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SOME CLASSES OF $L^p(0<p<1)$ CONVERGENCE OF TRIGONOMETRIC SERIES

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Abstract. We study here $L^p(0<p<1)$-convergence of complex trigonometric series, i.e. the extension is made for the V. B. Stanojevic Theorem [3], by considering the class $(BV)_r^m$, $m=1,2,3,\ldots$, $r=0,1,2,3,\ldots$ instead of $(BV)^m$, $m=1,2,3,\ldots$. Applying the Wang-Telyakovskii class $(\boldsymbol{BV})_r^\sigma$, $r=0,1,2,\ldots$, $\sigma\geq 1$ (see [11]), the extension of Uljanov's theorem [10] (case $r=0$, $\sigma=1$) for sine and cosine series with real coefficients is also given. For $\sigma=1$, some corollaryes of the main results are obtained.

1. Introduction

Let $\{c_k : k=0,\pm 1,\pm 2,\ldots\}$ be a sequence of complex numbers and the partial sums of the complex trigonometric series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{1.1}$$

be denoted by $S_n^*(t)=\sum_{k=-n}^{n} c_k e^{ikt}$, $t\in \boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{z}$ (1.2).
A sequence $\{c_k\}$ belongs to the class $(\boldsymbol{BV})^m$ (see [3]) if for some integer $m\geq 1$,

$$\sum_{|k|<\infty} \mid \triangle^m c_k \mid < \infty,$$

where $\triangle^m c_k=\triangle(\triangle^{m-1}c_k)=\triangle^{m-1}c_k-\triangle^{m-1}c_{k+1}$.
For $m=1$, the class $(BV)^1$ is the class of complex sequences with bounded variation. As an extension of the Uljanov's Theorem (see [10]), V. B. Stanojevic have obtained the following theorem for L^p, $0<p<1$-convergence of the series (1. 1).

Theorem A.([3]) *If for some integer $m\geq 1$, $\{c_n\}\in (BV)^m$, then the point-wise limit f^* of the partial sums* (1.2) *exists in $\boldsymbol{T}\backslash\{0\}$ and for any $0<p<\frac{1}{m}$.*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \mid S_n^*(t)-f^*(t)\mid^p dt=0$$

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On the other hand, Wang and Telyakovskii [11] have considered the following class of real sequences $\{a_n\}$. Namelly, a null-sequence $\{a_k\}$ belongs to the class $(\boldsymbol{BV})_r^{\sigma}$, $r = 0, 1, 2, \ldots$, $\sigma \geq 0$ if $\sum_{k=1}^{\infty} k^r \mid \triangle^{\sigma} a_k \mid < \infty$. If $\sigma = 1$, we denote $(\boldsymbol{BV})_r^{\sigma} = (\boldsymbol{BV})_r$ and if $\sigma = 1$, $r = 0$, then we denote $(\boldsymbol{BV})_r^{\sigma} = (\boldsymbol{BV})$.

Theorem B.([11]) *Let $\rho \geq 0$, $\sigma \geq 0$. Then for all $\gamma > \sigma$ the following embedding relation holds,*

$$(\boldsymbol{BV})_{\rho}^{\sigma} \subset (\boldsymbol{BV})_{\rho}^{\gamma}.$$

Let

$$\frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad (C)$$

$$\sum_{n=1}^{\infty} a_n \sin nx \qquad (S)$$

be the cosine and sine trigonometric series, and S_n, $\overline{S_n}$ denote the partial sums of the series (C) and (S) respectivelly.

Wang and Telyakovskii [11] considering the complex form of trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{inx}, \quad x \in (0, \pi]$$

have proved the following theorem.

