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A Characterization of Orthonormal Multilevel Wavelet Families in Sobolev Space over Local Fields of Positive Characteristic

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Abstract. In this paper, a characterization of orthonormal multilevel wavelet families in Sobolev space over a local fields of positive characteristic $(H^s(\mathbb{K}))$ is established. Finally an example is presented.

1 Introduction

The idea of wavelets as a family of function constructed by using translation and dilation of a single function was given by Morlet in 1982. The wavelet on $L^2(\mathbb{R})$ were characterized by Wang [27] and Gripenberg [17] with the help of Fourier transform of the wavelet. Bownik [21] obtained new concept to characterizing multiwavelets in $L^2(\mathbb{R}^n)$ by using the results of shift invariant systems and quasi-affine systems in ([8], [9], [14], [22]).

Jiang, Li and Jin discussed multiresolution analysis on local fields of positive characteristic and constructed corresponding orthonormal wavelets (see[18]).Behera and Jahan were first generalized the concept of multiresolution analysis (MRA) and wavelets in the space $L^2(\mathbb{K})$ and established orthonormal basis from Riesz basis (see [11]).They were also characterized the wavelets and MRA wavelets over a local fields of positive characteristic by affine frame systems and quasi-affine systems (see [12]). Pathak and Singh recently modified the classical definition of multiresolution analysis and construct the orthonormal wavelet in Sobolev space over local fields of positive characteristic $(H^s(\mathbb{K}))$ in [5].

In this paper, we characterized the orthonormal multilevel wavelet families in Sobolev space over a local fields of positive characteristic. This article is organized as follows. In section 2, we give some basic notations and definitions of local field and Sobolev space over local fields. In section 3, this section contains the basic definition of orthonormal wavelet in $H^s(\mathbb{K})$ and four lemmas, by using these lemmas we will prove the main result of this paper. Finally in this section we characterized the orthonormal multilevel wavelet families in Sobolev space over a local fields of positive characteristic $(H^s(\mathbb{K}))$ and an example is presented.

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2 Notation and definitions

A local field $\mathbb{K}$ means an algebraic field and a topological space with the properties of complete, locally compact, totally disconnected and non-discrete and Haar measure on $\mathbb{K}^+$ is denoted by dx. The absolute value or valuation of x is denoted by $|x|$ and have the properties (i) $|x| \geq 0$ and $|x| = 0$ iff $x = 0$;(ii) $|xy| = |x||y|$; (iii) $|x + y| \leq max(|x|, |y|)$.
Now, we recall some Notations which are used in this paper

- Throughout this paper $\mathbb{K}$ denotes the local field of positive characteristic.
- dx is the normalized Haar measure for $\mathbb{K}^+$.
- $|\alpha|$ is the valuation of $\alpha \in \mathbb{K}$ and it is non-archimedian norm.
- We will use following notations for the numbers, $\mathbb{Z}$ = the set of integers; $\mathbb{N}$ = the set of natural numbers; $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$.
- Let $\mathfrak{s}$ be a prime element in $\mathbb{K}$.
- For $k \in \mathbb{Z}$, $\mathfrak{P}^k = \{x \in \mathbb{K} : |x| \leq q^{-k}\}$ is a compact subgroup of $\mathbb{K}^+$. $\mathfrak{P}^0 = \mathfrak{D}$ is called ring of integres in $\mathbb{K}$.
- $|\mathfrak{P}^k| = q^{-k}$ and $|\mathfrak{D}| = 1$.
- χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is non trivial on $\mathfrak{P}^{-1}$. For $y \in \mathbb{K}$, $\chi_y(x) = \chi(yx)$, $x \in \mathbb{K}$.
- The "natural"order on the sequence is denoted by $\{\omega(k) \in \mathbb{K}\}_{k=0}^{\infty}$ and is described as follows.
  $\mathfrak{D}/\mathfrak{P} \cong GF(q) = \tau$, $q = p^s$, p is a prime, $s \in \mathbb{N}$ and $\Omega : \mathfrak{D} \to \tau$ the canonical homomorphism of $\mathfrak{D}$ on to τ. $\tau = GF(q)$ is a vector space over $GF(p) \subset \tau$. We consider a set $\{1 = \epsilon_0, \epsilon_1, ..., \epsilon_{s-1}\} \subset \mathfrak{D}^* = \mathfrak{D}\backslash\mathfrak{P}$ in such a way that $\{\Omega(\epsilon_k)\}_{k=0}^{s-1}$ is a basis of $GF(q)$ over $GF(p)$.
  For k, $0 \leq k < q$, $k = a_0 + a_1 p + ... + a_{s-1}p^{s-1}$, $0 \leq a_i < p$, $i = 0, 1, ..., s-1$, we define
  $$\omega(k) = (a_0 + a_1\epsilon_1 + ... + a_{s-1}\epsilon_{c-1})\mathfrak{s}^{-1} \ \ (0 \leq k < q).$$
  For $k = b_0 + b_1 q + ... + b_r q^r$, $0 \leq b_i < q$, $k \geq 0$, we set
  $$\omega(k) = \omega(b_0) + \mathfrak{s}^{-1}\omega(b_1) + ... + \mathfrak{s}^{-r}\omega(b_r).$$

