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Applications of Krasnoselskii-Dhage Type Fixed-Point Theorems to Fractional Hybrid Differential Equations

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Abstract. In this paper, we prove the existence of a solution of a fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators by utilizing a new version of Kransoselskii-Dhage type fixed-point theorem obtained in [13]. Moreover, we provide an example to support our result.

1 Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the vast number of applications in the existence theory of functional, fractional, differential, partial differential, and integral equations; see [22]-[25]. The differential equations involving Riemann-Liouville differential operators of fractional order $0 < \alpha < 1$ are very important in modeling several physical phenomena (see, for instance [19, 20]) and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations. In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. The importance of the investigations of hybrid differential equations lies in the fact that they include several dynamic systems as special cases. This class of hybrid differential equations includes the perturbations of original differential equations in different ways. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [12]-[18]. In 2013, Dhage [12] and Dhage and Lakshmikantham [14] proposed an important Krasnoselskii-type fixed-point theorems and applied them the following first-order hybrid differential equation with linear perturbations of first type:

$$\begin{cases} \frac{d}{dx}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad \text{a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $J=[t_0,t_0+a]$, for some fixed $t_0\in\mathbb{R}$, $a>0$ and $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g\in\mathcal{C}(J\times\mathbb{R},\mathbb{R})$. In the same year, Dhage and Jadhav [15] studied the existence of solution for hybrid differential equation with linear perturbations of second type:

$$\begin{cases} \frac{d}{dx}\Big[x(t)-f(t,x(t))\Big]=g(t,x(t)), & t\in J,\\ x(t_0)=x_0\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $f,g\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$. They established the existence and uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison result. In [25], Lu et al. proved an existence theorem for fractional hybrid differential equations under the φ-Lipschitz contraction condition and applying this theorem they develop the theory of fractional hybrid differential equations with linear perturbations of second type involving Riemann-Liouville differential operators of order $0<q<1$:

$$\begin{cases} D^q\Big[x(t)-f(t,x(t))\Big]=g(t,x(t)), & \text{a.e. } t\in J,\\ x(t_0)=x_0\in\mathbb{R}, \end{cases} \tag{1.3}$$

where $f,g\in\mathcal{C}(J\times\mathbb{R},\mathbb{R})$. Recently, Dhage [13] developed a new version of Kransoselskii-type fixed-point theorem under a nonlinear $\mathcal{D}$- contraction condition. We shall give the precise formulation of this theorem in the next section.

The purpose of this paper is to study the following fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators of orders $0<\alpha<1$ and $\beta>0$:

$$\begin{cases} D^\alpha[x(t)-f(t,x(t))]=g(t,x(t),I^\beta(x(t))), & \text{a.e. } t\in J,\ \beta>0,\\ x(t_0)=x_0, \end{cases} \tag{1.4}$$

where $J=[t_0,t_0+a]$, for some fixed $t_0\in\mathbb{R}$ and $a>0$ and $f\in C(J\times\mathbb{R},\mathbb{R})$, $g\in\mathcal{C}(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. We show the existence of a solution of equation (1.4) under a certain conditions of f and g. Moreover, we provide an example to illustrate the hypotheses and the abstract result of this paper.

2 Formulation of Main Theorem

To formulate our main theorem we need the following hypotheses.

2.0.1 Hypotheses for FHDE (1.4)

We assume:

(H1) The function $F_t : \mathbb{R} \to \mathbb{R}$ defined as $F_t(x) := x - f(t, x)$ is strictly increasing in $\mathbb{R}$ for all $t \in J$.

(H2) The function $f(t, \cdot)$ satisfies the following weak contraction condition

$$|f(t, x) - f(t, y)| \leq \tanh \big(|x - y|\big)$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(H3) There exists a continuous function $h \in C(J, \mathbb{R})$ such that

$$|g(t, x, y)| \leq h(t)$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

Now, we are in a position to formulate our main theorem.

Theorem 2.1. *Assume that hypotheses (H1)-(H3) hold. Then FHDE (1.4) has a solution in* $C(J, \mathbb{R})$.

3 Preliminaries

The following are discussions of some of the concepts we will need. Let $C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and let $\mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \to g(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$,

(ii) the map $x \to g(t, x, y)$ is continuous for each $x \in \mathbb{R}$,

(iii) the map $y \to g(t, x, y)$ is continuous for each $y \in \mathbb{R}$.

