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RAMANUJAN'S REMARKABLE SUMMATION FORMULA AS A 2-PARAMETER GENERALIZATION OF THE QUINTUPLE PRODUCT IDENTITY

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Abstract. It is well known that 'Ramanujan's remarkable summation formula' unifies and generalizes the q-binomial theorem and the triple product identity and has numerous applications. In this note we will demonstrate how, after a suitable transformation of the series side, it can be looked upon as a 2-parameter generalization of the quintuple product identity also.

1. Introduction

One of the famous identities of Ramanujan is his ${}_1\Psi_1$ summation:
If $|\beta q| < |z| < \frac{1}{|\alpha q|}$ and $|q| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{\left(\frac{1}{\alpha}; q^2\right)_n (-\alpha q z)^n}{(\beta q^2; q^2)_n} = \frac{(-qz; q^2)_\infty \left(\frac{-q}{z}; q^2\right)_\infty (q^2; q^2)_\infty (\alpha\beta q^2; q^2)_\infty}{(-\alpha q z; q^2)_\infty \left(\frac{-\beta q}{z}; q^2\right)_\infty (\alpha q^2; q^2)_\infty (\beta q^2; q^2)_\infty}, \tag{1.1}$$

where

$$(a)_0 := (a; q)_0 = 1,$$

$$(a)_\infty := (a; q)_\infty := \prod_0^\infty (1 - aq^n),$$

and

$$(a)_n := \frac{(a)_\infty}{(aq^n)_\infty}.$$

This appears as Entry 17 of Chapter 16 of his Second Notebook [4, p. 196]. A number of proofs and good many applications of (1.1) have been discovered since the time of G. H. Hardy [3, pp. 222, 223] who brought it to light. In one of his books B. C. Berndt [1, p. 32] has referred to 11 papers dedicated to proofs, 8 papers containing varied type of applications and several generalizations and multidimensional analogues of (1.1). One

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of the striking aspects of (1.1) is that it is a common generalization of the well known Jacobi's triple product identity and Euler-Cauchy q-binomial theorem.

The following identity, known as the quintuple product identity,

$$(-x)_\infty\left(\frac{-q}{x}\right)_\infty(q)_\infty(x^2q;q^2)_\infty\left(\frac{q}{x^2};q^2\right)_\infty=\sum_{-\infty}^{\infty}(-1)^nq^{\frac{3n^2+n}{2}}(x^{3n+1}+x^{-3n}), \tag{1.2}$$

for $|q|<1$, and $x\neq 0$ has been discovered many times in the past. One may see, for instance, Berndt's work [1] for an account of several proofs and applications.

The purpose of this note is to bring out another feature of the remarkable formula (1.1). Aided by Rogers - Fine identity [2, p. 15], we will infact show how (1.1), after suitably transforming its series side, can be regarded as a 2-parameter generalization of the quintuple product identity (1.2).

2. Main Result

Theorem 2.1. *If* $|q|<1$ *and* $|\beta q|^{1/2}<|x|<\frac{1}{|\alpha q|^{1/2}}$, *then*

$$\begin{aligned}&\frac{(x^2)_\infty\left(\frac{q}{x^2}\right)_\infty(q)_\infty(\alpha\beta q)_\infty}{(\alpha x^2)_\infty\left(\frac{\beta q}{x^2}\right)_\infty(\alpha q)_\infty(\beta q)_\infty}\\&=\sum_0^\infty\frac{(-1)^n\left(\frac{\alpha}{q^{n-1}}\right)_n\left(\frac{x^2}{\beta}\right)_n(1-x^2q^{2n})\left(\frac{\beta}{x}\right)^nq^{\frac{3n^2-n}{2}}x^{3n}}{(\beta q)_n(\alpha x^2)_{n+1}}\\&-\sum_1^\infty\frac{(-1)^n\left(\frac{1}{\beta}\right)_n\left(\frac{\alpha x^2}{q^{n-1}}\right)_{n-1}\left(\frac{\beta}{x}\right)^n\left(1-\frac{q^{2n}}{x^2}\right)q^{\frac{3n^2-n}{2}}x^{-3n+2}}{(\alpha q)_n\left(\frac{\beta q}{x^2}\right)_n}.\end{aligned} \tag{2.1}$$