Remark 1. A function g defined on $\mathbb{K}$ is integral-periodic if $g(x + \omega(n)) = g(x)$ for all $n \in \mathbb{N}_0$.

Proposition 2.1. *For all $l, k \in \mathbb{N}_0$, $\chi_k(\omega(l)) = 1$.*

2.1 Distributions over local fields

The space $\mathscr{S}'(\mathbb{K})$ of continuous linear functional on $\mathscr{S}(\mathbb{K})$ (the space of all finite linear combinations of characteristics functions of ball of $\mathbb{K}$) is called the space of distributions.
The Fourier transform of $f \in \mathscr{S}(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by the

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} f(x)\chi(-\xi x)dx, \ \xi \in \mathbb{K}, \tag{2.1}$$

and the inverse Fourier transform by

$$f(x) = \int_{\mathbb{K}} \hat{f}(\xi)\chi_x(\xi)d\xi, \ x \in \mathbb{K}. \tag{2.2}$$

The Fourier transform and inverse Fourier transforms of a distributions $f \in \mathscr{S}'(\mathbb{K})$ is defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \ \langle f^\vee, \phi \rangle = \langle f, \phi^\vee \rangle, \ \text{for all } \phi \in \mathscr{S}(\mathbb{K}). \tag{2.3}$$

Definition 1. Sobolev space over local fields ($H^s(\mathbb{K})$).
Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{K})$ is the space of all $f \in \mathscr{S}'(\mathbb{K})$ such that

$$\hat{\nu}^{\frac{s}{2}}(\xi)\hat{f}(\xi) \in L^2(\mathbb{K}).$$

Obviously, for any real number s, $H^s(\mathbb{K})$ is a linear space. We equip $H^s(\mathbb{K})$ with the inner product

$$\langle f, g \rangle_s = \langle f, g \rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\nu}^s(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

which induces the norm

$$||f||^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{f}(\xi)|^2 d\xi.$$

Theorem 2.1. *The space $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$.*

Proof. See [5]. □

3 Main Result

The aim of this section to established the condition which enable us to characterize all orthonormal wavelet families in $H^s(\mathbb{K})$. Before characterization of orthonormal wavelet, we first define the orthonormal wavelets and recall four lemmas which will we used in the proof of main result.

Definition 2. Let $\{\psi_r^{(j)}\}_{j\in\mathbb{Z}, r\in D_1} \subseteq H^s(\mathbb{K})$.Then $\{\psi_r^{(j)}\}_{j\in\mathbb{Z}, r\in D_1}$ is said to be an orthonormal wavelet family in $H^s(\mathbb{K})$ if $H^s(\mathbb{K}) = \overline{span\{\psi_{r,j,k}^{(j)}\}}_{r\in D_1, j\in\mathbb{Z}, k\in\mathbb{N}_0}$ where $D_1 = \{1, 2, 3, ...q-1\}$ and $\langle \psi_{r,j,k}^{(j)}, \psi_{r,l,m}^{(l)} \rangle = \delta_{j,l}\delta_{k,m}$ for all $j, l \in \mathbb{Z}$ and $k, m \in \mathbb{N}_0$, where both $\delta_{j,l}$ and $\delta_{k,m}$ are the Kronecker delta functions.

Lemma 3.1. *Let a function f belonging to dense subset $\mathscr{S}(\mathbb{K})$ of Sobolev space $H^s(\mathbb{K})$ and a set $\{v_j : j = 1, 2, 3, ...\}$ is the system of vectors in $H^s(\mathbb{K})$ such that*

$$||f||^2_{H^s(\mathbb{K})} = \sum_{j=1}^{\infty} |\langle f, v_j\rangle|^2, \tag{3.1}$$

then equality (3.1) holds for every $f \in H^s(\mathbb{K})$ (see [16]).