Theorem C.([11]) *If $\{a_k\} \in (\boldsymbol{BV})_r^{\sigma}$, $r = 0, 1, 2, \ldots$, $\sigma \geq 0$, then the series (C) and (S) have continuous derivatives of r-th order on $(0, \pi]$.*

The Wang-Telyakovskii class $(\boldsymbol{BV})_r^{\sigma}$, $r = 0, 1, 2, \ldots$, $\sigma \geq 0$, motivated me to consider a further class $(\boldsymbol{BV})_r^{m}$, $r = 0, 1, 2, \ldots$ $m = 1, 2, 3, \ldots$, of complex null-sequences $\{c_n\}$ such that

$$\sum_{|k| < \infty} k^r \mid \triangle^m c_k \mid < \infty$$

In this paper we shall extend the Theorem A, by considering the class $(\boldsymbol{BV})_r^{m}$, $r = 0, 1, 2, \ldots$, $m = 1, 2, 3, \ldots$ instead of $(BV)^m$. In addition we obtain the extension of corresponding Uljanov's theorem [10], by considering the Wang-Telyakovskii class $(\boldsymbol{BV})_r^{\sigma}$, $r = 0, 1, 2, \ldots$, $\sigma \geq 1$ instead of $\boldsymbol{BV}$.

2. Main results

For the proof of our new results, we need the following Lemma.

Lemma 1. *If $m = 1, 2, 3, \ldots$, $r = 0, 1, 2, 3, \ldots$ and $0 < p < \frac{1}{m+r}$, then the following estimate holds*

$$\int_{-\pi}^{\pi} \left| \frac{d^r}{dt^r} \left(\frac{e^{it}}{e^{it}-1} \right)^m \right|^p dt = O_{r,p,m}(1),$$

where $O_{r,p,m}$ depends on r, p and m.

Proof. We denote $h(t) = (\frac{e^{it}}{e^{it}-1})^m$. After some elementary calculations, this function can be written in the form

$$h(t) = \frac{e^{im\frac{t}{2}}}{(2i)^m \sin^m \frac{t}{2}}$$

Then, $$h^{(r)}(t) = \frac{1}{(2i)^m} \sum_{k=0}^{r} \binom{r}{k} \left(\sin^{-m} \frac{t}{2}\right)^{(k)} \left(e^{im\frac{t}{2}}\right)^{(r-k)}.$$

The equality $(\sin^{-m} \frac{t}{2})^{(k)} = \frac{P_k(\cos \frac{t}{2})}{\sin^{m+k} \frac{t}{2}}$, where P_k is some cosine polynomial of degree k, can be proved by mathematical induction.

Really, for $k = 1$, we obtain $(\sin^{-m} \frac{t}{2})' = \frac{(\frac{-m}{2}) \cos \frac{t}{2}}{\sin^{m+1} \frac{t}{2}} = \frac{P_1(\cos \frac{t}{2})}{\sin^{m+1} \frac{t}{2}}$.

Suppose that the equality is true for some k and consider the $k+1$-th derivate.

$$\begin{aligned}
\left(\sin^{-m} \frac{t}{2}\right)^{k+1} &= \frac{\left[P_k\left(\cos \frac{t}{2}\right)\right]' \sin^{m+k} \frac{t}{2} - P_k\left(\cos \frac{t}{2}\right)\left(\sin^{m+k} \frac{t}{2}\right)'}{\sin^{2m+2k} \frac{t}{2}} \\
&= \frac{\tilde{P}_{k-1}\left(\cos \frac{t}{2}\right)\left(-\frac{1}{2}\sin^2 \frac{t}{2}\right) - \frac{m+k}{2} P_k\left(\cos \frac{t}{2}\right) \cdot \cos \frac{t}{2}}{\sin^{m+k+1} \frac{t}{2}} \\
&= \frac{-\frac{1}{2}\tilde{P}_{k-1}\left(\cos \frac{t}{2}\right) + \frac{1}{2}\tilde{P}_{k-1}\left(\cos \frac{t}{2}\right)\cos^2 \frac{t}{2} - \frac{m+k}{2} \mathcal{Q}_{k+1}\left(\cos \frac{t}{2}\right)}{\sin^{m+k+1} \frac{t}{2}} \\
&= \frac{R_{k+1}\left(\cos \frac{t}{2}\right) - \frac{m+k}{2}\mathcal{Q}_{k+1}\left(\cos \frac{t}{2}\right)}{\sin^{m+k+1} \frac{t}{2}} = \frac{T_{k+1}\left(\cos \frac{t}{2}\right)}{\sin^{m+k+1} \frac{t}{2}}.
\end{aligned}$$

Here $\tilde{P}_{k-1}$, R_{k+1}, $\mathcal{Q}_{k+1}$, T_{k+1} are some cosine polynomials of degree $k-1$ and $k+1$ respectivelly.

