TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 3, Autumn 2002

DEGREE OF APPROXIMATION OF CONJUGATE OF $LIP(\alpha, p)$ FUNCTION BY $(C,1)$ $(E,1)$ MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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Abstract: An estimate of degree of approximation of conjugates of Lip(α, p) functions by $(C,1)$ $(E,1)$ product means of conjugate series of a Fourier Series is obtained.

1. Definition and Notations

$$1f \quad E_n^1 = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}S_k \to S, \qquad as\ n\to\infty$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ with the partial sums S_n is said to be summable $(E,1)$ to the definite number s. The $(C,1)$ transform of the $(E,1)$ transform E_n^1 defines the $(C,1)(E,1)$ transform of the partial sum S_n of the series $\sum_{n=0}^{\infty} u_n$,thus if

$$(CE)_n^1 = \frac{1}{(n+1)}\sum_{k=0}^{n} E_k^1 \to S, \qquad \text{as } n\to\infty$$

Where E_n^1 denotes the $(E,1)$ transform of S_n, then the Series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(C,1)(E,1)$ means or simply summable $(C,1)(E,1)$ to s.

We define $\|\ \|_p$ by

$$\| f \|_p = \left(\int_0^{2\pi}[f(x)]^p dx\right)^{1/p}, \qquad p\geq 1$$

and let the degree of approximation $M_n(f)$ be given by

$M_n(f) = Min \| f - T_n \|_p$, where T_n is trigonametic polynomial of degree n.

Received July 23, 2000.
2000 *Mathematics Subject Classification.* Primary 42B05, 42B08.
Key words and phrases. Degree of approximation, Lipschitz function, Lip (α, p) Function, Fourier series, conjugate series of a Fourier series (C,1) (E,1) means.

Let f: $R \to R$ be 2π periodic and Lip α, $0 < \alpha \leq 1$

so that $|f(x+t) - f(x)| = O(|t|^{\alpha})$ for all x, t,

$f \in Lip(\alpha, p)$, for $a \leq x \leq b$, if

$$\left[\int_a^b |f(x+t) - f(x)|^p dx\right]^{1/p} = O(t^{\alpha}), 0 < \alpha \leq 1, p \geq 1 \quad (1.1)$$

definition 5.38 of Mc Fadden(1942)

If $p \to \infty$ then Lip(α, p) coincides with Lip α class.

The function f has its Fourier series as following.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \ \cos n \ x + b_n \sin n \ x)$$

The conjugate series of above Fourier series is given by:

$$\sum_{n=1}^{\infty}(a_n \sin n \ x - b_n \cos n \ x) \quad (1.2)$$

where a_n b_n are Fourier coefficients of f over $[-\pi, \pi]$.
Writing,

$\psi_x(t) = f(x+t) - f(x-t)$ for all x, t.

f has also its conjugate function $\bar{f}$[8] given by

$$\bar{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi_x(t) \cot(\frac{t}{2})dt$$

2. Main Theorem

For the function $f \in$ Lip α and Lip(α, p) the degree of approximation by Cesaro means and by Nörlund means of Fourier series of f have been studied by Alexits [1], (Sahney) and Goel [9], Chandra [2], Qureshi [4,5], Qureshi and Neha [6] and many others. But till now no work seems to have been done in the direction of determining the degree of approximation of conjugate of a function belonging to $Lip(\alpha, p)$ class by product summability means of the form $(C, 1)$ $(E, 1)$. In an attempt to make advance study in this direcation, in this paper the degree of approximation of conjugate function by $(C, 1)$ $(E, 1)$ means of the conjugate series of a Fourier series of $f \in Lip(\alpha, p)$ class has been determined in the following form:

Theorem. *If $f:R\to R$ is 2π periodic and $Lip(\alpha,p)$ function then the degree of approximation of its conjugate function $\bar{f}$ by the $(C,1)(E,1)$ product means of conjugate series of Fourier series of f satisfies, for $n=0,1,2,\ldots$,*

$$M_n(\bar{f})=\text{Min}\,\|(CE)_n^1-\bar{f}\|_P=O\left(\frac{1}{(n+1)^{\alpha-1/p}}\right)$$

where $(CE)_n^1=\frac{1}{n+k}\sum_{k=0}^{n}(\frac{1}{2^k}\sum_{i=0}^{\infty}\binom{k}{i}S_i)$ *is* $(C,1)$ $(E,1)$ *means of series* (1.2)