Lemma 3.2. *Let a set $\{v_j : j = 1, 2, 3, ...\}$ is the system of vectors in $H^s(\mathbb{K})$ satisfying equality (3.1). If $||v_j|| \geq 1$ for $i = 1, 2, 3, ...,$ then $\{v_j : j = 1, 2, 3, ...\}$ is an orthonormal basis for $H^s(\mathbb{K})$ (see [16]).*

Lemma 3.3. *If $f \in \mathscr{S}(\mathbb{K})$ and $\psi_r^{(j)} \in H^s(\mathbb{K})$, then*

$$\sum_{r\in D_1}\sum_{k\in\mathbb{N}_0} |\langle f, \psi^{(j)}_{r,j,k}\rangle_{H^s(\mathbb{K})}|^2 = \sum_{r\in D_1}\int_{\mathbb{K}} \hat{\nu}^s(\xi)\overline{\hat{f}(\xi)}\hat{\psi}_r^{(j)}(\mathfrak{s}^{-j}\xi)\{\sum_{k\in\mathbb{N}_0}\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(k)) \times\hat{f}(\xi+\mathfrak{s}^{-j}\omega(k))\overline{\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi+\omega(k))}\}d\xi. \tag{3.2}$$

Proof. See [4]. □

Lemma 3.4. *Let f be in $\mathscr{S}(\mathbb{K})$ and $\psi_r^{(j)} \in H^s(\mathbb{K})$. If ess sup $\left\{\hat{\nu}^s(\xi)\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|^2 : \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\right\} < \infty$, then*

$$\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f, \psi^{(j)}_{r,j,k}\rangle|^2 = \int_{\mathbb{K}}\hat{\nu}^{2s}(\xi)|\hat{f}(\xi)|^2\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|^2 d\xi + T_2, \tag{3.3}$$

where

$$T_2 = \sum_{r\in D_1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\overline{\hat{f}(\xi)}\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)\Big[\sum_{l\in\mathbb{N}}\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(l))\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l)) \times\overline{\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi+\omega(l))}\Big]d\xi. \tag{3.4}$$

Furthermore, the iterated series in (3.4) is absolutely convergent.

Proof. See [4]. □

Theorem 3.1. *Let $\{\psi_r^{(j)}\}_{j\in\mathbb{Z},r\in D_1} \subseteq H^s(\mathbb{K})$, with $\|\psi_r^{(j)}\|_{H^s(\mathbb{K})} = 1, j \in \mathbb{Z}, r \in D_1$. Then the collection $\{\psi_r^{(j)}\}_{j\in\mathbb{Z},r\in D_1}$ is an orthonormal wavelet families in $H^s(\mathbb{K})$ if and only if*

$$\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)| = \nu^{-s}(\xi) \;\; for \;\; a.e.\xi \in \mathbb{K}, \tag{3.5}$$

and for every $s \in \mathbb{N}_0\backslash q\mathbb{N}_0$,

$$\sum_{r\in D_1}\sum_{i=0}^{\infty}\overline{\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)}\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\mathfrak{s}^j\xi+\omega(s))) = 0, \tag{3.6}$$

for a.e. $\xi \in \mathbb{K}$ and for all $s \in \mathbb{N}_0\backslash q\mathbb{N}_0$, where $q\mathbb{N}_0 = \{qk : k \in \mathbb{N}_0\}$.

Proof. Suppose that equalities (3.5) and (3.6) holds. Then by Lemma 3.1 and 3.2, it is sufficient to show that

$$\sum_{r\in D_1}\sum_{k\in\mathbb{N}_0}\sum_{j\in\mathbb{Z}}|\langle f,\psi^{(j)}_{r,j,k}\rangle_{H^s(\mathbb{K})}|^2=\|f\|^2_{H^s(\mathbb{K})} \tag{3.7}$$

hold for each f belonging to $\mathscr{S}(\mathbb{K})$.