Thus $$h^{(r)}(t) = \frac{1}{(2i)^m} \sum_{k=0}^{r} \binom{r}{k} \left(\frac{im}{2}\right)^{r-k} \left(e^{im\frac{t}{2}}\right) \frac{P_k\left(\cos \frac{t}{2}\right)}{\sin^{m+k} \frac{t}{2}}.$$

We note that $|P_k(\cos \frac{t}{2})| = O_{k,m}(1)$.

Hence, $|h^{(r)}(t)| \le \frac{1}{2^m} \sum_{k=0}^{r} O_{k,m}(1) \cdot \binom{r}{k} \left(\frac{m}{2}\right)^{r-k} \cdot \frac{1}{|\sin^{m+k} \frac{t}{2}|}$

Applying the well-known inequality

$$\left(\sum \alpha_i\right)^{\lambda} \le \left(\sum \alpha_i^{\lambda}\right), \text{where } \alpha_i \ge 0,\ 0 < \lambda \le 1, \tag{2.1}$$

we obtain

$$|h^{(r)}(t)|^p \leq \frac{1}{2^{mp}} \sum_{k=0}^{r} O_{k,m,p}(1) \binom{r}{k}^p \left(\frac{m}{2}\right)^{(r-k)p} \frac{1}{\left|\sin^{(m+k)p} \frac{t}{2}\right|}.$$

Finally, for $0 < p < \frac{1}{m+r}$, we get

$$\int_{-\pi}^{\pi} |h^{(r)}(t)|^p dt \leq \frac{1}{2^{mp-2}} \sum_{k=0}^{r} O_{k,m,p}(1) \binom{r}{k}^p \left(\frac{m}{2}\right)^{(r-k)p} \int_0^{\pi/2} \frac{dt}{\sin^{(m+k)p} \frac{t}{2}}$$
$$\leq \frac{1}{2^{mp-2}} \sum_{k=0}^{r} O_{k,m,p}(1) \binom{r}{k}^p \left(\frac{m}{2}\right)^{(r-k)p} \pi^{(m+k)p} \int_0^{\pi/2} \frac{dt}{t^{(m+k)p}} = O_{r,p,m}(1).$$

Theorem 1. *Let $\{c_n\} \in (BV)_r^m$, for some integer $m \geq 1$ and $r = 0, 1, 2, 3, \ldots$. Then the point-wise limit $f^{*(r)}$ of the r-th derivate of the sums (1.2) exists in $\mathbf{T} \setminus \{0\}$ and for any $0 < p < \frac{1}{m+r}$,*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |S_n^{*(r)}(t) - f^{*(r)}(t)|^p dt = 0. \tag{2.2}$$

Proof. We consider the identity, obtained by V. B. Stanojevic in [3].

$$S_n^*(t) = \left(\frac{e^{it}}{e^{it}-1}\right)^m \sum_{|k|\leq n} (\Delta^m c_k) e^{ikt}$$
$$-\left(\frac{e^{it}}{e^{it}-1}\right)^m \sum_{j=0}^{m-1} \left(\frac{e^{it}-1}{e^{it}}\right)^j (\Delta^{m-1-j} c_{-n+j}) e^{i(-n+j)t}$$
$$+\left(\frac{e^{it}}{e^{it}-1}\right)^m \sum_{j=0}^{m-1} \left(\frac{e^{it}-1}{e^{it}}\right)^j (\Delta^{m-1-j} c_{n+j+1}) e^{i(n+j+1)t}$$
$$+\sum_{j=-n}^{-n+m-1} c_j e^{ijt} - \sum_{j=n+1}^{n+m} c_j e^{ijt}.$$

For the $r-th$ derivate of the partial sums of S_n^*, we have:

$$S_n^{*(r)}(t) = \sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left(\frac{e^{it}}{e^{it}-1}\right)^m i^{r-v} \sum_{|k|\leq n} k^{r-v} (\Delta^m c_k) e^{ikt}$$
$$-\sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left(\frac{e^{it}}{e^{it}-1}\right)^m \sum_{j=0}^{m-1} \sum_{q=0}^{j} (-1)^q \binom{j}{q} (j-q-n)^{r-v} i^{r-v}$$
$$e^{it(j-q-n)} (\Delta^{m-1-j} c_{-n+j}) + \sum_{v=0}^{r} \binom{r}{v} \frac{d^v}{dt^v} \left(\frac{e^{it}}{e^{it}-1}\right)^m \sum_{j=0}^{m-1} \sum_{q=0}^{j} (-1)^q \binom{j}{q}$$

$$\times(j-q+n+1)^{r-v}i^{r-v}e^{it(i-q+n+1)}(\Delta^{m-1-j}c_{n+j+1})$$
$$+\sum_{j=-n}^{-n+m-1}c_j j^r i^r e^{ijt}-\sum_{j=n+1}^{n+m}c_j j^r i^r e^{ijt}.$$

