SOME CLASSES OF $L^p(0 < p < 1)$ CONVERGENCE OF TRIGONOMETRIC SERIES

ŽIVORAD TOMOVSKI

Abstract. We study here $L^p(0 < p < 1)$-convergence of complex trigonometric series, i.e., the extension is made for the V. B. Stanojevic Theorem [3], by considering the class $(BV)^m_1$, $m = 1, 2, 3, \ldots$, instead of $(BV)^m$, $m = 1, 2, 3, \ldots$. Applying the Wang-Telyakovski class $(BV)^r_1$, $r = 0, 1, 2, \ldots, m \geq 1$ (see [11]), the extension of Uljanov’s theorem [10] (case $r = 0, m = 1$) for sine and cosine series with real coefficients is also given. For $m = 1$, some corollaries of the main results are obtained.

1. Introduction

Let $\{c_k : k = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of complex numbers and the partial sums of the complex trigonometric series

$$
\sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{1.1}
$$

be denoted by $S_n^*(t) = \sum_{k=-n}^{n} c_k e^{ikt}$, $t \in T = \mathbb{R}/2\pi\mathbb{Z}$ (1.2).

A sequence $\{c_k\}$ belongs to the class $(BV)^m$ (see [3]) if for some integer $m \geq 1$,

$$
\sum_{|k|<\infty} |\Delta^m c_k| < \infty,
$$

where $\Delta^m c_k = \Delta(\Delta^{m-1} c_k) = \Delta^{m-1} c_k - \Delta^{m-1} c_{k+1}$.

For $m = 1$, the class $(BV)^1$ is the class of complex sequences with bounded variation.

As an extension of the Uljanov’s Theorem (see [10]), V. B. Stanojevic have obtained the following theorem for $L^p$, $0 < p < 1$-convergence of the series (1.1).

Theorem A.[3] If for some integer $m \geq 1$, $\{c_n\} \in (BV)^m$, then the point-wise limit $f^*$ of the partial sums (1.2) exists in $T \setminus \{0\}$ and for any $0 < p < \frac{1}{m}$,

$$
\lim_{n \to \infty} \int_{-\pi}^{\pi} |S_n^*(t) - f^*(t)|^p dt = 0
$$

Received July 25, 2001;
2000 Mathematics Subject Classification. 26D15, 42A20, 42A32.
Key words and phrases. $L^p(0 < p < 1)$-approximation, complex trigonometric series, Wang-Telyakovski class, Uljanov’s theorem.
On the other hand, Wang and Telyakovskii [11] have considered the following class of real sequences \( \{a_n\} \). Namely, a null-sequence \( \{a_k\} \) belongs to the class \( (BV)_r^\sigma \), \( r = 0, 1, 2, \ldots, \sigma \geq 0 \) if \( \sum_{k=1}^\infty k^r | \Delta^\sigma a_k | < \infty \). If \( \sigma = 1 \), we denote \( (BV)_r = (BV)_r^1 \) and if \( r = 0, \sigma = 1 \), then we denote \( (BV)_r^1 = (BV) \).

**Theorem B.** ([11]) Let \( \rho \geq 0, \sigma \geq 0 \). Then for all \( \gamma > \sigma \) the following embedding relation holds,

\[
(BV)_\rho^\sigma \subset (BV)_\rho^\gamma.
\]

Let

\[
a_0 + \sum_{n=1}^\infty a_n \cos nx \quad (C)
\]

\[
\sum_{n=1}^\infty a_n \sin nx \quad (S)
\]

be the cosine and sine trigonometric series, and \( S_n, \overline{S}_n \) denote the partial sums of the series (C) and (S) respectively.


\[
a_0 + \sum_{n=1}^\infty a_n e^{inx}, \quad x \in (0, \pi]
\]

have proved the following theorem.

**Theorem C.** ([11]) If \( \{a_k\} \in (BV)_r^\sigma \), \( r = 0, 1, 2, \ldots, \sigma \geq 0 \), then the series (C) and (S) have continuous derivatives of \( r \)-th order on \( (0, \pi] \).