We have

$$\begin{aligned} T &= \sum_{r\in D_1}\sum_{k\in\mathbb{N}_0}\sum_{j\in\mathbb{Z}}|\langle f,\psi^{(j)}_{r,j,k}\rangle_{H^s(\mathbb{K})}|^2\\ &= \sum_{r\in D_1}\sum_{k\in\mathbb{N}_0}\sum_{j\in\mathbb{Z}}\Big|\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)q^{\frac{j}{2}}\overline{\hat{\psi}^{(j)}_r(\mathfrak{s}^{-j}\xi-\omega(k))}d\xi\Big|^2\\ &= \sum_{r\in D_1,k\in\mathbb{N}_0,j\in\mathbb{Z}}\Big|\int_{\mathbb{K}}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)\hat{f}(\mathfrak{s}^{-j}\xi)q^{\frac{j}{2}}\hat{\psi}^{(j)}_r(\xi)\chi_k(\xi)d\xi\Big|^2\\ &= \sum_{r\in D_1,k\in\mathbb{N}_0,j\in\mathbb{Z}}q^j\int_{\mathbb{K}}\{\sum_{l=0}^{\infty}\int_{\mathfrak{D}}\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(l)))\hat{f}(\mathfrak{s}^{-j}(\xi+\omega(l)))\chi_k(\xi+\omega(l))\\ &\qquad\times\overline{\hat{\psi}^{(j)}_r(\xi+\omega(l))}d\xi\}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)\overline{\hat{f}(\mathfrak{s}^{-j}\xi)}\hat{\psi}^{(j)}_r(\xi)\bar{\chi}_k(\xi)d\xi. \end{aligned}$$

Since $f\in\mathscr{S}(\mathbb{K})$ so that $\sum_{l=0}^{\infty}$ contains only finite non-zero terms and $\chi_k(\omega(l))=1$ for $k,l\in\mathbb{N}_0$, then we get

$$\begin{aligned} T &= \sum_{r\in D_1,k\in\mathbb{N}_0,j\in\mathbb{Z}}q^j\int_{\mathbb{K}}\{\int_{\mathfrak{D}}\left(\sum_{l=0}^{\infty}\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(l)))\hat{f}(\mathfrak{s}^{-j}(\xi+\omega(l)))\overline{\hat{\psi}^{(j)}_r(\xi+\omega(l))}\right)\\ &\qquad\times\chi_k(\xi)d\xi\}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)\overline{\hat{f}(\mathfrak{s}^{-j}\xi)}\hat{\psi}^{(j)}_r(\xi)\bar{\chi}_k(\xi)d\xi. \end{aligned} \tag{3.8}$$

By convergence theorem of Fourier series on $\mathfrak{D}$, we obtain

$$\begin{aligned} T &= \sum_{r\in D_1,j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\overline{\hat{f}(\xi)}\hat{\psi}^{(j)}_r(\xi)\left\{\sum_{l=0}^{\infty}\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(l))\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l))\overline{\hat{\psi}^{(j)}_r(\mathfrak{s}^j\xi+\omega(l))}\right\}d\xi\\ &= \sum_{r\in D_1,j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^{2s}(\xi)|\hat{f}(\xi)|^2|\hat{\psi}^{(j)}_r(\xi)|^2d\xi+\int_{\mathbb{K}}\sum_{r\in D_1,j\in\mathbb{Z}}\sum_{l=1}^{\infty}\hat{\nu}^s(\xi)\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(l))\\ &\qquad\times\hat{f}(\xi)\overline{\hat{\psi}^{(j)}_r(\mathfrak{s}^j\xi)}\overline{\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l))}\hat{\psi}^{(j)}_r(\mathfrak{s}^j\xi+\omega(l))d\xi \qquad (3.9)\\ &= T_1+T_2, \end{aligned}$$

where

$$T_1=\int_{\mathbb{K}}\hat{\nu}^{2s}(\xi)|\hat{f}(\xi)|^2\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}|\hat{\psi}^{(j)}_r(\mathfrak{s}^j\xi)|^2d\xi, \tag{3.10}$$

and

$$T_2=\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\overline{\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)}\sum_{l=1}^{\infty}\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(l))\overline{\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l))}\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi+\omega(l))d\xi. \tag{3.11}$$

Now by equality (3.5) and (3.10), we get

$$T_1=\int_{\mathbb{K}}\hat{\nu}^s(\xi)|\hat{f}(\xi)|^2d\xi=\|f\|^2_{H^s(\mathbb{K})}. \tag{3.12}$$