Hence, $\lim_{n\to\infty} S_n^{*(r)}(t)=f^{*(r)}(t)$ exists in $\mathbf{T}\setminus\{0\}$.

Then, we shall prove that $S_n^{*(r)}$ converges to $f^{*(r)}$ in $L^p(\boldsymbol{T})$-metric, for any $0<p<\frac{1}{m+r}$.

For $t\neq 0$, we consider

$$\begin{aligned}
&S_n^{*(r)}(t)-f^{*(r)}(t)\\
&=\sum_{v=0}^{r}\binom{r}{v}\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m i^{r-v}\sum_{|k|<\infty}k^{r-v}(\Delta^m c_k)e^{ikt}\\
&\quad-\sum_{v=0}^{r}\binom{r}{v}\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m i^{r-v}\sum_{|k|\le n}k^{r-v}(\Delta^m c_k)e^{ikt}\\
&\quad+\sum_{v=0}^{r}\binom{r}{v}\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\sum_{j=0}^{m-1}\sum_{q=0}^{j}(-1)^q\binom{j}{q}(j-q-n)^{r-v}i^{r-v}e^{it(j-q-n)}(\Delta^{m-1-j}c_{-n+j})\\
&\quad-\sum_{v=0}^{r}\binom{r}{v}\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\sum_{j=0}^{m-1}\sum_{q=0}^{j}(-1)^q\binom{j}{q}(j-q+n+1)^{r-v}i^{r-v}e^{it(j-q+n+1)}\\
&\quad(\Delta^{m-1-j}c_{n+j+1})-\sum_{j=-n}^{-n+m-1}c_j j^r i^r e^{ijt}+\sum_{j=n+1}^{n+m}c_j j^r i^r e^{ijt}.
\end{aligned}$$

By inequality (2.1), we obtain

$$\begin{aligned}
&|S_n^{*(r)}(t)-f^{*(r)}(t)|^p\\
&\le\left(\sum_{v=0}^{r}\binom{r}{v}^p\left|\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\right|^p\right)\Big(\sum_{|k|\ge n+1}k^r|\Delta^m c_k|\Big)^p\\
&\quad+\left(\sum_{v=0}^{r}\binom{r}{v}^p\left|\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\right|^p\right)\left(\sum_{j=0}^{m-1}\sum_{q=0}^{j}\binom{j}{q}(j-q-n)^r|\Delta^{m-1-j}c_{-n+j}|\right)^p\\
&\quad+\left(\sum_{v=0}^{r}\binom{r}{v}^p\left|\frac{d^v}{dt^v}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\right|^p\right)\left(\sum_{j=0}^{m-1}\sum_{q=0}^{j}\binom{j}{q}(j-q+n+1)^r|\Delta^{m-1-j}c_{n+j+1}|\right)^p\\
&\quad+\left(\sum_{j=-n}^{-n+m-1}j^r|c_j|\right)^p+\left(\sum_{j=n+1}^{n+m}j^r|c_j|\right)^p.
\end{aligned}$$

Applying the Lemma 1, we obtain

$$\int_{-\pi}^{\pi} |S_n^{*(r)}(t) - f^{*(r)}(t)|^p dt$$
$$\le O_{r,p,m}(1) \left(\sum_{|k| \ge n+1} k^r |\Delta^m c_k| \right)^p$$
$$+ O_{r,p,m}(1) \left(\sum_{j=0}^{m-1} \sum_{q=0}^{j} \binom{j}{q} (j-q-n)^r |\Delta^{m-1-i} c_{-n+j}| \right)^p$$
$$+ O_{r,p,m}(1) \left(\sum_{j=0}^{m-1} \sum_{q=0}^{j} \binom{j}{q} (j-q+n+1)^r |\Delta^{m-1-j} c_{n+j+1}| \right)^p$$
$$+ \left(\sum_{j=-n}^{-n+m-1} j^r |c_j| \right)^p + \left(\sum_{j=n+1}^{n+m} j^r |c_j| \right)^p .$$

The second and thirth sums on the right-hand side of the last inequality are finite sums of $o(1)$ terms as $n \to \infty$.