3. Proof of the Theorem

Following Lal(1997) the partial sums of conjugate series (1.2) can be written as

$$S_n(x)=\bar{f}(x)+\frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt\qquad(n=0,1,2,\ldots)$$

So the $(E,1)$ means (see[3], of the series (1.2) are

$$\begin{aligned}E_n^1(x)&=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k(x)\qquad(n=0,1,2)\\&=\bar{f}(x)+\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}\Big\{\sum_{k=0}^{n}\binom{n}{k}\cos(k+\frac{1}{2})t\Big\}dt\\&=\bar{f}(x)+\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}Re\{e^{it/2}(1+e^{it})^n dt\\&=\bar{f}(x)+\frac{1}{2\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}\cos^n(\frac{t}{2})\cos(n+1)\frac{t}{2}dt\end{aligned}$$

Then the $(C,1)(E,1)$ product means of the series (1.2) are,

$$\begin{aligned}(CE)_n^1(x)&=\frac{1}{n+1}\sum_{k=0}^{n}E_k^1(x)\qquad(n=0,1,2\ldots)\\&=\bar{f}(x)+\frac{1}{2\pi(n+1)}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}(\sum_{k=0}^{n}\cos^k(\frac{t}{2})\cos(k+1)\frac{t}{2})dt\\&=\bar{f}(x)+\frac{1}{2\pi(n+1)}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}Re(\sum_{k=0}^{n}\cos^k(\frac{t}{2})e^{-i(k+1)t/2})dt\\&=\bar{f}(x)+\frac{1}{2\pi(n+1)}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}Re[\frac{e^{it/2}\{1-\cos^{n+1}(t/2)\}e^{i(n+1)t/2}}{1-\cos(t/2)e^{it/2}}]dt\\&=\bar{f}(x)+\frac{1}{2\pi(n+1)}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}\frac{\cos^{n+1}(t/2)\sin(n+1)t/2}{\sin(t/2)}dt\end{aligned}$$

Since here $\sin(t/2) \geq t/\pi$ and $|\sin\theta| \leq \theta$ it follows that

$$
\begin{aligned}
|(CE)^1_n(x) - \bar{f}(x)| &\leq \frac{1}{2\pi(n+1)}\int_0^{\frac{1}{n+1}} \frac{\psi_x(t)}{\sin(t/2)}\frac{\cos^{n+1}(t/2)\sin(n+1)t/2}{\sin(t/2)}dt \\
&+ \frac{1}{2\pi(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{\psi_x(t)}{\sin(t/2)}\frac{\cos^{n+1}(t/2)\sin(n+1)t/2}{\sin(t/2)}dt \\
&= I_1 + I_2 \\
I_1 &= \frac{1}{2\pi(n+1)}\int_0^{\frac{1}{n+1}} \frac{\psi_x(t)}{\sin(t/2)}\frac{\cos^{n+1}(t/2)\sin\,(n+1)t/2}{\sin(t/2)}dt \\
&\leq \frac{1}{2\pi(n+1)}\int_0^{\frac{1}{n+1}} \frac{\psi_x(t)}{|\sin(t/2)|}\frac{|\cos^{n+1}(t/2)|\,|\sin(n+1)(t/2)|}{|sin(t/2)|}dt \\
&= \frac{1}{2\pi(n+1)}\int_0^{\frac{1}{n+1}} \frac{\psi_x(t)}{t}dt \qquad \therefore |\sin\theta| \leq \theta \text{ and } |\cos\theta| \leq 1 \\
&= \frac{1}{2\pi(n+1)}\Big[\int_0^{\frac{1}{n+1}} \Big\{\frac{|t\psi_x(t)|}{t^\alpha}\Big\}^p dt\Big]^{1/p}\Big[\int_0^{\frac{1}{n+1}} |t^{\alpha-2}|^q dt\Big]^{\frac{1}{q}} \\
&= \frac{1}{2\pi}(\frac{1}{n+1})\Big[\Big\{\frac{t^{\alpha q-2q+1}}{\alpha q-2q+1}\Big\}_0^{\frac{1}{n+1}}\Big]^{\frac{1}{q}} \\
&= O\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{(n+1)^{\alpha q-2q+1}}\Big)^{\frac{1}{q}} \\
&= O\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{(n+1)^{\alpha-2+1/q}}\Big) \\
&= O\Big(\frac{1}{(n+1)^{\alpha-1+1/q}}\Big) \\
&= O\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big) \qquad (\therefore \frac{1}{p}+\frac{1}{q}=1)
\end{aligned}
$$