Now we need only to show that $T_2=0$. Since T_2 is absolutely convergent. In the expression of T_2, $l\in\mathbb{N}$,for given l there exist $i\in\mathbb{N}_0$ and a unique $m\in q\mathbb{N}_0+\tilde{Q}$ such that $l=q^im$ where $\tilde{Q}=\{1,2,3,...q-1\}$. Thus by Definition 2, we have $\{\omega(l)\}_{l\in\mathbb{N}}=\{\mathfrak{s}^i\omega(m)\}_{(i,m)\in\mathbb{N}_0\times\{q\mathbb{N}_0+\tilde{Q}\}}$. Since the series in (3.11) is absolutely convergent. Therefore equation (3.11) become

$$\begin{aligned}T_2 &= \sum_{r\in D_1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\overline{\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)}\{\sum_{i\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\hat{\nu}^s(\xi+\mathfrak{s}^{-j-i}\omega(m))\overline{\hat{f}(\xi+\mathfrak{s}^{-j-i}\omega(m))}\\ &\qquad\times\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi+\mathfrak{s}^{-i}\omega(m))\}d\xi\\ &= \int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\{\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(m))\overline{\hat{f}(\xi+\mathfrak{s}^{-j}\omega(m))}\\ &\qquad\times\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)}\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi+\mathfrak{s}^{-i}\omega(m))\}d\xi.\end{aligned}$$

Now using equation (3.6), we get $T_2=0$.

Conversely, suppose that $\{\psi_r^{(j)}:j\in\mathbb{Z},r\in D_1\}$ is the orthonormal wavelet family in $H^s(\mathbb{K})$. Then we show that both equations (3.5) and (3.6) are satisfied.
Since $\{\hat{\psi}_r^{(j)}\}_{j\in\mathbb{Z},r\in D_1}$ is orthonormal wavelet. Therefore for all $f\in\mathscr{S}(\mathbb{K})$, we have

$$\sum_{r\in D_1}\sum_{k\in\mathbb{N}_0}\sum_{j\in\mathbb{Z}}|\langle f,\psi_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}|^2=\|f\|^2_{H^s(\mathbb{K})}=T_1+T_2 \tag{3.13}$$

Using the fact that $|T_2|<\infty$, we have

$$T_1=\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\hat{\nu}^{2s}(\xi)|\hat{f}(\xi)|^2|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|^2d\xi<\infty. \tag{3.14}$$

By the assumption of $f\in\mathscr{S}(\mathbb{K})$,we can obtain that $\hat{\nu}^s(\xi)|\hat{f}(\xi)|$ is a bounded function. So $\hat{\nu}^s(\xi)\sum_{r\in D_1}\sum_{j\in\mathbb{Z}}|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|^2$ is localy integrable in $\mathbb{K}\backslash\bigcup_{j\in\mathbb{Z}}E_j^c$, where E_j is the set of regular points of $|\psi_r^{(j)}(\mathfrak{s}^j\xi)|$ (see[23], pp. 29). Therefore each $\xi_0\in\mathbb{K}\backslash\bigcup_{j\in\mathbb{Z}}E_j^c$, we consider

$$\hat{g}(\xi)=q^{\frac{M}{2}}Q_m(\xi-\xi_0)\hat{\nu}^{\frac{-s}{2}}(\xi) \tag{3.15}$$

where $f = g$ and $Q_m(\xi - \xi_0)$ is the characteristic function of $\xi_0 + \mathfrak{P}^M$. Then it follows that $\bar{\hat{f}}(\xi)\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l)) = 0$ for $l \in \mathbb{N}$. Since ξ and $\xi+\mathfrak{s}^{-j}\omega(l)$ can not be in $\xi_0+\mathfrak{P}^M$ simultaneously, and hence $\|g\|_{H^s(\mathbb{K})} = 1$. Further more, we have

$$1 = \|g\|^2_{H^s(\mathbb{K})} = \int_{\xi_0+\mathfrak{P}^M} \hat{\nu}^s(\xi) \sum_{r\in D_1} \sum_{j\in\mathbb{Z}} q^M |\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|^2 d\xi + T_2 \tag{3.16}$$

and let $M \to \infty$, we obtain

$$1 = \hat{\nu}^s(\xi_0) \sum_{r\in D_1} \sum_{j\in\mathbb{Z}} |\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi_0)|^2 + lim_{M\to\infty} T_2. \tag{3.17}$$