The class $\mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R} \times \mathbb{R}$, which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

Definition 1 ([21])**.** The form of the Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} f(s) ds.$$

Definition 2 ([21])**.** The Riemann-Liouville derivative of fractional order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-s)^{m-\alpha-1} f(s) ds,$$

where $m = [\alpha] + 1$.

The following lemma will be used in the sequel.

Lemma 3.1 ([21, 26])**.** *Let $0 < \alpha < 1$ and $f \in L^1(0,1)$. Then*

(1) the equality $D^{\alpha} I^{\alpha} f(x) = f(x)$ holds;

(2) the equality

$$I^{\alpha} D^{\alpha} f(x) = f(x) - \frac{[D^{\alpha-1} f(x)]_{x=0}}{\Gamma(\alpha)} x^{\alpha-1}$$

holds for almost everywhere on J.

The following lemma is useful in the proof of main theorem.

Lemma 3.2 ([25])**.** *Assume that hypothesis (H1) holds. Then, for any $y \in C(J, \mathbb{R})$ and $\alpha \in (0,1)$ the function $x \in C(J, \mathbb{R})$ is a solution of FHDE*

$$D^{\alpha}[x(t) - f(t, x(t))] = y(t), \ t \in J, \tag{3.1}$$

with the initial condition $x(t_0) = x_0$ if and only if $x(t)$ satisfies the hybrid integral equation (HIE)

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} y(s) ds, \ \ t \in J. \tag{3.2}$$

4 A Kransoselskii-Dhage type Theorem

Throughout this paper $X := (X, \|\cdot\|)$ stands for a real Banach space. Next, we will borrow some definitions and facts from Dhage [7]-[11] which is well-known in the metric fixed point theory and has been widely used in the literature on applications to the theory of nonlinear differential and integral equations.

Definition 3 ([7]-[11])**.** An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Definition 4 ([7]-[11])**.** An operator $A : X \to X$ is called a nonlinear $\mathcal{D}$-contraction on X if there exists a $\mathcal{D}$-function $\psi_A \in \mathfrak{D}$ such that

$$\|Ax - Ay\| \leq \psi_A\Big(\|x - y\|\Big)$$

for all elements $x, y \in X$, where $0 < \psi_A(r) < r$ for all $r > 0$.

Next, we formulate a new version of Kransoselskii-Dhage type fixed-point theorem obtained by Dhage in [13]. This theorem will be used in the proof of main theorem.

Theorem 4.1 ([13]). *Let S be a closed, convex, and bounded subset of X. Let $A : X \to X$ and $B : S \to X$ be two operators verifying the following conditions:*

(a) A is nonlinear $\mathcal{D}$-contraction,

(b) B is completely continuous and

(c) $x = Ax + By \Rightarrow x \in S$ for all $y \in S$.

Then the operator $A + B$ has a fixed point in S.

Remark 1. Note that a weaker contraction condition (so called (ψ, θ, φ)-weak contraction condition) has been successfully applied in multidimensional fixed point theorems and their applications to the system of matrices equations and nonlinear integral equations (see, for instance [1]-[6] and [27]). Our previous studies encourage us to believe that it should not be so hard to obtain a multidimension version of Kransoselskii-Dhage type fixed-point theorem under (ψ, θ, φ)-weak contraction condition.

5 Proof of Main Theorem

Proof. Let $X = C(J, \mathbb{R})$ and $\|\cdot\|$ be a uniform norm in X, that is, $\|x\| = \max_{t\in J} |x(t)|$. Obviously $(X, \|\cdot\|)$ is a Banach space. Let S be a subset of X defined as follows.

$$S = \Big\{x \in X : \ \ \|x\| \le M\Big\}$$

where $M = |x_0 - f(t_0, x_0)| + 1 + L + \frac{|J|^\alpha}{\Gamma(\alpha+1)}\|h\|$ and $L = \max_{t\in J} f(t, 0)$. Clearly, S is a closed, convex and bounded subset of the Banach space X. Let $y \in S$. In order to show the existence of a solution of equation (1.4) we consider the following generalized fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators of orders $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^\alpha[x(t) - f(t, x(t))] = g(t, y(t), I^\beta(y(t))), \ \ \text{a.e. } t \in J, \ \beta > 0, \\ x(t_0) = x_0 \end{cases} \tag{5.1}$$

where $J = [t_0, t_0 + a]$, for some fixed $t_0, a \in \mathbb{R}^+$ and $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Assume f and g satisfy hypotheses (H1)-(H3). From Lemma 3.2 it implies that the equation (5.1) is equivalent to the nonlinear hybrid integral equation (HIE)

