TAMKANG JOURNAL OF MATHEMATICS
Volume 50, Number 3, 207-221, September 2019
doi:10.5556/j.tkjm.50.2019.3347

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

AN INVERSE SPECTRAL PROBLEM FOR STURM–LIOUVILLE-TYPE INTEGRO-DIFFERENTIAL OPERATORS WITH ROBIN BOUNDARY CONDITIONS

SERGEY BUTERIN

Abstract. The perturbation of the Sturm–Liouville differential operator on a finite interval with Robin boundary conditions by a convolution operator is considered. The inverse problem of recovering the convolution term along with one boundary condition from the spectrum is studied, provided that the Sturm–Liouville potential as well as the other boundary condition are known a priori. The uniqueness of solution for this inverse problem is established along with necessary and sufficient conditions for its solvability. The proof is constructive and gives an algorithm for solving the inverse problem.

1. Introduction

Let $\{\lambda_n\}_{n\geq 0}$ be the spectrum of the boundary value problem $L = L(q, M, h, H)$ of the form

$$\ell y := -y'' + q(x)y + \int_0^x M(x-t)y(t)\,dt = \lambda y, \quad 0 < x < \pi, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0, \tag{2}$$

where λ is the spectral parameter, $q(x)$, $M(x)$ are complex-valued functions, $q(x) \in L_2(0,\pi)$, $(\pi - x)M(x) \in L_2(0,\pi)$ and $h, H \in \mathbb{C}$. By the standard method involving Rouché's theorem one can prove the following assertion (for more details see Section 2 below).

Theorem 1. *Eigenvalues λ_n, $n \geq 0$, of the problem L have the form*

$$\lambda_n = \Big(n + \frac{\omega}{n+1} + \frac{\kappa_n}{n+1}\Big)^2, \quad \{\kappa_n\}_{n\geq 0} \in l_2. \tag{3}$$

Moreover,

$$\omega = \frac{h+H}{\pi} + \frac{1}{2\pi}\int_0^\pi q(x)\,dx. \tag{4}$$

We study the following inverse problem.

2010 *Mathematics Subject Classification*. 34A55, 45J05, 34B10, 45G10.
Key words and phrases. Integro-differential operator, nonlocal operator, convolution, inverse spectral problem, nonlinear integral equation.

Inverse Problem 1. *Given the spectrum* $\{\lambda_n\}_{n\ge 0}$, *find the function* $M(x)$ *along with the coefficient* H, *provided that* $q(x)$ *and* h *are known a priori.*

In this statement, by virtue of (3) and (4), one can alternatively consider the coefficient h as unknown while H as given.

The greatest success in the inverse spectral theory has been achieved for the classical Sturm–Liouville operator (see [1]-[6] and the references therein) and afterwards for higher-order differential operators and other classes of differential operators and systems [5]-[8]. However, the classical methods of the inverse spectral theory (such as the transformation operator method [2]-[5] and the method of spectral mappings [4]-[7]), which allow to obtain global solutions of inverse problems for differential operators, are not applicable for integro-differential operators as well as for other classes of nonlocal operators. At the same time, integro-differential operators are often more adequate for modelling various processes in physics, biology, economics and engineering [9].

Various aspects of inverse problems for some classes of integro-differential operators were studied in [10]-[32] and other works. One of the first substantial studies in this direction was undertaken in [13], where it was established, in particular, that specification of the spectrum of the boundary value problem for equation (1) with Dirichlet boundary conditions uniquely determines the function $M(x)$, provided that the potential $q(x)$ is known a priori. Moreover, developing Borg's method [1] (see also [4, 34]) *local* solvability and stability of the corresponding inverse problem were proved.

In [16] the *global* solvability of the inverse problem in [13] was established by another approach based on reducing the inverse problem to solving the so-called main nonlinear integral equation with a singularity, which was solved globally. That main equation was equivalent to finding the unknown function $M(x)$ from a trace of the transformation operator kernel for the sine-type solution of equation (1). In the case $q(x) \equiv const$, the main equation can be presented in an explicit form [14]. Developing this approach allowed obtaining global solutions of inverse problems also for integro-differential Dirac systems [22, 24], for integro-differential operators with discontinuities [25, 26] and for the operators on a geometrical graph [27], for integro-differential operators of fractional orders [30, 31], as well as the so-called half inverse problems [21, 32], when the convolution kernel is to be recovered on a part of its domain of definition from a part of the spectrum. For different classes of operators, the corresponding main equations take different forms, which makes it necessary to provide the proof of their solvability in all new cases. In order to make it more convenient, in [33] a general approach has been developed for solving nonlinear equations of this type by introducing some abstract equation and proving its global solvability. Moreover, in [33] stability of such nonlinear equations was established, which has not been studied before even in simple cases.