Now we need to show that $T_2 \to 0$ as $M \to \infty$ .Now ,we estimate T_2 as follows

$$\begin{aligned} |T_2| &\le \sum_{r\in D_1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{f}(\xi)||\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi)|\{\sum_{l=1}^{\infty} \hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(l))|\hat{f}(\xi+\mathfrak{s}^{-j}\omega(l))| \\ &\qquad \times|\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi+\omega(l))|\}d\xi \\ &= \sum_{r\in D_1} \sum_{j\in\mathbb{Z}} \sum_{l=1}^{\infty} q^j \int_{\mathbb{K}} \hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)||\hat{\psi}_r^{(j)}(\xi)|\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(l)))|\hat{f}(\mathfrak{s}^{-j}(\xi+\omega(l)))| \\ &\qquad \times|\hat{\psi}_r^{(j)}(\xi+\omega(l))|d\xi \\ &\le \sum_{r\in D_1} \sum_{j\in\mathbb{Z}} \sum_{l=1}^{\infty} q^j \int_{\mathbb{K}} \hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)|\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi+\omega(l)))|\hat{f}(\mathfrak{s}^{-j}(\xi+\omega(l)))| \\ &\qquad \times\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j)}(\xi)|^2 d\xi. \end{aligned} \tag{3.18}$$

Since $\omega(l) \neq 0 (l \in \mathbb{N})$ and $f \in \mathscr{S}(\mathbb{K})$, there exist a constant $J > 0$ such that $\hat{f}(\mathfrak{s}^{-j}\xi)\hat{f}(\mathfrak{s}^{-j}\xi + \mathfrak{s}^{-j}\omega(l)) = 0$ for $|j| > J$.

Therefore series in equation (3.18) have only a finite number of terms are nonzero so there exist a constant K such that

$$\begin{aligned} |T_2| &\le K\|\hat{\nu}^s(\xi)\hat{f}(\xi)\|^2_\infty \|\hat{\psi}_r^{(j)}(\xi)\|^2_{H^s(\mathbb{K})} \\ &= Kq^m\|\hat{\psi}_r^{(j)}(\xi)\|^2_{H^s(\mathbb{K})} \end{aligned}$$

and implies that $T \to 0$ as $M \to \infty$. Hence equation (3.17) becomes

$$\sum_{r\in D_1} \sum_{j\in\mathbb{Z}} |\hat{\psi}_r^{(j)}(\mathfrak{s}^j\xi_0)| = \hat{\nu}^{-s}(\xi_0). \tag{3.19}$$

Now we show that if

$$\sum_{r\in D_1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{r,j,k}^{(j)} \rangle_{H^s(\mathbb{K})}|^2 = \|f\|^2_{H^s(\mathbb{K})} \tag{3.20}$$

holds for all $f \in \mathscr{S}(\mathbb{K})$, then the equation (3.6) holds. Now by using (3.7),(3.9) and (3.11) we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(m))\overline{\hat{f}(\xi+\mathfrak{s}^{-j}\omega(m))}$$
$$\times\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(j-i)}}(\mathfrak{s}^{j-i}\xi)\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\mathfrak{s}^j\xi+\omega(m)))\}d\xi=0. \quad (3.21)$$

By Polarisation, we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(m))\overline{\hat{g}(\xi+\mathfrak{s}^{-j}\omega(m))}$$
$$\times\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(j-i)}}(\mathfrak{s}^{j-i}\xi)\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\mathfrak{s}^j\xi+\omega(m)))\}d\xi=0. \quad (3.22)$$

Let us fix $k_0 \in q\mathbb{N}_0+\tilde{Q}$ and $\xi_0 \in \mathbb{K}\backslash\bigcup_{j\in\mathbb{Z}}E_j^c$ such that neither $\xi_0 \neq 0$ nor $\xi_0+\omega(k_0)\neq 0$. Consider $\hat{f}(\xi) = q^{\frac{M}{2}}\phi(\xi-\xi_0)\hat{\nu}^{\frac{-s}{2}}(\xi)$ and $\hat{g}(\xi) = \hat{f}(\xi-\omega(k_0))\hat{\nu}^{\frac{-s}{2}}(\xi-\omega(k_0))$. We obtain $\hat{f}(\xi)\hat{g}(\xi+\omega(k_0)) = q^{\frac{M}{2}}\phi_M(\xi-\xi_0)\hat{\nu}^{-s}(\xi)\hat{\nu}^{-s}(\xi-\omega(k_0))$.