The Wang-Telyakovskii class \( (BV)_r^\sigma \), \( r = 0, 1, 2, \ldots, \sigma \geq 0 \), motivated me to consider a further class \( (BV)_r^m \), \( r = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots \), of complex null-sequences \( \{c_n\} \) such that

\[
\sum_{|k|<\infty} k^r | \Delta^m c_k | < \infty
\]

In this paper we shall extend the Theorem A, by considering the class \( (BV)_r^m \), \( r = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots \) instead of \( (BV)^m \). In addition we obtain the extension of corresponding Ul'janov’s theorem [10], by considering the Wang-Telyakovskii class \( (BV)_r^\sigma \), \( r = 0, 1, 2, \ldots, \sigma \geq 1 \) instead of \( BV \).

2. Main results

For the proof of our new results, we need the following Lemma.
Lemma 1. If \( m = 1, 2, 3, \ldots, r = 0, 1, 2, 3, \ldots \) and \( 0 < p < \frac{1}{m+r} \), then the following estimate holds
\[
\int_{-\pi}^{\pi} \left| \frac{d^r}{dt^r} \left( e^{it} - 1 \right)^m \right|^p dt = O_{r,p,m}(1),
\]
where \( O_{r,p,m} \) depends on \( r, p \) and \( m \).

Proof. We denote \( h(t) = (\frac{e^{it}}{e^{it} - 1})^m \). After some elementary calculations, this function can be written in the form
\[
h(t) = \frac{e^{imt}}{(2i)^m \sin^m \frac{t}{2}}.
\]
Then,
\[
h^{(r)}(t) = \frac{1}{(2i)^m} \sum_{k=0}^{r} \binom{r}{k} \left( \sin^{-m} \frac{t}{2} \right)^{(k)} \left( e^{imt} \right)^{(r-k)}.
\]
The equality \( (\sin^{-m} \frac{t}{2})^{(k)} = \frac{P_k(\cos \frac{t}{2})}{\sin^{m+k+1} \frac{t}{2}} \), where \( P_k \) is some cosine polynomial of degree \( k \), can be proved by mathematical induction.

Really, for \( k = 1 \), we obtain
\[
(\sin^{-m} \frac{t}{2})' = \frac{P_1(\cos \frac{t}{2})}{\sin^{m+1} \frac{t}{2}} = \frac{P_1(\cos \frac{t}{2})}{\sin^{m+1} \frac{t}{2}}.
\]
Suppose that the equality is true for some \( k \) and consider the \( k+1 \)-th derive.
\[
(\sin^{-m} \frac{t}{2})^{k+1} = \frac{P_k(\cos \frac{t}{2})'}{\sin^{2m+2k+1} \frac{t}{2}} \sin^{-m+k+1} \frac{t}{2} - \frac{P_k(\cos \frac{t}{2})}{\sin^{m+k+1} \frac{t}{2}} \cos \frac{t}{2}
\]
\[
= \frac{\tilde{P}_{k+1}(\cos \frac{t}{2})}{\sin^{m+k+1} \frac{t}{2}} + \frac{1}{2} \tilde{P}_{k+1}(\cos \frac{t}{2}) \cos^{2} \frac{t}{2} - \frac{m+2}{2} Q_{k+1}(\cos \frac{t}{2})
\]
\[
= \frac{R_{k+1}(\cos \frac{t}{2}) - \frac{m+k}{2} Q_{k+1}(\cos \frac{t}{2})}{\sin^{m+k+1} \frac{t}{2} = T_{k+1}(\cos \frac{t}{2})\sin^{m+k+1} \frac{t}{2}.
\]
Here \( \tilde{P}_{k+1}, R_{k+1}, Q_{k+1}, T_{k+1} \) are some cosine polynomials of degree \( k-1 \) and \( k+1 \) respectively.

Thus
\[
h^{(r)}(t) = \frac{1}{(2i)^m} \sum_{k=0}^{r} \binom{r}{k} \left( \frac{im}{2} \right)^{r-k} \left( e^{imt} \right) \frac{P_k(\cos \frac{t}{2})}{\sin^{m+k+1} \frac{t}{2}}.
\]
We note that \( |P_k(\cos \frac{t}{2})| = O_{k,m}(1) \).