In the present paper, uniqueness of solution of Inverse Problem 1 is established along with necessary and sufficient conditions for its solvability. Note that the case of Robin boundary conditions (2) brings additional difficulties in the study of the inverse problem (see [20]). The main results of the paper are contained in the following theorem.

Theorem 2. *Let a complex-valued function $q(x) \in L_2(0,\pi)$ and a complex number h be given. Then for any sequence of complex numbers $\{\lambda_n\}_{n\ge 0}$ of the form (3) there exists a unique (up to values on a set of measure zero) function $M(x)$, $(\pi - x)M(x) \in L_2(0,\pi)$, and a unique number $H \in \mathbb{C}$ such that $\{\lambda_n\}_{n\ge 0}$ is the spectrum of the corresponding boundary value problem $L(q,M,h,H)$ of the form (1), (2).*

The paper is organized as follows. In the next section, we study the transformation operator related to the cosine-type solution of equation (1) as well as the characteristic function of the boundary value problem L, and derive the main nonlinear integral equation of the inverse problem. In Section 3, we study dependence of the transformation operator kernel on the function $M(x)$. In Section 4, we prove global solvability of the main equation. In Section 5, we prove Theorem 2 and provide a constructive procedure for solving the inverse problem (Algorithm 1).

2. Characteristic function and transformation operator

Let $y = \varphi(x,\lambda) = \varphi(x,\lambda; h, q, M)$ be a solution of equation (1) under the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h. \tag{5}$$

Here and below, in order to emphasize dependence of a function $f(x_1,\dots,x_n)$ on some constants or functions $f_1,\dots,f_m$, sometimes we write $f(x_1,\dots,x_n; f_1,\dots,f_m)$.

Thus, $U(\varphi(x,\lambda)) = 0$ and, by virtue of uniqueness of the solution $\varphi(x,\lambda)$, eigenvalues of L coincide with zeros with account of multiplicity of the entire function

$$\Delta(\lambda) := V(\varphi(x,\lambda)), \tag{6}$$

which is called the *characteristic function* of the problem L.

For obtaining an appropriate representation of the function $\varphi(x,\lambda)$, we need the following auxiliary assertion.

Lemma 1. *The integral equation*

$$G(x,t,\tau) = G_0(x,t,\tau) + \frac{1}{2}\Big(\int_\tau^t \Big(\int_s^{\frac{t+s}{2}} q(\xi)G(\xi,s,\tau)\,d\xi + \int_s^{x-\frac{t-s}{2}} q(\xi)G(\xi,s,\tau)\,d\xi\Big)ds$$

$$+\int_0^{t-\tau} M(s)\,ds \int_\tau^{t-s} \Big(\int_\xi^{\frac{t-s+\xi}{2}} G(\eta,\xi,\tau)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} G(\eta,\xi,\tau)\,d\eta\Big)d\xi\Big),$$

$$0 \le \tau \le t \le x \le \pi, \qquad (7)$$

with a continuous free term $G_0(x,t,\tau)$ *has a unique solution* $G(x,t,\tau) = G(x,t,\tau;q,M)$ *being, in turn, a continuous function too.*

Proof. The method of successive approximations gives

$$G(x,t,\tau) = \sum_{k=0}^{\infty} G_k(x,t,\tau), \qquad (8)$$

where

$$G_{k+1}(x,t,\tau) = \frac{1}{2}\Big(\int_\tau^t \Big(\int_s^{\frac{t+s}{2}} q(\xi)G_k(\xi,s,\tau)\,d\xi + \int_s^{x-\frac{t-s}{2}} q(\xi)G_k(\xi,s,\tau)\,d\xi\Big)ds$$
$$+\int_0^{t-\tau} M(s)\,ds \int_\tau^{t-s} \Big(\int_\xi^{\frac{t-s+\xi}{2}} G_k(\eta,\xi,\tau)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} G_k(\eta,\xi,\tau)\,d\eta\Big)d\xi\Big), \quad k \ge 0. \qquad (9)$$

Put

$$G_0 := \max_{0\le\tau\le t\le x\le\pi} |G_0(x,t,\tau)|, \quad A := \int_0^\pi |q(s)|\,ds + \frac{1}{2}\int_0^\pi (\pi - s)|M(s)|\,ds$$

and let us show that

$$|G_k(x,t,\tau)| \le G_0\frac{(At)^k}{k!}, \quad 0 \le \tau \le t \le x \le \pi, \quad k \ge 0. \qquad (10)$$

Indeed, for $k = 0$ estimate (10) is obvious. Supposing that it holds for $k = j$ with some $j \ge 0$, let us prove it for $k = j+1$. According to (9) and (10) for $k = j$, we get

$$|G_{j+1}(x,t,\tau)| \le G_0\frac{A^j}{2j!}\Big(\int_\tau^t \Big(\int_s^{\frac{t+s}{2}} |q(\xi)|\,d\xi + \int_s^{x-\frac{t-s}{2}} |q(\xi)|\,d\xi\Big)s^j\,ds$$
$$+\int_0^{t-\tau} |M(s)|\,ds \int_\tau^{t-s} \Big(\int_\xi^{\frac{t-s+\xi}{2}} d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} d\eta\Big)\xi^j\,d\xi\Big)$$
$$\le G_0\frac{(At)^{j+1}}{(j+1)!}.$$

Hence, the series in (8) converges uniformly for $0 \le \tau \le t \le x \le \pi$ to the solution of equation (7).