Now, equation (3.22) can be written as

$$0=q^M\int_{\xi_0+\mathfrak{P}^M}\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(-i)}}(\mathfrak{s}^{-i}\xi)\hat{\psi}_r^{(-i)}(\mathfrak{s}^{-i}\xi+\mathfrak{s}^{-i}\omega(k_0))d\xi+I, \quad (3.23)$$

where

$$I = \sum_{\substack{j\in\mathbb{Z}\\(j,m)\neq(0,k_0)}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(m))\overline{\hat{g}(\xi+\mathfrak{s}^{-j}\omega(m))}$$
$$\times\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(j-i)}}(\mathfrak{s}^{j-i}\xi)\times\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi+\mathfrak{s}^{-i}\omega(m))\}d\xi. \quad (3.24)$$

Since the first term in (3.23) is zero.So we only to show that $I\to 0$ as $M\to\infty$. Since $\omega(m)\neq 0 (m\in\mathbb{N})$ and $f,g\in\mathscr{S}(\mathbb{K})$, there exist a constant $J_0>0$ such that

$$\hat{f}(\xi)\overline{\hat{g}(\xi+\omega(m))}=0 \ \forall \ j>J_0. \quad (3.25)$$

Therefore we have

$$I = \sum_{j\leq J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\hat{f}(\xi)\hat{\nu}^s(\xi+\mathfrak{s}^{-j}\omega(m))\overline{\hat{g}(\xi+\mathfrak{s}^{-j}\omega(m))}\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\overline{\hat{\psi}_r^{(j-i)}}(\mathfrak{s}^{j-i}\xi)$$
$$\times\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi+\mathfrak{s}^{-i}\omega(m))\}d\xi,$$
$$|I| \leq \sum_{j\leq J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}q^j\int_{\mathbb{K}}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)|\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))|$$

$$\begin{aligned}
& \times\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)||\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\xi+\omega(m)))|\}d\xi \\
\leq & \sum_{j\leq J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j\int_{\mathbb{K}}\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)|\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))|\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0} \\
& \hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2+\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\xi+\omega(m)))|^2\}d\xi \\
= & I_1+I_2, \qquad (3.26)
\end{aligned}$$

where

$$\begin{aligned}
I_1 = & \sum_{j\leq J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j\int_{\mathbb{K}}\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)|\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))| \\
& \times\left\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2\right\}d\xi. \qquad (3.27)
\end{aligned}$$

We claim for fixed j, $\int_{\mathbb{K}}\left(\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2\right)d\xi<\infty$.

$$\begin{aligned}
\int_{\mathbb{K}}\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2d\xi &= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\int_{\mathbb{K}}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2d\xi \\
&= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\int_{\mathbb{K}}\hat{\nu}^s(\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)|^2q^{-j}d\xi \\
&= q^{-j}\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\frac{q^{i-j}}{q^{i-j}}|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)|^2d\xi \\
&= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\int_{\mathbb{K}}\hat{\nu}^s(\xi)\frac{q^{i-j}}{q^{i}}|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)|^2d\xi \\
&= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\frac{1}{q^i}\int_{\mathbb{K}}\hat{\nu}^s(\xi)|q^{\frac{(i-j)}{2}}\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)|^2d\xi \\
&= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\frac{1}{q^i}\|q^{\frac{(i-j)}{2}}\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{j-i}\xi)\|_{H^s(\mathbb{K})} \\
&= \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\frac{1}{q^i}\times 1 \quad <\infty,
\end{aligned}$$

and

$$\begin{aligned}
I_2 = & \sum_{j\leq J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j\int_{\mathbb{K}}\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}\xi)|\hat{f}(\mathfrak{s}^{-j}\xi)|\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))| \\
& \times\left\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}(\xi+\omega(m)))|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}(\xi+\omega(m)))|^2\right\}d\xi
\end{aligned}$$

$$= \sum_{j\le J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j \int_{\mathbb{K}} \hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi-\omega(m)))|\hat{f}(\mathfrak{s}^{-j}(\xi-\omega(m)))|\hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{j}\xi)|\hat{g}(\mathfrak{s}^{-j}\xi)| \times\left\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2\right\}d\xi.$$