Hence,
\[
|h^{(r)}(t)| \leq \frac{1}{2^m} \sum_{k=0}^{r} O_{k,m}(1) \cdot \left( \frac{m}{2} \right)^{r-k} \frac{1}{|\sin^{m+k+1} \frac{t}{2}|}.
\]
Applying the well-known inequality
\[
\left( \sum \alpha_t \right)^\lambda \leq \left( \sum \alpha_t^\lambda \right), \text{ where } \alpha_t \geq 0, \ 0 < \lambda \leq 1,
\]
(2.1)
we obtain
\[ |h^{(r)}(t)|^p \leq \frac{1}{2^{mp}} \sum_{k=0}^{r} O_{k,m,p}(1) \left( \frac{r}{k} \right)^p \left( \frac{m}{2} \right)^{(r-k)p} \frac{1}{\sin^{(m+k)p} \frac{t}{2}} \]

Finally, for \( 0 < p < \frac{1}{m+r} \), we get
\[ \int_{-\pi}^{\pi} |h^{(r)}(t)|^p \, dt \leq \frac{1}{2^{mp+2}} \sum_{k=0}^{r} O_{k,m,p}(1) \left( \frac{r}{k} \right)^p \left( \frac{m}{2} \right)^{(r-k)p} \int_{0}^{\pi/2} \frac{dt}{\sin^{(m+k)p} \frac{t}{2}} \]
\[ \leq \frac{1}{2^{mp+2}} \sum_{k=0}^{r} O_{k,m,p}(1) \left( \frac{r}{k} \right)^p \left( \frac{m}{2} \right)^{(r-k)p} \pi^{(m+k)p} \int_{0}^{\pi/2} \frac{dt}{t^{(m+k)p}} = O_{r,p,m}(1). \]

**Theorem 1.** Let \( \{c_n\} \in (BV)^m \), for some integer \( m \geq 1 \) and \( r = 0, 1, 2, 3, \ldots \). Then the point-wise limit \( f^{(r)} \) of the \( r \)-th derivate of the sums \( (1.2) \) exists in \( T \setminus \{0\} \) and for any \( 0 < p < \frac{1}{m+r} \),
\[ \lim_{n \to \infty} \int_{-\pi}^{\pi} |S_n^{(r)}(t) - f^{(r)}(t)|^p \, dt = 0. \quad (2.2) \]

**Proof.** We consider the identity, obtained by V. B. Stanojevic in [3].
\[ S_n^{(r)}(t) = \left( \frac{e^{it}}{e^{it} - 1} \right)^m \sum_{|k| \leq n} (\Delta^m c_k) e^{ikt} \]
\[ - \left( \frac{e^{it}}{e^{it} - 1} \right)^m \sum_{j=0}^{m-1} \left( \frac{e^{it} - 1}{e^{it}} \right)^j (\Delta^{m-1-j} c_{-n+j}) e^{i(-n+j)t} \]
\[ + \left( \frac{e^{it}}{e^{it} - 1} \right)^m \sum_{j=0}^{m-1} \left( \frac{e^{it} - 1}{e^{it}} \right)^j (\Delta^{m-1-j} c_{n+j+1}) e^{i(n+j+1)t} \]
\[ + \sum_{j=-n}^{-n+m-1} c_j e^{ijt} - \sum_{j=n+1}^{n+m} c_j e^{ijt}. \]

For the \( r \)-th derivate of the partial sums of \( S_n \), we have:
\[ S_n^{(r)}(t) = \sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left( \frac{e^{it}}{e^{it} - 1} \right)^m \left( 1 - r - v \right)^{m-v} \sum_{|k| \leq n} k^{r-v} (\Delta^m c_k) e^{ikt} \]
\[ - \sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left( \frac{e^{it}}{e^{it} - 1} \right)^m \sum_{j=0}^{m-1} \sum_{q=0}^{j} (-1)^q \binom{j}{q} (j-q-n)^{r-v} r-v \]
\[ e^{it(j-q-n)} (\Delta^{m-1-j} c_{-n+j}) + \sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left( \frac{e^{it}}{e^{it} - 1} \right)^m \sum_{j=0}^{m-1} \sum_{q=0}^{j} (-1)^q \binom{j}{q} (j-q-n)^{r-v} r-v \]
\[ e^{it(j-q-n)} (\Delta^{m-1-j} c_{n+j+1}) + \sum_{j=-n}^{-n+m-1} c_j e^{ijt} - \sum_{j=n+1}^{n+m} c_j e^{ijt}. \]
\[ x(j - q + n + 1) - n + m - 1 \sum_{j=-n}^{j=n+1} c_{j} \mathcal{F} \mathcal{I} e^{ij} + \sum_{j=n+1}^{n+m} c_{j} \mathcal{F} \mathcal{I} e^{ij}. \]

Hence, \( \lim_{n \to \infty} S_{n}^{(r)}(t) = f^{(r)}(t) \) exists in \( T \setminus \{0\} \).