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} g(s, y(s), I^\beta(y(s)))ds. \tag{5.2}$$

Consider the operators $A : X \to X$ and $B : S \to X$ defined as follow

$$Ax(t) = x_0 - f(t_0, x_0) + f(t, x(t)), \quad t \in J, \tag{5.3}$$

and

$$By(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds, \quad t \in J. \tag{5.4}$$

It is easy to see that HIE (5.2) can be transformed into the operator equation as

$$x(t) = Ax(t) + By(t), \quad t \in J. \tag{5.5}$$

We will show that the operators A and B satisfy all hypotheses of Theorem 4.1. First, we show that the operator A is a nonlinear $\mathcal{D}$-contraction with $\psi_A(t) = \tanh(t)$. Let $x, y \in X$. By hypothesis (H2) we have

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq \tanh(|x(t) - y(t)|) \leq \tanh(\|x - y\|).$$

Taking maximum over t from the left hand side we obtain

$$\|Ax - Ay\| \leq \tanh(\|x - y\|).$$

Consider the function

$$k(r) = (r-1)e^r + \frac{r+1}{e^r}.$$

One can easily see that $k'(r) = r(e^r - e^{-r}) > 0$ for $r > 0$. Thus the function $k(r)$ is strictly increasing and $k(r) > 0$ for $r > 0$. This implies

$$r - \psi_A(r) = r - \tanh(r) = \frac{k(r)}{e^r + e^{-r}} > 0$$

for $r > 0$. Hence the operator A is a nonlinear contraction. Next, we show that the operator B is a continuous and compact on S into X. First, we show that B is continuous on S. Let $\{y_n\}$ be a sequence in S converging to a point $y \in S$. By Lebesgue dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty} By_n &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \lim_{n\to\infty} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds = By(t)
\end{aligned}$$

for all $t \in J$. Hence the operator B is continuous. Next, we show that B is a compact operator on S. For this, it is sufficient to show that BS is a uniformly bounded and equicontinuous set in S. By hypothesis (H3), we obtain

$$\begin{aligned}|By(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds\Big|\\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big|g(s,y(s),I^{\beta}(y(s)))\Big|ds\\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}h(s)ds \le \frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\|h\|\end{aligned}$$

for all $t \in J$. Taking maximum over t from the left hand side we get

$$\|By\| \le \frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\|h\|.$$

Thus the operator B is uniformly bounded on S. Let $t_1, t_2 \in J$ with $t_1 < t_2$. For any $x \in S$, one has

$$\begin{aligned}&|Bx(t_1) - Bx(t_2)| =\\ &\quad \Big|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds\Big|\\ &\quad \le \Big|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_2-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds\Big|\\ &\quad + \Big|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_2-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}g(s,y(s),I^{\beta}(y(s)))ds\Big|\\ &\quad \le \frac{\|h\|}{\Gamma(\alpha+1)}\Big[|(t_2-t_0)^{\alpha} - (t_1-t_0)^{\alpha} + (t_2-t_1)^{\alpha}| + (t_2-t_1)^{\alpha}\Big].\end{aligned}$$

Hence, for any ε, there exists a $\delta > 0$ such that

$$|t_1 - t_2| \le \delta \quad \Rightarrow \quad |Bx(t_1) - Bx(t_2)| \le \varepsilon,$$

for all $t_1, t_2 \in J$ and $x \in S$. This shows that BS is an equicontinuous set in S. Therefore the set BS is a uniformly bounded and equicontinuous set in X, so it is compact by the Arzela-Ascoli theorem. Finally, we show that hypothesis (c) of Theorem 4.1 is satisfied. Assume $x \in X$ and