Since $j^r|c_j| \le \sum_{l=j}^{\infty} l^r |\Delta c_l| = o(1),\ j \to \infty$ the last two terms are trivially $o(1)$. Hence, $\int_{-\pi}^{\pi} |S_n^{*(r)}(t) - f^{*(r)}(t)|^p dt \le O_{r,p,m}(1) (\sum_{|k| \ge n+1} k^r |\Delta^m c_k|)^p = o(1),\ n \to \infty$.

Theorem 2. *Let $\{a_n\} \in (\boldsymbol{BV})_r^{\sigma}$, where $\sigma \ge 1$ and $r = 1, 2, 3, \ldots$. Then the point-wise limits $f^{(r)}$ and $\bar{f}^{(r)}$ of the $r-th$ derivatives of the sums S_n and $\bar{S}_n$ exist in $(0, \pi]$ and for any $0 < p < \frac{1}{\sigma + r}$, the following limits hold:*

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f^{(r)}(t) - S_n^{(r)}(t)|^p dt = 0 \tag{2.3}$$

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |\bar{f}^{(r)}(t) - \bar{S}_n^{(r)}(t)|^p dt = 0 \tag{2.4}$$

Proof. Let m is integer such that $m \ge \sigma$. Then by Theorem B, we obtain $\{a_n\} \in (\boldsymbol{BV})_r^m$, and by Theorem C, the point-wise limits $f^{(r)}$ and $\bar{f}^{(r)}$ of the $r-th$ derivatives of the sums S_n and $\bar{S}_n$ exist in $(0, \pi]$. Applying the same technique for series (C) and (S) as in the proof of Theorem 1, we obtain (2.3) and (2.4).

3. Some Corollaryes for $\sigma = 1$

A null sequence $\{a_n\}$ belongs to the class $H_{q\alpha r},\ 0 < q \le 1,\ \alpha \ge 0, r \in \{0, 1, 2, \ldots, [\alpha]\}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^{\alpha} A_k < \infty$ and $\frac{1}{n^{q(\alpha - r) + q}} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1)$.

Corollary 3.1. *Let* $\{a_n\} \in H_{q\alpha r}$, $0 < q \le 1$, $\alpha \ge 0$, $r \in \{0, 1, 2, \dots, [\alpha]\}$. *Then the point-wise limits* $f^{(r)}$ *and* $\bar{f}^{(r)}$ *of the* $r - th$ *derivatives of the sums* S_n *and* $\bar{S}_n$ *exist in* $(0, \pi]$ *and for any* $0 < p < 1$, *the limits* (2.3) *and* (2.4) *hold.*

Proof. By Theorem 2 (case $\sigma = 1$), it suffices to show that $H_{q\alpha r}$ is a subclass of $(\boldsymbol{BV})_r$.

Applying first Abel's transformation, then inequality (2.1), we obtain:

$$\begin{aligned}
\sum_{k=1}^{n} k^r |\Delta a_k| &= \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left(\frac{1}{k^{\alpha+1}} \sum_{j=1}^{k} j^r \frac{|\Delta a_j|}{A_j} \right) + n^{\alpha+1} A_n \left(\frac{1}{n^{\alpha+1}} \sum_{j=1}^{n} j^r \frac{|\Delta a_j|}{A_j} \right) \\
&\le \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left(\frac{1}{k^{\alpha-r+1}} \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \right) + n^{\alpha+1} A_n \left(\frac{1}{n^{\alpha-r+1}} \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \right) \\
&\le \sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) \left(\frac{1}{k^{q(\alpha-r)+q}} \sum_{j=1}^{k} \frac{|\Delta a_j|^q}{A_j^q} \right)^{1/q} \\
&\quad + n^{\alpha+1} A_n \left(\frac{1}{n^{q(\alpha-r)+q}} \sum_{j=1}^{n} \frac{|\Delta a_j|^q}{A_j^q} \right)^{1/q} \\
&= O_q(1) \left[\sum_{k=1}^{n-1} k^{\alpha+1} (\Delta A_k) + n^{\alpha+1} A_n \right] \\
&= O_q(1) \left\{ \sum_{k=1}^{n} \left[k^{\alpha+1} - (k-1)^{\alpha+1} \right] A_k - n^{\alpha+1} A_n + n^{\alpha+1} A_n \right\} \\
&= O_q \left(\sum_{k=1}^{n} k^\alpha A_k \right).
\end{aligned}$$

