TAMKANG JOURNAL OF MATHEMATICS
Volume 50, Number 3, 199-206, September 2019
doi:10.5556/j.tkjm.50.2019.3343

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

INVERSE PROBLEMS FOR STURM-LIOUVILLE DIFFERENTIAL OPERATORS ON CLOSED SETS

V. YURKO

Abstract. Second-order differential operators on closed sets (time scales) are considered. Properties of their spectral characteristics are obtained and inverse problems are studied, which consists in recovering the operators from their spectral characteristics. We establish the uniqueness and develop constructive algorithms for the solution of the inverse problems.

1. Introduction

We study inverse spectral problems for Sturm-Liouville differential operators on a closed set of the real line (in literature it is sometimes called a time scale). Such problems often appear in natural sciences and engineering (see monograph [1]-[2] and the references therein). Inverse spectral problems consist in constructing operators with given spectral characteristics. For the classical Sturm-Liouville operators *on an interval* inverse problems have been studied fairly completely; the main results can be found in [3]-[6]. However, differential operators defined on closed sets are essentially more difficult for investigating, and nowadays there is no inverse problem theory for this class of operators. We mention only one paper [7] where an Ambarzumian-type theorem is proved for Sturm-Liouville differential operators on closed sets. We note that the theory of equations on closed sets has been created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations. Such unification is useful in many applied problems, for example in string theory, in biology for studying insect population, in spectral problems for spatial networks and others.

The statement and the study of inverse spectral problems essentially depend on the structure of the closed set. In this paper we will study inverse problems for an important subclass of closed sets, namely, for the so-called Y1-structure (the definition see below). In Section 2 we present the main notions of the time scale theory and establish properties of spectral characteristics of the Sturm-Liouville operator on a closed set. In Section 3 we study the inverse

2010 *Mathematics Subject Classification.* 34A55, 34B24, 47E05.
Key words and phrases. differential operators, closed sets, inverse spectral problems.

problem of recovering the operator from the given Weyl-type function. The main results of the paper are Theorem 1 and Algorithm 1, where a global constructive procedure for solving the inverse problems is provided, and the uniqueness of the solution is proved.

2. Differential equations on closed sets

Before presenting our main results, we recall some notions of the time scale theory; see [1]-[2] for more details (we use a little bit different notations).

Let T be a closed subset of the real line; it is called sometimes the time scale. We define the so-called jump functions σ and σ_0 on T as follows:
$\sigma(x) = \inf\{s \in T: \ s > x\}$ for $x \neq \sup T$, and $\sigma(\sup T) = \sup T$,
$\sigma_0(x) = \sup\{s \in T: \ s < x\}$ for $x \neq \inf T$, and $\sigma_0(\inf T) = \inf T$.
A point $x \in T$ is called left-dense, left-isolated, right-dense and right-isolated, if $\sigma_0(x) = x$, $\sigma_0(x) < x$, $\sigma(x) = x$ and $\sigma(x) > x$, respectively. If $\sigma_0(x) < x < \sigma(x)$, then x is called isolated; if $\sigma_0(x) = x = \sigma(x)$, then x is called dense. A function f on T is called T-continuous, if it is continuous at all right-dense points and has left-sided limits at all left-dense points in T. The set of T-continuous functions is denoted by C_T. Put $T^0 := T \setminus \{\sup T\}$, if $\sup T$ is left-isolated, and $T^0 := T$, otherwise.

A function f on T is called delta-differentiable at $x \in T^0$, if for any $\varepsilon > 0$ there exist a neighborhood $U = (x-\delta, x+\delta) \cap T$ such that

$$|(f(\sigma(x)) - f(s)) - f^{\Delta}(x)(\sigma(x) - s)| \leq \varepsilon|\sigma(x) - s|$$

for all $s \in U$. We will call $f^{\Delta}(x)$ the delta-derivative of f at x.

Example 1. If x is a right-isolated point, then

$$f^{\Delta}(x) = \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x}.$$

In particular, if $T = \{x = hk: \ k \in \mathbf{Z}\}$, then

$$f^{\Delta}(x) = \frac{f(x+h) - f(x)}{h}.$$

Example 2. If $x \in T$ is a right-dense point, and f is a delta-differentiable at x, then

$$f^{\Delta}(x) = \lim_{s \to x,\, s > x} \frac{f(x) - f(s)}{x - s}.$$

In particular, if $x \in T$ is a dense point, and f is a delta-differentiable at x, then f is differentiable at x, and $f^{\Delta}(x) = f'(x)$.