$y \in S$ satisfy the equation $x = Ax + By$. By hypothesis (H2), we have

$$\begin{aligned}
|x(t)| &= |Ax(t) - By(t)| \leq |Ax(t)| + |By(t)| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t))| + \left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds\right| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |g(s, y(s), I^{\beta}(y(s)))| ds \\
&\leq |x_0 - f(t_0, x_0)| + \tanh(\|x\|) + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\| \\
&\leq |x_0 - f(t_0, x_0)| + 1 + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\|.
\end{aligned}$$

Taking maximum over t from the left hand side we obtain

$$\|x\| \leq |x_0 - f(t_0, x_0)| + 1 + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\| = M.$$

Hence $x \in S$. Thus all hypotheses of Theorem 4.1 are satisfied and so the operator $A + B$ has a fixed point in S, that is, there exists $z^* \in S$ such that $Az^* + Bz^* = z^*$. As a result, FHDE (1.4) has a solution in S. This completes the proof of Theorem 2.1.

6 Illustrative example

Let $J = [0, 1]$. Denote by X the set of continuous and non-negative functions $f : J \to [0, \infty)$. In X consider the following fractional hybrid differential equation:

$$\begin{cases} D^{\frac{1}{2}} \left[x(t) + \frac{1-t}{2t+1} \tanh\big(x(t) + 1\big)\right] = t^3 \ln(1+t) \cos\big(x(t)\big) \frac{4I^{\beta}\left(x(t)\right)+1}{11 I^{\beta}\left(x(t)\right)+1}, \\ x(0) = 0, \end{cases} \tag{6.1}$$

where $t \in J$ and $\beta > 0$. It is observe that this equation is a special case of the FHDE (1.4) if we set

$$f(t, x(t)) = -\frac{1-t}{2t+1} \tanh\big(x(t) + 1\big)$$

and

$$g(t, x(t), I^{\beta}(x(t))) = t^3 \ln(1+t) \cos\big(x(t)\big) \frac{4I^{\beta}\big(x(t)\big)+1}{11 I^{\beta}\big(x(t)\big)+1}.$$

We will show that the equation (6.1) has a solution. For this we will show that this equation satisfies hypotheses (H1)-(H3). We claim that the function

$$F_t(x) := x - f(t, x) = x + \frac{1-t}{2t+1} \tanh\big(x(t) + 1\big)$$

is strictly increasing in $\mathbb{R}^+ \cup \{0\}$ for all $t \in J$. Indeed one can see that

$$\frac{\partial F_t}{\partial x} = 1 + \frac{1-t}{2t+1}\sinh^2\big(x(t)+1\big) > 0.$$

for $t \in J$. Hence, hypothesis (H1) is satisfied. Next we show that f satisfies the hypothesis (H2). Let $x, y \in \mathbb{R}^+ \cup \{0\}$. By using the subtraction formula for the hyperbolic tangent function we obtain

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{1-t}{2t+1}\Big|\tanh(y+1) - \tanh(x+1)\Big| \\ &= \frac{1-t}{2t+1}\Big|\tan(y-x)\big(1-\tanh(y+1)\tanh(x+1)\big)\Big| \\ &\leq \Big|\tan(y-x)\Big| \leq \tanh\big(|x-y|\big) \end{aligned} \tag{6.2}$$

for all $t \in J$. Hence, f satisfies the hypothesis (H2). Finally, we show that g satisfies the hypothesis (H3) with $h(t) = t^3\ln(1+t)$. It is easy to see that $I^\beta\big(x(t)\big) \geq 0$ since $x(t) \geq 0$. On the other side since

$$|\cos\big(x(t)\big)| \leq 1 \quad \text{and} \quad \frac{4I^\beta\big(x(t)\big)+1}{11I^\beta\big(x(t)\big)+1} \leq 1$$

we have

$$|g(t,x(t),I^\beta(x(t)))| \leq t^3\ln(1+t)|\cos\big(x(t)\big)|\frac{4I^\beta\big(x(t)\big)+1}{11I^\beta\big(x(t)\big)+1} \leq t^3\ln(1+t).$$

Thus

$$|g(t,x(t),I^\beta(x(t)))| \leq t^3\ln(1+t).$$

So, g satisfies hypothesis (H3). It follows from Theorem 2.1 that the hybrid differential equation (6.1) has a solution. □

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