Letting $n \to \infty$, we obtain $\{a_n\} \in (\boldsymbol{BV})_r$.

Next, we shall define some known classes of real sequences introduced in [5], [6], [7], [8], [9].

A null-sequence $\{a_k\}$ belongs to the class S_r, $r = 0, 1, 2, \dots$ (see [6]) if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=0}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k.

When $r = 0$, we obtain the Sidon-Telyakovskii class S (see [4]). It is obvious that $S_r \subset (\boldsymbol{BV})_r$.

A null sequence $\{a_k\}$ belongs to the class S_{qr}, $q > 1$, $r = 0, 1, 2, \dots$ (see [5]), if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and $\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1)$.

In [5], we proved that $S_{qr} \subset (\boldsymbol{BV})_r$.

Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$, for $m \ge 1$.

A null sequence $\{a_k\}$ belongs to the class F_{qr}, $q > 1$, $r = 0, 1, 2, \ldots$ if

$$\sum_{m=1}^{\infty} 2^{m(1+r)} \left(\frac{1}{2^m} \sum_{k \in I_m} |\Delta a_k|^q \right)^{1/q} < \infty \qquad \text{(see [7])}$$

It is obvious that for $r = 0$, we obtain the Fomin's class F_q (see [7]).

But, in [7] we verified the embedding relation $F_{qr} \subset (\boldsymbol{BV})_r$. On the other hand, in [8], [9] we defined an equivalent form of the Sheng's class $S'_{q\alpha r}$, $q > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$ (see [2]) as follows: a null sequence $\{a_k\}$ belongs to the class $S_{q\alpha r}$, $q > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^{\alpha} A_k < \infty$ and $\frac{1}{n^{q(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1)$.

The following embedding relation holds $S_{q\alpha r} \subset (\boldsymbol{BV})_r$. (see [8]).

However, we can to formulate the following corollaryes of the Theorem 2.

Corollary 3.2. *Let* $\{a_n\} \in S_r$, $r = 0, 1, 2, \ldots$. *Then the point-wise limits* $f^{(r)}$ *and* $\bar{f}^{(r)}$ *of the* $r-th$ *derivatives of the sums* S_n *and* $\bar{S}_n$ *exist in* $(0, \pi]$ *and for any* $0 < p < 1$, *the limits* (2.3) *and* (2.4) *hold.*

Corollary 3.3. *Let* $\{a_n\} \in S_{qr}$, $q > 1$, $r = 0, 1, 2, \ldots$. *Then the point-wise limits* $f^{(r)}$ *and* $\bar{f}^{(r)}$ *of the* $r-th$ *derivatives of the sums* S_n *and* $\bar{S}_n$ *exist in* $(0, \pi]$ *and for any* $0 < p < 1$, *the limits* (2.3) *and* (2.4) *hold.*

Corollary 3.4. *Let* $\{a_n\} \in F_{qr}$, $q > 1$, $r = 0, 1, 2, \ldots$. *Then the point-wise limits* $f^{(r)}$ *and* $\bar{f}^{(r)}$ *of the* $r-th$ *derivatives of the sums* S_n *and* $\bar{S}_n$ *exist in* $(0, \pi]$ *and for any* $0 < p < 1$, *the limits* (2.3) *and* $(2, 4)$ *hold.*

Corollary 3.5. *Let* $S_{q\alpha r}$, $q > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$. *Then the point-wise limits* $f^{(r)}$ *and* $\bar{f}^{(r)}$ *of the* $r-th$ *derivatives of the sums* S_n *and* $\bar{S}_n$ *exist in* $(0, \pi]$ *and for any* $0 < p < 1$, *the limits* (2.3) *and* (2.4) *hold.*

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