Then, we shall prove that \( S_{n}^{(r)} \) converges to \( f^{(r)} \) in \( L^{p}(T) \)-metric, for any \( 0 < p < \frac{1}{m+1} \).

For \( t \neq 0 \), we consider

\[
S_{n}^{(r)}(t) - f^{(r)}(t) = \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \sum_{|k| \leq n} k^{v} \left( \Delta^{m} c_{k} \right) e^{ikt} - \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \sum_{|k| \leq n} k^{v} \left( \Delta^{m} c_{k} \right) e^{ikt} + \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \sum_{j=0}^{j=n+1} \sum_{q=0}^{q=m} (-1)^{q} (j-q-n)^{r-v} \left( \Delta^{m} c_{n+j} \right) \left( j-q+n+1 \right)^{r-v} \left( \Delta^{m} c_{n+j+1} \right) \]

By inequality (2.1), we obtain

\[
|S_{n}^{(r)}(t) - f^{(r)}(t)|^{p} \leq \left( \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \right)^{p} \left( \sum_{|k| \leq n+1} k^{v} \left( \Delta^{m} c_{k} \right) \right)^{p} + \left( \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \right)^{p} \left( \sum_{j=0}^{j=n+1} \sum_{q=0}^{q=m} (-1)^{q} (j-q-n)^{r} \left( \Delta^{m-1-j} c_{n+j} \right) \right)^{p} + \left( \sum_{e_{0} \in \{0\}}^{r} \left( \frac{d^{v}}{dt} \left( \frac{e^{it}}{e^{it} - 1} \right) \right)^{m} \right)^{p} \left( \sum_{j=0}^{j=n+1} \sum_{q=0}^{q=m} (-1)^{q} (j-q+n+1)^{r} \left( \Delta^{m-1-j} c_{n+j+1} \right) \right)^{p} + \left( \sum_{j=-n}^{j=n+1} \mathcal{F} \mathcal{I} |c_{j}| \right)^{p} + \left( \sum_{j=n+1}^{n+m} \mathcal{F} \mathcal{I} |c_{j}| \right)^{p}.
\]
Applying the Lemma 1, we obtain
\[
\int_{-\pi}^{\pi} |S_n^{(r)}(t) - f^{(r)}(t)|^p dt \\
\leq O_{r,p,m}(1) \left( \sum_{|k| \geq n+1} k^r |\Delta^m c_k| \right)^p \\
+ O_{r,p,m}(1) \left( \sum_{j=0}^{m-1} \sum_{q=0}^{j} (j-q-n)^r |\Delta^{m-1-j} c_{n+j}| \right)^p \\
+ O_{r,p,m}(1) \left( \sum_{j=0}^{m-1} \sum_{q=0}^{j} (j-q+n+1)^r |\Delta^{m-1-j} c_{n+j+1}| \right)^p \\
+ \left( \sum_{j=-n}^{n} j^r |c_j| \right)^p + \left( \sum_{j=n+1}^{n+m} j^r |c_j| \right)^p.
\]

The second and thirteenth sums on the right-hand side of the last inequality are finite sums of \(o(1)\) terms as \(n \to \infty\).

Since \(j^r |c_j| \leq \sum_{j=1}^{\infty} j^r |\Delta c_j| = o(1), \ j \to \infty\) the last two terms are trivially \(o(1)\).
Hence, \(\int_{-\pi}^{\pi} |S_n^{(r)}(t) - f^{(r)}(t)|^p dt \leq O_{r,p,m}(1)(\sum_{|k| \geq n+1} k^r |\Delta^m c_k|)^p = o(1), \ n \to \infty\).

