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Line Graph Associated to Graph of a Near-Ring with Respect to an Ideal

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Abstract. Let N be a near-ring and I be an ideal of N. The graph of N with respect to I is a graph with $V(N)$ as vertex set and any two distinct vertices x and y are adjacent if and only if $xNy \subseteq I$ or $yNx \subseteq I$. This graph is denoted by $G_I(N)$. We define the line graph of $G_I(N)$ as a graph with each edge of $G_I(N)$ as vertex and any two distinct vertices are adjacent if and only if their corresponding edges share a common vertex in the graph $G_I(N)$. We denote this graph by $L(G_I(N))$. We have discussed the diameter, girth, clique number, dominating set of $L(G_I(N))$. We have also found conditions for the graph $L(G_I(N))$ to be a cycle graph.

1 Introduction

Let N be a right near-ring. Let I be an ideal of N. A graph was defined by S. Bhavanari et.al.on a near-ring N with respect to an ideal I of N, denoted by $G_I(N)$. $G_I(N)$ is defined by considering all the elements of N as vertices and any two distinct vertices are adjacent if and only if $xNy \subseteq I$ or $yNx \subseteq I$.

In this paper we define the line graph of $G_I(N)$, denoted by $L(G_I(N))$. $L(G_I(N))$ is the graph where vertex set is represented by all the edges of $G_I(N)$ and any two distinct vertices are adjacent if and only if their corresponding edges share a common vertex in the graph $G_I(N)$. If x, y be two vertices adjacent in $G_I(N)$, then the corresponding vertex in the line graph $L(G_I(N))$ is denoted by $[x, y]$.

A near-ring N is called integral if it has no nonzero zero-divisors. N is called simple if its ideals are $\{0\}$ and N. An ideal I of N is called prime, if for ideals A, B of N, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$. An ideal I is called semiprime if for any ideal J of N, $J^2 \subseteq I$ implies that $J \subseteq I$. An ideal I is called 3-prime if for $a, b \in N$ and $aNb \subseteq I$ either $a \in I$ or $b \in I$. N is called 3-prime near-ring if $\{0\}$ is a 3-prime ideal of N.

Let G be a graph. The graph G is said to be connected if there is a path between any two distinct vertices of G. On the other side, the graph G is called totally disconnected if no two

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vertices of G are adjacent. For vertices x and y of G, the distance between x and y denoted by $d(x, y)$ is defined as the length of the shortest path from x to y; $d(x, y) = \infty$, if there is no such path. The diameter of G is $diam(G) = sup\{d(x, y)\colon x, y$ are vertices of $G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G; $gr(G) = \infty$ if G contains no cycle. A graph is said to be a cycle graph if it consists of a single cycle.

A dominating set for a graph G is a subset D of the vertex set such that every vertex not in D is adjacent to at least one member of D.In a graph G the maximal complete subgraph is called a *clique*. The number of vertices in a clique is called the *clique number*, denoted by $\omega(G)$.

For usual graph-theoretic terms and definitions, one can look at [1]. General references for the algebraic part of this paper are [3, 4, 5, 6].

Example 1. Let us consider $\mathbb{Z}_4$ the ring of integers modulo 4. The ideals of $\mathbb{Z}_4$ are $I = \{0\}$, $J = \{0, 2\}$ and $K = \mathbb{Z}_4$. The graphs of $\mathbb{Z}_4$ with respect to ideals I, J and K and their corresponding line graphs are shown below in Figure 1 and Figure 2 respectively.

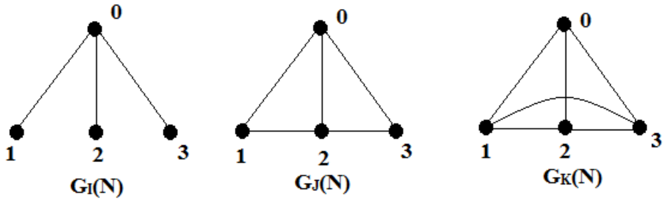

Figure 1: The graphs of $N = \mathbb{Z}_4$ with respect to ideals I, J and K.