And

$$
\begin{aligned}
I_2 &\leq \frac{1}{2\pi(n+1)}\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\psi_x(t)}{t^2}\Big)dt \\
&= \frac{1}{2\pi(n+1)}\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\psi_x(t)}{t^2}\Big)dt \\
&= \frac{1}{2\pi(n+1)}\Big[\int_{\frac{1}{n+1}}^{\pi}\Big\{\frac{t^{-\delta}\psi_x(t)}{t^\alpha}\Big\}^{1/p}\Big]^{1/p}\Big[\int_{\frac{1}{n+1}}^{\pi}(\frac{t^{-\delta+\alpha}}{t^2})^q dt\Big]^{1/q} \\
&= \frac{1}{(n+1)}\frac{1}{(n+1)^{-\delta}}\Big[\int_{\frac{1}{n+1}}^{\pi} t^{(\delta+\alpha-2)q}dt\Big]^{\frac{1}{q}} \\
&= \frac{1}{(n+1)^{-\delta+1}}\Big[\Big(\frac{t^{(\delta+\alpha-2)q+1}}{(\delta+\alpha-2)q+1}\Big)^{\pi}_{\frac{1}{n+1}}\Big]^{1/q} \\
&= O\Big(\frac{1}{(n+1)^{-\delta+1}}\Big(\frac{1}{(n+1)^{(\delta+\alpha-2)q+1}}\Big)^{1/q}\Big)
\end{aligned}
$$

$$= O\Big(\frac{1}{(n+1)^{-\delta+1}}\Big(\frac{1}{(n+1)^{\delta+\alpha-2+1/q}}\Big)\Big)$$

$$= O\Big(\frac{1}{(n+1)^{\alpha-1+1/q}}\Big)$$

$$I_2 = O\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big) \qquad \Big(\therefore \frac{1}{p}+\frac{1}{q}=1\Big)$$

By(1) and (2)

$$|(CE)_n^1 - \bar{f}(x)| = O\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big)$$

$$\text{or} \quad ||(CE)_n^1 - \bar{f}||_p = O\Big[\Big\{\int_0^{2\pi}\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big)^p dx\Big\}\Big]^{\frac{1}{p}}$$

$$= O\Big[\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big)\Big(\int_0^{2\pi} dx\Big)\Big]^{\frac{1}{p}}$$

$$= O\Big(\frac{1}{(n+1)^{\alpha-1/p}}\Big)$$

This completes the proof of theorem.

4. Corollary

Following corollary can be derived from our theorem

Corollary. *If $p \to \infty$, $0 < \alpha < 1$ then the degree of approximation of the function $\bar{f}$, conjugate to 2π periodic function f belonging to Lip α is given by*

$$|(CE)_n^1 - \bar{f}(x)| = O\Big(\frac{1}{(n+1)^{\alpha}}\Big).$$

Remark. An independent proof of corollary can be derived along the same lines as the theorem.

Acknowledgement

The authors are grateful to professor L. M. Tripathi for suggesting this problem and to Prof. T. Pati Ex. Vice Chancellor, University of Allahabad, Allahabad-211002(INDIA) Who has taken the pain to see the manuscript of this paper. Shyam Lal. one of the authors, is thankful to the University Grant Commission New Delhi for providing financial assistance in the form of a minor research project letter No.3.3/58/199-2000/MRP/NR dated 31.3.2000.

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