**Theorem 2.** Let \(\{a_n\} \in (BV)_r^m\), where \(\sigma \geq 1\) and \(r = 1, 2, 3, \ldots\). Then the point-wise limits \(f^{(r)}\) and \(\tilde{f}^{(r)}\) of the \(r - \text{th}\) derivatives of the sums \(S_n\) and \(\tilde{S}_n\) exist in \((0, \pi]\) and for any \(0 < p < \frac{1}{\sigma + r}\) the following limits hold:
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} |f^{(r)}(t) - S_n^{(r)}(t)|^p dt = 0 \quad (2.3)
\]
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} |\tilde{f}^{(r)}(t) - \tilde{S}_n^{(r)}(t)|^p dt = 0 \quad (2.4)
\]

**Proof.** Let \(m\) is integer such that \(m \geq \sigma\). Then by Theorem B, we obtain \(\{a_n\} \in (BV)_r^m\), and by Theorem C, the point-wise limits \(f^{(r)}\) and \(\tilde{f}^{(r)}\) of the \(r - \text{th}\) derivatives of the sums \(S_n\) and \(\tilde{S}_n\) exist in \((0, \pi]\). Applying the same technique for series (C) and (S) as in the proof of Theorem 1, we obtain (2.3) and (2.4).

3. Some Corollaries for \(\sigma = 1\)

A null sequence \(\{a_n\}\) belongs to the class \(H_{q,r}, \ 0 < q \leq 1, \ \alpha \geq 0, r \in \{0, 1, 2, \ldots, [\alpha]\}\) if there exists a monotonically decreasing sequence \(\{A_k\}\) such that \(\sum_{k=1}^{\infty} k^\alpha A_k < \infty\) and \(\frac{1}{n^{q+1-r}} \sum_{k=1}^{n} \frac{1}{A_k^{1+q/r}} = O(1)\).
Corollary 3.1. Let \( \{a_n\} \in H_{q,r}, \ 0 < q \leq 1, \alpha \geq 0, \ r \in \{0, 1, 2, \ldots, [a]\} \). Then the point-wise limits \( f^{(r)} \) and \( \bar{f}^{(r)} \) of the \( r \)-th derivatives of the sums \( S_n \) and \( \bar{S}_n \) exist in \( (0, \pi) \) and for any \( 0 < p < 1, \) the limits (2.3) and (2.4) hold.

Proof. By Theorem 2 (case \( \sigma = 1 \)), it suffices to show that \( H_{q,r} \) is a subclass of \((BV)_r\).

Applying first Abel’s transformation, then inequality (2.1), we obtain:

\[
\sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left( \frac{1}{k^{\alpha+1}} \sum_{j=1}^{k} j^r \frac{|\Delta a_j|}{A_j} \right) + n^{\alpha+1} A_n \left( \frac{1}{n^{\alpha+1}} \sum_{j=1}^{n} j^r \frac{|\Delta a_j|}{A_j} \right)
\]

\[
\leq \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left( \frac{1}{k^{\alpha-r+1}} \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \right) + n^{\alpha+1} A_n \left( \frac{1}{n^{\alpha-r+1}} \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \right)
\]

\[
\leq \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left( \frac{1}{k^{\alpha-r+1}} \sum_{j=1}^{k} \frac{|\Delta a_j|^q}{A_j^q} \right)^{1/q} + n^{\alpha+1} A_n \left( \frac{1}{n^{\alpha-r+1}} \sum_{j=1}^{n} \frac{|\Delta a_j|^q}{A_j^q} \right)^{1/q} = O_q(1) \left[ \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) + n^{\alpha+1} A_n \right]
\]

\[
= O_q(1) \left\{ \sum_{k=1}^{n} \left[ k^{\alpha+1} - (k-1)^{\alpha+1} \right] A_k - n^{\alpha+1} A_n + n^{\alpha+1} A_n \right\}
\]

\[
= O_q \left( \sum_{k=1}^{n} k^\alpha A_k \right).
\]

Letting \( n \to \infty \), we obtain \( \{a_n\} \in (BV)_r \).

Next, we shall define some known classes of real sequences introduced in [5], [6], [7], [8], [9].

A null-sequence \( \{a_k\} \) belongs to the class \( S_r, \ r = 0, 1, 2, \ldots \) (see [6]) if there exists a monotonically decreasing sequence \( \{ A_k \} \) such that \( \sum_{k=0}^{\infty} k^r A_k < \infty \) and \( |\Delta a_k| \leq A_k \), for all \( k \).