Consider the Sturm-Liouville equation on T:

$$-y^{\Delta\Delta}(x)+q(x)y(\sigma(x))=\lambda y(\sigma(x)),\quad x\in T. \tag{1}$$

Here λ is the spectral parameter, $q(x)\in C_T$ is a complex-valued function. A function y is called a solution of Eq. (1), if $y\in C_T^2$ and satisfies Eq. (1). The statement and the study of inverse spectral problems essentially depend on the structure of the time scale T. It is necessary to choose and describe subclasses of time scales for which the inverse problem theory can be constructed adequately. In this paper we consider one of such subclasses, namely, the so-called $Y1$–structure. More precisely, we consider the time scale of the form

$$T=\bigcup_{k=1}^{N}[a_k,b_k],\ N\ge 2,\ b_{k-1}<a_k\le b_k<a_{k+1},\ a_1<b_1,\ a_N<b_N,\ a_k=b_k,\ k=\overline{2,N-1}.$$

For $Y1$–structure one has

$$y^{\Delta}(b_k)=\frac{y(a_{k+1})-y(b_k)}{a_{k+1}-b_k},\ k=\overline{1,N-1},\quad y^{\Delta}(x)=y'(x),\ x\in[a_1,b_1]\cup[a_N,b_N]. \tag{2}$$

In particular, this yields $y^{\Delta}(b_1)=y'(b_1)$, and consequently,

$$y(a_2)=y(b_1)+(a_2-b_1)y'(b_1). \tag{3}$$

Using (1) and (2) we obtain

$$\begin{aligned}
-y''(x) &= q(x)y(x)=\lambda y(x),\quad x\in[a_1,b_1]\cup[a_N,b_N], \qquad (4)\\
y^{\Delta\Delta}(b_k) &= \frac{1}{a_{k+1}-b_k}\left(\frac{y(a_{k+2})-y(b_{k+1})}{a_{k+2}-b_{k+1}}-\frac{y(a_{k+1})-y(b_k)}{a_{k+1}-b_k}\right)\\
&= (q(b_k)-\lambda)y(a_{k+1}),\ k=\overline{1,N-2},\\
y^{\Delta\Delta}(b_{N-1}) &= \frac{1}{a_N-b_{N-1}}\left(y'(a_N)-\frac{y(a_N)-y(b_{N-1})}{a_N-b_{N-1}}\right)=(q(b_{N-1})-\lambda)y(a_N).
\end{aligned}$$

Therefore

$$\begin{aligned}
y(a_{k+2}) &= y(b_{k+1})+\frac{a_{k+2}-b_{k+1}}{a_{k+1}-b_k}\Big(y(a_{k+1})-y(b_k)\Big)\\
&\quad +(a_{k+1}-b_k)(a_{k+2}-b_{k+1})(q(b_k)-\lambda)y(a_{k+1}),\ k=\overline{1,N-2}, \qquad (5)\\
y'(a_N) &= \frac{y(a_N)-y(b_{N-1})}{a_N-b_{N-1}}+(a_N-b_{N-1})(q(b_{N-1})-\lambda)y(a_N). \qquad (6)
\end{aligned}$$

Let $\lambda=\rho^2$. It follows from (3) and (5)-(6) that

$$\left.\begin{aligned}
y(a_N)&=\alpha_{11}(\rho)y(b_1)+\alpha_{12}(\rho)y'(b_1),\\
y'(a_N)&=\alpha_{21}(\rho)y(b_1)+\alpha_{22}(\rho)y'(b_1),
\end{aligned}\right\} \tag{7}$$

where $\alpha_{jk}(\rho)$ are polynomials with respect to ρ of degree $2(N+j-3)$, and they depend on $q(b_1),\dots,q(b_{N+j-3})$. Moreover,

$$\alpha_{jk}(\rho)=(i\rho)^{2(N+j-3)}\alpha^0_{jk}[1],\ |\rho|\to\infty, \tag{8}$$

where $[1]=1+O(\rho^{-1})$,

$$\alpha^0_{12}=(a_2-b_1)\alpha^0_{11},\quad \alpha^0_{21}=(a_N-b_{N-1})\alpha^0_{11},\quad \alpha^0_{22}=(a_2-b_1)(a_N-b_{N-1})\alpha^0_{11},$$

$$\alpha^0_{11}=(a_2-b_1)(a_N-b_{N-1})\prod_{k=2}^{N-2}(a_{k+1}-b_k)^2$$

($\alpha^0_{11}=1$ for $N=2$, and $\alpha^0_{11}=(a_2-b_1)(a_3-b_2)$ for $N=3$). Without loss of generality we assume that $a_1=0$.