Proof. On replacing q by $q^{1/2}$ and z by $-x^2q^{-1/2}$ the remarkable formula (1.1) can be written as

$$\frac{(x^2)_\infty\left(\frac{q}{x^2}\right)_\infty(q)_\infty(\alpha\beta q)_\infty}{(\alpha x^2)_\infty\left(\frac{\beta q}{x^2}\right)_\infty(\alpha q)_\infty(\beta q)_\infty}=\sum_0^\infty\frac{\left(\frac{1}{\alpha}\right)_n(\alpha x^2)^n}{(\beta q)_n}+\sum_1^\infty\frac{\left(\frac{1}{\beta}\right)_n\left(\frac{\beta q}{x^2}\right)^n}{(\alpha q)_n}. \tag{2.2}$$

Denoting the first sum on the right side of (2.2) by $F(\alpha,\beta,x)$ and applying Rogers-Fine identity [2, p. 15], namely

$$\sum_0^\infty\frac{(a)_n\tau^n}{(b)_n}=\sum_0^\infty\frac{(a)_n\left(\frac{a\tau q}{b}\right)_nb^n\tau^nq^{n^2-n}(1-a\tau q^{2n})}{(b)_n(\tau)_{n+1}},$$

(with $|q| < 1$ and $|\tau| < 1$) we have the transform

$$F(\alpha,\beta,x) = \sum_{0}^{\infty} \frac{\left(\frac{1}{\alpha}\right)_n \left(\frac{x^2}{\beta}\right)_n \alpha^n \beta^n q^{n^2} x^{2n}(1 - x^2 q^{2n})}{(\beta q)_n (\alpha x^2)_{n+1}} \tag{2.3}$$

$$= \sum_{0}^{\infty} \frac{(-1)^n \left(\frac{\alpha}{q^{n-1}}\right)_n \left(\frac{x^2}{\beta}\right)_n (1 - x^2 q^{2n}) \left(\frac{\beta}{x}\right)^n q^{\frac{3n^2-n}{2}} x^{3n}}{(\beta q)_n (\alpha x^2)_{n+1}}. \tag{2.4}$$

Similarly, the second sum on the right side of (2.2) can be transformed as

$$\frac{q(\beta-1)}{(1-\alpha q)x^2} F\left(\frac{\beta}{q}, \alpha q, \frac{q}{x}\right)$$

$$= -\sum_{1}^{\infty} \frac{(-1)^n \left(\frac{1}{\beta}\right)_n \left(\frac{\alpha x^2}{q^{n-1}}\right)_{n-1} \left(\frac{\beta}{x}\right)^n q^{\frac{3n^2-n}{2}} x^{-3n+2} \left(1 - \frac{q^{2n}}{x^2}\right)}{(\alpha q)_n \left(\frac{\beta q}{x^2}\right)_n}. \tag{2.5}$$

Using (2.4) and (2.5) in (2.2), we have (2.1).

Remark.

(i) Putting $\alpha = 0$, and $\beta = x$, in (2.1), we obtain

$$\frac{(x^2)_\infty \left(\frac{q}{x^2}\right)_\infty (q)_\infty}{\left(\frac{q}{x}\right)_\infty (xq)_\infty} = \sum_{-\infty}^{\infty} (-1)^n (1 + xq^n) q^{\frac{3n^2-n}{2}} x^{3n}$$

$$= \sum_{-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} (x^{3n+1} + x^{3n} q^{-n}),$$

which is same as the quintuple product identity (1.2).

(ii) The identity (2.1) contains other elegant special cases also namely the case $\beta = x$ or $\alpha = 0$.

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