conclude the desired result. □

5 Example and Numerical results

In this section, we give numerical results to illustrate the main theorem.

Example 2. Let $\mathbb{R}$ be a set of real number. Let $S : [0,10] \to [0,1]$ be a mapping defined by $Sx = \frac{2x^2}{x+2x}$, for all $x \in [0,1]$ and let $A : [0,10] \to \mathbb{R}$ defined by $Ax = 3x^2$ for all $x \in [0,10]$. Suppose the sequence $\{x_n\}$ generated by (3.1), where $\alpha_n = \frac{1}{60n}$, $\rho = \frac{1}{100}$, and $a = \frac{1}{80}$. Then the sequence $\{x_n\}$ converses strongly to 0.
Solution. It is obvious that S is $\frac{1}{50}$-strictly pseudo contractive mapping and A is $\frac{1}{60}$-inverse strongly accretive operator with $F(S)\bigcap S_2(C,A) = \{0\}$. Since $\{x_n\}$ generated by (3.1), we have

$$x_{n+1} = \frac{1}{60n}u + \left(1 - \frac{1}{60n}\right) Q_{[0,10]} \left(I - \frac{1}{100}A\right)\left(\frac{1}{80}I + (1-\frac{1}{80})S\right)x_n, \tag{5.1}$$

where $u, x_1 \in [0,10]$. It is easy to see that α_n, for all $n \geq 1$, a, ρ satisfied all condition in Theorem 3.1. From Theorem 3.1, we have the sequence $\{x_n\}$ coonvergence strongly to 0.
Putting $u = 0.55$ and $x_1 = 0.99$ in (5.1), we have the numerical results as shown in the following Figure 1 and Table 1.

n	x_n
1	0.990000
2	0.649212
3	0.430824
4	0.287969
5	0.193551
$\vdots$	$\vdots$
46	0.000650
47	0.000635
48	0.000621
49	0.000607
50	0.000594

Table 1: The values of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

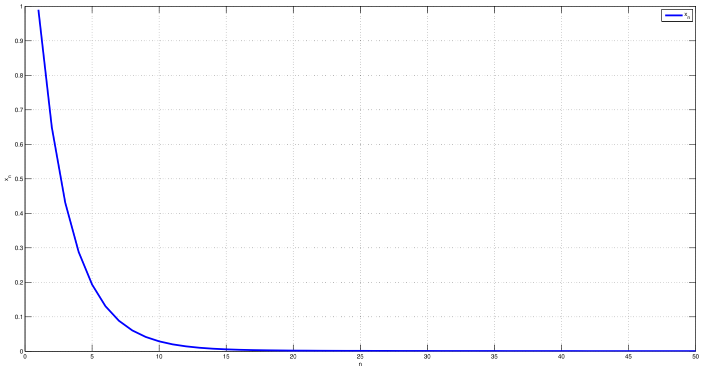

Figure 1: The behavior of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

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