Denote by L_0 the boundary value problem for Eq. (1) on T with the boundary conditions $y(0)=y(b_N)=0$. Let $S(x,\lambda)$ and $C(x,\lambda)$ be solutions of Eq. (1) on T satisfying the initial conditions

$$C(0,\lambda)=S^{\Delta}(0,\lambda)=1,\ S(0,\lambda)=C^{\Delta}(0,\lambda)=0.$$

For each fixed x, the functions $S(x,\lambda)$ and $C(x,\lambda)$ are entire in λ of order $1/2$. Denote

$$\Delta_0(\lambda):=S(b_N,\lambda).$$

The eigenvalues $\{\lambda_{n0}\}_{n\ge1}$ of the boundary value problem L_0 coincide with the zeros of the entire function $\Delta_0(\lambda)$. The function $\Delta_0(\lambda)$ is called the characteristic function for L_0.

Let $\Phi(x,\lambda)$ be the solution of Eq. (1) on T satisfying the boundary conditions

$$\Phi(0,\lambda)=1,\quad \Phi(b_N,\lambda)=0. \tag{9}$$

Put $M(\lambda):=\Phi^{\Delta}(0,\lambda)$. The function $M(\lambda)$ is called the Weyl-type function or simply Weyl function. Clearly,

$$\Phi(x,\lambda)=C(x,\lambda)+M(\lambda)S(x,\lambda), \tag{10}$$

$$M(\lambda)=-\Delta_1(\lambda)/\Delta_0(\lambda), \tag{11}$$

where $\Delta_1(\lambda):=C(b_N,\lambda)$ is the characteristic function for the boundary value problem L_1 for Eq. (1) on T with the boundary conditions $y^{\Delta}(0)=y(b_N)=0$. The zeros $\{\lambda_{n1}\}_{n\ge1}$ of $\Delta_1(\lambda)$ coincide with the eigenvalues of L_1.

Now we need to study the asymptotical behavior of the solutions $\Phi(x,\lambda)$ and $S(x,\lambda)$. For this purpose we extent the function $q(x)$ on the whole segment $[a_1,b_N]$ such that $q(x)\in C[a_1,b_N]$ and arbitrary in the rest. Consider the Sturm-Liouville equation

$$-y''(x)+q(x)y(x)=\lambda y(x),\quad x\in[0,b_N]. \tag{12}$$

It is known (see, for example, [5]) that there exists a fundamental system of solutions of Eq. (12) $\{Y_1(x,\rho), Y_2(x,\rho)\}$, $x\in[0,b_N]$, $Im\,\rho\ge 0$, $|\rho|>\rho_0$, having the following asymptotical behavior for each fixed $x\in[0,b_N]$ as $|\rho|\to\infty$:

$$Y_1^{(\nu)}(x,\rho)=(i\rho)^\nu\exp(i\rho x)[1],\quad Y_2^{(\nu)}(x,\rho)=(-i\rho)^\nu\exp(-i\rho x)[1],\ \nu=0,1. \tag{13}$$

The function $\Phi(x,\lambda)$ is the solution of Eq. (4) satisfying the boundary conditions (9) and the jump conditions

$$\left.\begin{aligned}\Phi(a_N,\lambda)&=\alpha_{11}(\rho)\Phi(b_1,\lambda)+\alpha_{12}(\rho)\Phi'(b_1,\lambda),\\ \Phi'(a_N,\lambda)&=\alpha_{21}(\rho)\Phi(b_1,\lambda)+\alpha_{22}(\rho)\Phi'(b_1,\lambda).\end{aligned}\right\} \tag{14}$$

Using the fundamental system of solutions $\{Y_1(x,\rho), Y_2(x,\rho)\}$, one has

$$\left.\begin{aligned}\Phi(x,\lambda)&=A_1(\rho)Y_1(x,\rho)+A_2(\rho)Y_2(x,\rho),\ x\in[0,b_1],\\ \Phi(x,\lambda)&=B_1(\rho)Y_1(x,\rho)+B_2(\rho)Y_2(x,\rho),\ x\in[a_N,b_N].\end{aligned}\right\} \tag{15}$$