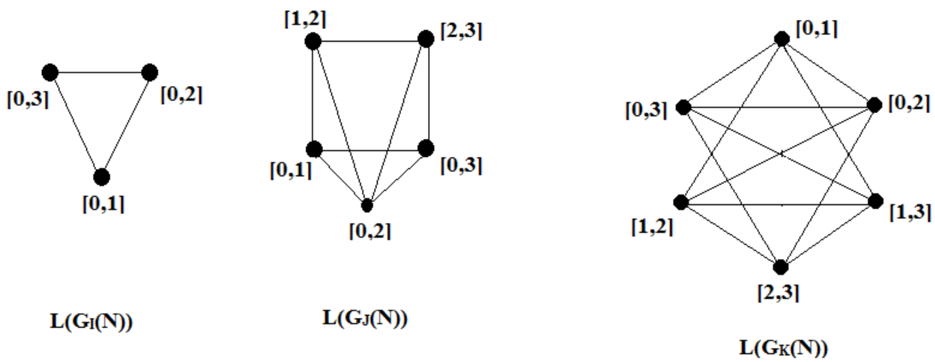

Figure 2: The Line graphs of $N = \mathbb{Z}_4$ with respect to ideals I, J and K.

Example 2. Let us consider a near-ring $N = \{0, a, b, c\}$ under the two binary operations $'+'$ and $'.'$ defined in the following tables:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	b	b	b	b
c	b	c	b	c

Here the ideals of N are $I = \{0\}$, $J = \{0, b\}$, $K = \{0, b\}$ and $P = N$. The graphs of N with respect to these ideals and their corresponding line graphs are shown below in Figure 3 and Figure 4 respectively.

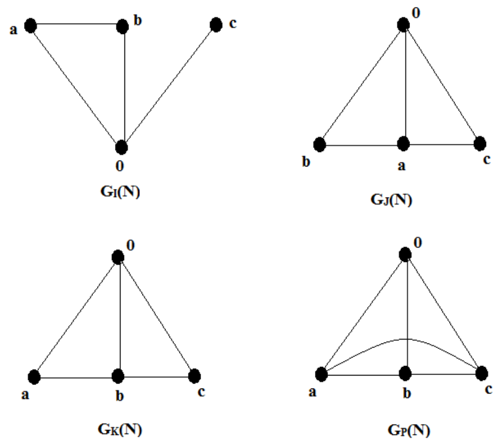

Figure 3: The graphs of $N = \{0, a, b, c\}$ with respect to ideals I, J, K and P.

Remark 1. Every ideal is itself a near-ring and so every ideal of a near-ing N is a subnear-ring of N.

Theorem 1.1 ([2]). *Let N be integral as well as simple. Let I be an ideal of N. Then $G_I(N) = K_n$ or $G_I(N)$ is a rooted tree with root vertex* 0.

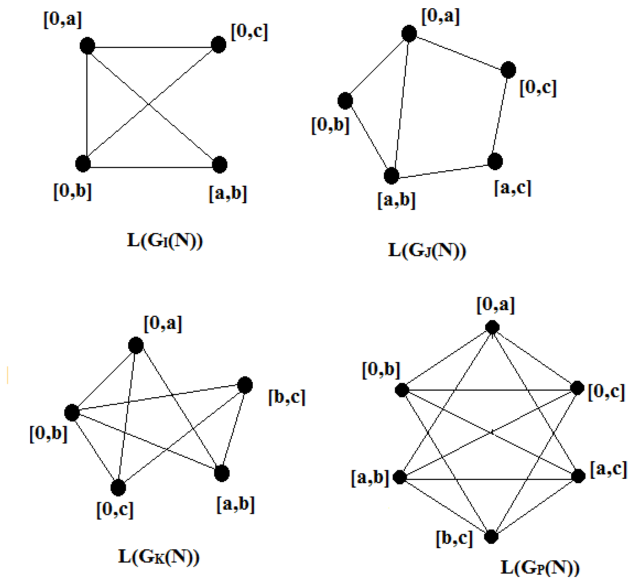

Figure 4: The Line graphs of $N = \{0, a, b, c\}$ with respect to ideals I, J, K and P.

2 Main Results

In this section we present some main results.Let N be a right-near ring with $\mid N \mid = n$, where $\mid N \mid$ denotes the cardinality of N, n may be of infinite cardinality also.
We begin our section with the following observation.