Thus I_2 has same form as I_1 with the roles of $\hat{f}$ and $\hat{g}$ interchanged. Now we only to show that $I_1\to 0$ as $M\to\infty$. We can write equation (3.27)

$$I_1 = \sum_{j\le J_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j q^{\frac{M}{2}} \int_{\mathfrak{s}^{-j}\xi_0+\mathfrak{P}^{-j+M}} \hat{\nu}^{\frac{s}{2}}(\mathfrak{s}^{-j}(\xi+\omega(m))|\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))| \times\left\{\sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2\right\}d\xi. \quad (3.28)$$

Now ,if $\hat{g}(\mathfrak{s}^{-j}(\xi+\omega(m)))\neq 0$ then we must obtain $\mathfrak{s}^{-j}\xi+\mathfrak{s}^{-j}\omega(m)\in\xi+\mathfrak{P}^M+\omega(k_0)$ and $|\mathfrak{s}^{-j}\omega(m)|\le q^{-M}$, and hence $|\omega(m)|\le q^{-M-j}$. Then the summation index is bounded by q^{-M-j}, so using this we get

$$\begin{aligned} I_1 &\le \sum_{j\le J_0} q^j q^{\frac{M}{2}} \int_{\mathfrak{s}^{-j}\xi_0+\mathfrak{P}^{-j+M}} \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2 q^{\frac{M}{2}}q^{-M-j}d\xi \\ &\le \sum_{j\le J_0}\int_{\mathfrak{s}^{-j}\xi_0+\mathfrak{P}^{-j+M}} \sum_{r\in D_1}\sum_{i\in\mathbb{N}_0}\hat{\nu}^s(\mathfrak{s}^{-j}\xi)|\hat{\psi}_r^{(j-i)}(\mathfrak{s}^{-i}\xi)|^2 d\xi. \end{aligned}$$

For given ξ_0, for any $\epsilon>0$, choose $k<0$ enough small satisfies the following two inequalities $q^k<|\xi|=q^\rho$ such that $k+\rho<0$, and

$$\int_{\mathfrak{P}^{-k-\rho}} \hat{\nu}^s(\mathfrak{s}^{-k}\xi)|\hat{\psi}_r^{(k-i)}(\mathfrak{s}^{-i}\xi)|^2 d\xi<\epsilon. \quad (3.29)$$

We have $\mathfrak{s}^{-j}\xi_0+\mathfrak{P}^{-j+M}\subset\mathfrak{P}^{-k-\rho}$ for all $j\le k$. Since $|\mathfrak{s}^{-j}\xi_0|\le q^kq^q$ and $\mathfrak{P}^{-j+M}\subseteq\mathfrak{P}^{-k-\rho}$ Hence $I_1\to 0$ as $M\to\infty$. □

Example 1. Assume that the function
$\psi_r^{(j)}=\frac{q^j}{q-1}\{\nu^{\frac{-s}{2}}(\mathfrak{s}^{-j}x)\}*(\chi_{\mathfrak{D}}(x)-\phi(x))$ for all $j\in\mathbb{Z},\ r=1,2,3,4,....(q-1)$,
where $\chi_{\mathfrak{D}}$ is the characteristic function on $\mathfrak{D}$ and $\phi(x)=\begin{cases} q^{-1}, & \text{if } x\in\mathfrak{P}^{-1},\\ 0, & \text{otherwise .}\end{cases}$

Thus, $$\hat{\psi}_r^{(j)}(\xi)=\begin{cases}\frac{1}{q-1}\hat{\nu}^{\frac{-s}{2}}(\mathfrak{s}^j\xi), & \text{if } \xi\in\mathfrak{P}^{-1}\backslash\mathfrak{D},\\ 0, & \text{otherwise .}\end{cases}$$

For $\xi\neq 0$, we note that

$$\sum_{r=1}^{q-1}\sum_{j\in\mathbb{Z}}|\hat{\psi}_r^{(j)}(\mathfrak{s}^{-j}\xi)|^2=\frac{1}{q-1}\sum_{r=1}^{q-1}\hat{\nu}^{-s}(\xi)=\hat{\nu}^{-s}(\xi).$$

Now, it is to see that both $\mathfrak{s}^{-j}\xi$ and $\mathfrak{s}^{-j}(\xi + w(s'))$ are not lies in $\mathfrak{P}^{-1} \setminus \mathfrak{D}$ simultaneously. Therefore, equation (3.6) holds. Hence the system $\{\psi_r^{(j)} : j \in \mathbb{Z}, r = 1, 2, 3, ...q - 1\}$ forms orthonormal wavelet of $H^s(\mathbb{K})$.

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