Substituting (15) into (9) and (14) and using (13), we obtain the following linear algebraic system with respect to $A_k(\rho)$ and $B_k(\rho)$:

$$\begin{aligned}&A_1(\rho)[1]+A_2(\rho)[1]=1,\ B_1(\rho)\exp(i\rho b_N)[1]+B_2(\rho)\exp(-i\rho b_N)[1]=0,\\ &B_1(\rho)\exp(i\rho a_N)[1]+B_2(\rho)\exp(-i\rho a_N)[1]\\ &\quad=\alpha_{11}(\rho)\Big(A_1(\rho)\exp(i\rho b_1)[1]+A_2(\rho)\exp(-i\rho b_1)[1]\Big)\\ &\qquad+\alpha_{12}(\rho)\Big(A_1(\rho)(i\rho)\exp(i\rho b_1)[1]+A_2(\rho)(-i\rho)\exp(-i\rho b_1)[1]\Big),\\ &B_1(\rho)(i\rho)\exp(i\rho a_N)[1]+B_2(\rho)(-i\rho)\exp(-i\rho a_N)[1]\\ &\quad=\alpha_{21}(\rho)\Big(A_1(\rho)\exp(i\rho b_1)[1]+A_2(\rho)\exp(-i\rho b_1)[1]\Big)\\ &\qquad+\alpha_{22}(\rho)\Big(A_1(\rho)(i\rho)\exp(i\rho b_1)[1]+A_2(\rho)(-i\rho)\exp(-i\rho b_1)[1]\Big).\end{aligned}$$

Taking (8) into account we deduce that the determinant $D(\rho)$ of this system has the form

$$\begin{aligned}D(\rho)&=(i\rho)\alpha_{22}(\rho)\Big(\exp(i\rho(b_N-a_N))[1]-\exp(-i\rho(b_N-a_N))[1]\Big)\\ &\quad\times\Big(\exp(i\rho(b_1-a_1))[1]-\exp(-i\rho(b_1-a_1))[1]\Big),\quad |\rho|\to\infty,\ Im\,\rho\ge 0.\end{aligned} \tag{16}$$

Denote $\Omega_\delta:=\{\rho:\ \arg\rho\in[\delta,\pi-\delta]\}$. Solving this algebraic system by Cramer's rule and using (16), we get for $|\rho|\to\infty$, $\rho\in\Omega_\delta$:

$$A_1(\rho)=[1],\quad A_2(\rho)=\exp(2i\rho b_1)[1],$$

$$B_1(\rho)=\exp(-i\rho(a_N-b_1))O(\rho^{2N-4})[1],\quad B_2(\rho)=\exp(-i\rho(a_N-b_1))\exp(2i\rho b_N)O(\rho^{2N-4})[1].$$

In particular this yields for each fixed $x \in [0, b_1)$:

$$\Phi^{(\nu)}(x,\lambda) = (i\rho)^{\nu} \exp(i\rho x)[1], \quad \nu = 0,1, \ |\rho| \to \infty, \ \rho \in \Omega_\delta. \tag{17}$$

Similarly, we obtain

$$S^{(\nu)}(x,\lambda) = -\frac{(-i\rho)^{\nu}}{2i\rho} \exp(-i\rho x)[1], \quad \nu = 0,1, \ |\rho| \to \infty, \ \rho \in \Omega_\delta, \tag{18}$$

for each fixed $x \in (0, b_1]$.

3. Solution of the inverse problem

Let the numbers $q(b_2), \dots, q(b_{N-1})$ be known a priori. The inverse problem is formulate as follows.

Inverse problem 1. Given $M(\lambda)$, construct q on T.

In order to solve this inverse problem we will use the ideas of the method of spectral mappings [6]. Let us prove the uniqueness theorem for the solution of Inverse problem 1. For this purpose together with L_0 we consider a boundary value problem $\tilde{L}_0$ of the same form but with another potential $\tilde{q}$. We agree that if a certain symbol θ denotes an object related to L_0, then $\tilde{\theta}$ will denote an analogous object related to $\tilde{L}_0$.

Theorem 1. *If $M(\lambda) = \tilde{M}(\lambda)$, then $q = \tilde{q}$ on T. Thus, the specification of the Weyl function $M(\lambda)$ uniquely determines the potential q.*

Proof. For $x \in (0, b_1)$, we consider the functions

$$Q_1(x,\lambda) = \Phi(x,\lambda)\tilde{S}'(x,\lambda) - \tilde{\Phi}'(x,\lambda)S(x,\lambda), \ Q_2(x,\lambda) = \tilde{\Phi}(x,\lambda)S(x,\lambda) - \Phi(x,\lambda)\tilde{S}(x,\lambda).$$

It follows from (17)-(18) that for each fixed $x \in (0, b_1)$,

$$Q_1(x,\lambda) = 1 + O(\rho^{-1}), \ Q_2(x,\lambda) = O(\rho^{-1}), \ |\rho| \to \infty, \ \rho \in \Omega_\delta. \tag{19}$$