Remark 2. : In the graph $G_I(N)$, 0 is always adjacent to all the other vertices resulting at least one edge and as a result its corresponding line graph will contain at least one vertex. Hence $L(G_I(N))$ can never be an empty graph.

Theorem 2.1. *For any near-ring N, the graph $L(G_I(N))$ is always connected and $diam(L(G_I(N))) \leq 3$.*

Proof. In $G_I(N)$ the vertex 0 is adjacent to each vertex, so $G_I(N)$ is always connected and hence its line graph $L(G_I(N))$ is always a connected graph.
Let us now prove the second part. Let $[x, y], [z, w]$ be any two vertices in $L(G_I(N))$. Then we can always construct a path of length 3 as $[x, y] - [y, 0] - [0, z] - [z, w]$. Thus $diam(L(G_I(N))) \leq 3$. □

Remark 3. If N is simple and integral, then it is clear from Theorem 1.1. that the line graph $L(G_I(N))$ is either a regular graph of degree $2n - 4$ or a complete graph K_{n-1}.

The following results give the girth of $L(G_I(N))$ under various conditions.

Theorem 2.2. *For any near ring N let $I = \{0\}$. Then $gr(L(G_I(N))) = \infty$ if and only if $N \cong \mathbb{Z}_2$ or $N \cong \mathbb{Z}_3$.*

Proof. Let $N \cong \mathbb{Z}_2$. Then $G_{\{0\}}(N)$ is an edge $0 - 1$. Thus $L(G_I(N))$ contains only one vertex and thus $gr(L(G_I(N))) = \infty$. Again let $N \cong \mathbb{Z}_3$. In this case $G_{\{0\}}(N)$ is a path of length 2, $1 - 0 - 2$ and so $gr(L(G_I(N))) = \infty$.
Conversely let $gr(L(G_I(N))) = \infty$. If possible let N contains more than 3 elements. Suppose $| N |= 4$. If $I = \{0\}$,then in $G_I(N)$ we will get an induced subgraph which is a star graph $K_{1,3}$. Thus the corresponding line graph will contain a 4-cycle, which is a contradiction. Hence $N \cong \mathbb{Z}_2$ or $N \cong \mathbb{Z}_3$. □

Theorem 2.3. *For any near-ring N if $G_I(N)$ contains a cycle, then $gr(L(G_I(N))) = 3$,*

Proof. If $G_I(N)$ contains a 3 cycle, then we are done.
Let $G_I(N)$ contains a 4 cycle. Thus N contains atleast 4 elements. Since 0 is adjacent to every other vertex in $G_I(N)$, so there will be a star graph $K_{1,3}$ with centre vertex 0. Hence the corresponding line graph will contain a subgraph isomorphic to K_3. Thus $L(G_I(N))$ will contain a 3-cycle and so $gr(L(G_I(N))) = 3$.
Next let $G_I(N)$ contains a 5-cycle. Then by the similar arguments given above $G_I(N)$ contains a star graph isomorphic to $K_{1,4}$. Hence in the corresponding line graph we will get a subgraph isomorphic to K_4. Therefore $gr(L(G_I(N))) = 3$.
By The similar arguments if $G_I(N)$ contains a n cycle, n is any non zero positive integer, in $L(G_I(N))$ there will be a complete subgraph isomorphic to K_{n-1}. Thus $L(G_I(N))$ contains a 3 cycle. Hence $gr(L(G_I(N))) = 3$. Thus the theorem is proved. □

Remark 4. The converse of the Theorem 2.3. does not hold always. For example the line graph $L(G_I(N))$ in Figure 2 has girth 3, but its corresponding graph $G_I(N)$ in Figure 1 does not contain any cycle.