When \( r = 0 \), we obtain the Sidon-Telyakovskii class \( S \) (see [4]). It is obvious that \( S \subset (BV)_r \).

A null sequence \( \{a_k\} \) belongs to the class \( S_{q,r}, \ q > 1, \ r = 0, 1, 2, \ldots \) (see [5]), if there exists a monotonically decreasing sequence \( \{ A_k \} \) such that \( \sum_{k=0}^{\infty} k^r A_k < \infty \) and

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1).
\]

In [5], we proved that \( S_{q,r} \subset (BV)_r \).

Denote by \( I_m \) the dyadic interval \( \left( 2^{m-1}, 2^m \right) \), for \( m \geq 1 \).
A null sequence \( \{a_k\} \) belongs to the class \( F_{qr} \), \( q > 1, r = 0, 1, 2, \ldots \) if
\[
\sum_{m=1}^{\infty} 2^{m(1+r)} \left( \frac{1}{2^m} \sum_{k \in I_m} |\Delta a_k|^r \right)^{1/r} < \infty \quad \text{(see [7])}
\]

It is obvious that for \( r = 0 \), we obtain the Fomin’s class \( F_q \) (see [7]).

But, in [7] we verified the embedding relation \( F_{qr} \subset (BV)_{r'} \). On the other hand, in [8], [9] we defined an equivalent form of the Sheng’s class \( S_{qr} \), \( q > 1, \alpha \geq 0, r \in \{0, 1, 2, \ldots, [\alpha]\} \) (see [2]) as follows: a null sequence \( \{a_k\} \) belongs to the class \( S_{qr} \), \( q > 1, \alpha \geq 0, r \in \{0, 1, 2, \ldots, [\alpha]\} \) if there exists a monotonically decreasing sequence \( \{A_k\} \) such that \( \sum_{k=1}^{\infty} k^\alpha A_k < \infty \) and
\[
\frac{1}{q(\pi/n)^{1/r}} \sum_{k=1}^{n} |\Delta a_k|^{r} = O(1).
\]

The following embedding relation holds \( S_{qr} \subset (BV)_{r'} \) (see [8]). However, we can to formulate the following corollaries of the Theorem 2.

**Corollary 3.2.** Let \( \{a_n\} \in S_r \), \( r = 0, 1, 2, \ldots \). Then the point-wise limits \( f^{(r)} \) and \( \tilde{f}^{(r)} \) of the \( r \)-th derivatives of the sums \( S_n \) and \( \tilde{S}_n \) exist in \( (0, \pi] \) and for any \( 0 < p < 1 \), the limits (2.3) and (2.4) hold.

**Corollary 3.3.** Let \( \{a_n\} \in S_{qr} \), \( q > 1, r = 0, 1, 2, \ldots \). Then the point-wise limits \( f^{(r)} \) and \( \tilde{f}^{(r)} \) of the \( r \)-th derivatives of the sums \( S_n \) and \( \tilde{S}_n \) exist in \( (0, \pi] \) and for any \( 0 < p < 1 \), the limits (2.3) and (2.4) hold.

**Corollary 3.4.** Let \( \{a_n\} \in F_{qr} \), \( q > 1, r = 0, 1, 2, \ldots \). Then the point-wise limits \( f^{(r)} \) and \( \tilde{f}^{(r)} \) of the \( r \)-th derivatives of the sums \( S_n \) and \( \tilde{S}_n \) exist in \( (0, \pi] \) and for any \( 0 < p < 1 \), the limits (2.3) and (2.4) hold.

**Corollary 3.5.** Let \( S_{qr} \), \( q > 1, \alpha \geq 0, r \in \{0, 1, 2, \ldots, [\alpha]\} \). Then the point-wise limits \( f^{(r)} \) and \( \tilde{f}^{(r)} \) of the \( r \)-th derivatives of the sums \( S_n \) and \( S_n \) exist in \( (0, \pi] \) and for any \( 0 < p < 1 \), the limits (2.3) and (2.4) hold.

**References**


Faculty of Mathematical and Natural Sciences, P. O. Box 162, 1000 Skopje, MACEDONIA
E-mail: tomovski@iunona.pmf.ukim.edu.mk