On the other hand, using (10) and the assumption of the theorem, we get

$$Q_1(x,\lambda) = C(x,\lambda)\tilde{S}'(x,\lambda) - \tilde{C}'(x,\lambda)S(x,\lambda), \ Q_2(x,\lambda) = \tilde{C}(x,\lambda)S(x,\lambda) - C(x,\lambda)\tilde{S}(x,\lambda),$$

and consequently, for each fixed $x \in (0, b_1)$, the functions $Q_1(x,\lambda)$ and $Q_2(x,\lambda)$ are entire in λ of order 1/2. Together with (19) this yields $Q_1(x,\lambda) \equiv 1$ and $Q_2(x,\lambda) \equiv 0$. Since $\Phi(x,\lambda)S'(x,\lambda) - \Phi'(x,\lambda)S(x,\lambda) \equiv 1$, it follows that

$$Q_1(x,\lambda)\tilde{\Phi}(x,\lambda) + Q_2(x,\lambda)\tilde{\Phi}'(x,\lambda) = \Phi(x,\lambda),$$

$$Q_1(x,\lambda)\tilde{S}(x,\lambda) + Q_2(x,\lambda)\tilde{S}'(x,\lambda) = S(x,\lambda).$$

Therefore,

$$\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda), \tag{20}$$

and consequently, $q(x) = \tilde{q}(x)$ for $x \in [0, b_1]$. Using the method of spectral mappings [6] we also obtain an algorithm for constructing the potential $q(x)$ for $x \in [0, b_1]$.

Denote

$$\Phi_1(x,\lambda) := \frac{\Phi(x,\lambda)}{\Phi(a_N,\lambda)}, \quad M_1(\lambda) := \Phi_1'(a_N,\lambda) = \frac{\Phi'(a_N,\lambda)}{\Phi(a_N,\lambda)}. \tag{21}$$

Since $\Phi_1(a_N,\lambda) = 1$, $\Phi_1(b_N,\lambda) = 0$, it follows that the function $M_1(\lambda)$ is the Weyl function for Eq. (4) on the segment $[a_N, b_N]$. Taking (14), (20) and (21) into account we infer $M_1(\lambda) = \tilde{M}_1(\lambda)$. The specification of the Weyl function $M_1(\lambda)$ uniquely determines the potential $q(x)$ for $x \in [a_N, b_N]$. This means that Theorem 1 is proved, and the solution of Inverse problem 1 can be found by the following algorithm.

Algorithm 1. *Let the function $M(\lambda)$ be given.*

1) *Construct $q(x)$ and $\Phi(x,\lambda)$ for $x \in [a_1, b_1]$ using (17)-(19) and the method of spectral mappings.*
2) *Find $\Phi(a_N,\lambda)$ and $\Phi'(a_N,\lambda)$ via (14).*
3) *Calculate $M_1(\lambda)$ by (21).*
4) *Construct $q(x)$ and $\Phi(x,\lambda)$ for $x \in [a_N, b_N]$ by the method of spectral mappings.*

Remark. The inverse problem of recovering the potential $q(x)$ from the given two spectra $\{\lambda_{nj}\}_{n\ge1}$, $j = 0,1$, can be reduced to the solution of Inverse problem 1. Indeed, using Hadamard's factorization theorem one can uniquely reconstruct the characteristic functions $\Delta_j(\lambda)$, $j = 0,1$, and then calculate $M(\lambda)$ by (11).

Acknowledgement

This work was supported in part by Grant 1.1660.2017/4.6 of the Russian Ministry of Education and Science and by Grant 19-01-00102 of Russian Foundation for Basic Research.

References

[1] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhäuser, Boston, MA, 2001.

[2] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhaäuser, Boston, MA, 2003.

[3] V. A. Marchenko, Sturm-Liouville Operators and Their Applications, Naukova Dumka, Kiev, 1977; English transl., Birkhäuser, 1986.

[4] B. M. Levitan, Inverse Sturm-Liouville Problems, Nauka, Moscow, 1984; Engl. transl., VNU Sci.Press, Utrecht, 1987.

[5] G. Freiling and V. A. Yurko, Inverse Sturm-Liouville Problems and Their Applications, NOVA Science Publishers, New York, 2001.
[6] V. A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2002.
[7] S. Ozkan, *Ambarzumian-type theorem on a time scale*, Journal of Inverse and Ill-Posed Problems, 2018.

Department of Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: yurkova@info.sgu.ru