Theorem 2.4. *The line graph $L(G_I(N))$ is a cycle graph if and only if either $N = I \cong \mathbb{Z}_3$ or $N \cong \mathbb{Z}_4$ and $I = \{0\}$.*

Proof. Suppose $N = I \cong \mathbb{Z}_3$. It is clear from the definition that $G_I(N)$ is a cycle of length 3 and hence the corresponding line graph is a cycle graph that is C_3. Next suppose $N \cong \mathbb{Z}_4$ and $I = \{0\}$. In this case also $G_I(N)$ is a complete bipartite graph $K_{1,3}$ and it implies that the corresponding line graph is a cycle of length 3.
Conversely let the line graph $L(G_I(N))$ be a cycle graph. If possible let $N = I \ncong \mathbb{Z}_3$. From the definition it is clear that $G_I(N)$ is a complete graph if $N = I$ that is $G_I(N) = K_n$. Therefore

if $N = I \ncong \mathbb{Z}_3$, then $G_I(N) = K_n, n \geq 4$. Hence the corresponding line graph contains more than one cycles, a contradiction. Again let $I = \{0\}$ and $N \ncong \mathbb{Z}_4$. If $N \cong \mathbb{Z}_3$ or $\mathbb{Z}_2$, then the corresponding line graph does not contain any cycle, in fact the line graph is a path. Next, let $N \ncong \mathbb{Z}_n, n \geq 5$. Thus in $G_I(N)$ we get a spanning subgraph $K_{1,n-1}$ which will induce a complete graph K_{n-1} in $L(G_I(N))$. Thus $L(G_I(N))$ contains more than one cycle which is a contradiction. Thus the theorem is proved. □

Next let us find the clique number in the graph $L(G_I(N)), \omega(L(G_I(N)))$.

Theorem 2.5. *For any ideal I of N, we have $\omega(L(G_I(N))) = n - 1$, where $n =| N |$.*

Proof. Let $n =| N |$. For any ideal I of N, in the graph $G_I(N)$, 0 is adjacent to every other vertex inducing a spanning subgraph $K_{1,n-1}$. Thus in $L(G_I(N)), n - 1$ vertices are adjacent to each other resulting a complete graph K_{n-1}. For suppose $L(G_I(N))$ contains a subgraph isomorphic to K_n. Thus n vertices are connected to each other and hence n edges share a common vertex in $G_I(N)$. Therefore that common vertex has degree n which is a contradiction since the maximum connectivity of any vertex in $G_I(N)$ is $n - 1$. It implies that any other set having more than $n - 1$ edges, the corresponding vertices in the line graph are not all adjacent to each other. Hence $\omega(L(G_I(N))) = n - 1$. □

Theorem 2.6. *For any near-ring N if $G_I(N)$ does not contain a cycle then $L(G_I(N))$ is a complete graph K_{n-1}, where $n =| N |$.*

Proof. Let $G_I(N)$ contains no cycle. It implies that any two nonzero elements $x, y \in N$ are not adjacent in the graph $G_I(N)$, otherwise these elements together with 0 will form a cycle in $G_I(N)$ and hence same in $L(G_I(N))$. Thus $G_I(N)$ is a star graph with 0 as a centre vertex and the corresponding line graph is a complete graph. Since In $G_I(N)$ there are $n - 1$ edges, so $L(G_I(N))$ is a complete graph with $n - 1$ vertices that is K_{n-1}. □

Theorem 2.7. *For any ideal I of a near ring N, if $[x, y]$ is a vertex in $L(G_I(N))$, then $deg([x, y])$ is atleast two.*

Proof. Let $[x, y]$ be a vertex in $L(G_I(N))$. Since 0 is adjacent to all other vertices in $G_I(N)$, thus there exist at least two edges $0 - x$ and $0 - y$ and as a result in the corresponding line graph we get two vertices $[0, x], [0, y]$. These two vertices are adjacent to $[x, y]$. Thus the result is proved. □

Theorem 2.8. *Let I be a 3-prime ideal of N. Then the set $D = \{[x, n] : x \in I, n \in N\}$ is a dominating set in $L(G_I(N))$.*

Proof. Let I be a 3-prime ideal of the near-ring N. Then if $x - y$ is an edge in $G_I(N)$, it implies either $xNy \subseteq I$ or $yNx \subseteq I$. Without loss of generality, let $xNy \subseteq I$. since I is a 3-prime ideal, so either x or $y \in I$. let $x \in I$. Thus for any edge in $G_I(N)$ at least one end vertex of that edge will belong to I. It implies that I is a dominating set for $G_I(N)$. Thus the edge set $\{(x - n) : x \in I, n \in N\}$ will be an edge dominating set and thus $\{[x, n] : x \in I, n \in N\}$ is a vertex dominating set for the corresponding line graph $L(G_I(N))$. □

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