For the uniqueness it is sufficient to show that $G_0(x,t,\tau) \equiv 0$ implies $G(x,t,\tau) \equiv 0$. Indeed, assuming the zero free term, we determine $G_k(x,t,\tau)$ by formulae (9) with $G_0(x,t,\tau) := G(x,t,\tau)$. Then $G_k(x,t,\tau) = G(x,t,\tau)$ for all $k \ge 0$ and, according to (10), we get $G(x,t,\tau) \equiv 0$. □

Note that in equation (7) the variable τ is, actually, a parameter. In other words, the assertion of Lemma 1 remains true, if $\tau \in [0,\pi)$ is fixed.

The following lemma gives the transformation operator for the function $\varphi(x,\lambda)$.

Lemma 2. *Put $\rho^2 = \lambda$. The following representation holds:*

$$\varphi(x,\lambda) = \cos\rho x + \int_0^x K(x,t)\cos\rho(x-t)\,dt, \quad 0 \le x \le \pi, \tag{11}$$

where the function

$$K(x,t) = K(x,t;h,q,M) = G(x,t,0) \tag{12}$$

is the solution of equation (7) *for $\tau = 0$ and with the free term*

$$G_0(x,t,0) = h + \frac{1}{2}\Big(\int_0^{x-\frac{t}{2}} q(s)\,ds + \int_0^{\frac{t}{2}} q(s)\,ds + \int_0^t (x-s)M(s)\,ds\Big). \tag{13}$$

The function $K(x,t)$ is continuous in the triangle $0 \le t \le x \le \pi$, $K(x,\cdot) \in W_2^1[0,x]$ for all $x \in (0,\pi]$ and $K(\cdot,t) \in W_2^1[t,\pi]$ for all $t \in [0,\pi)$. Moreover,

$$K(x,0) = h + \frac{1}{2}\int_0^x q(t)\,dt. \tag{14}$$

Proof. By substitution it is easy to check that the Cauchy problem (1), (5) for the function $y = \varphi(x,\lambda)$ is equivalent to the integral equation

$$\varphi(x,\lambda) = \cos\rho x + h\frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho}\Big(q(t)\varphi(t,\lambda) + \int_0^t M(t-s)\varphi(s,\lambda)\,ds\Big)\,dt. \tag{15}$$

Substituting (11) into equation (15), we arrive at

$$\int_0^x K(x,t)\cos\rho(x-t)\,dt = h\int_0^x \cos\rho(x-t)\,dt + \sum_{\nu=1}^4 \mathcal{K}_\nu(x,\lambda), \tag{16}$$

$$\mathcal{K}_1(x,\lambda) = \int_0^x q(t)\cos\rho t\,dt \int_0^{x-t} \cos\rho s\,ds,$$

$$\mathcal{K}_2(x,\lambda) = \int_0^x dt \int_0^t M(t-s)\cos\rho s\,ds \int_0^{x-t} \cos\rho\xi\,d\xi,$$

$$\mathcal{K}_3(x,\lambda) = \int_0^x q(t)\,dt \int_0^t K(t,s)\cos\rho(t-s)\,ds \int_0^{x-t} \cos\rho\xi\,d\xi,$$

$$\mathcal{K}_4(x,\lambda) = \int_0^x dt \int_0^t M(t-s)\,ds \int_0^s K(s,\xi)\cos\rho(s-\xi)\,d\xi \int_0^{x-t} \cos\rho\eta\,d\eta.$$

Using the relation $2\cos\rho t\cos\rho s = \cos\rho(t+s) + \cos\rho(t-s)$ and changing the variables and the order of integration, we get

$$\mathcal{K}_1(x,\lambda) = \frac{1}{2}\int_0^x \Big(\int_0^{x-\frac{t}{2}} q(s)\,ds + \int_0^{\frac{t}{2}} q(s)\,ds\Big)\cos\rho(x-t)\,dt,$$

$$\mathcal{K}_2(x,\lambda) = \frac{1}{2}\int_0^x \cos\rho(x-t)\,dt \int_0^t (x-s)M(s)\,ds,$$

$$\mathcal{K}_3(x,\lambda) = \frac{1}{2}\int_0^x \cos\rho(x-t)\,dt \int_0^t \Big(\int_s^{\frac{t+s}{2}} q(\xi)K(\xi,s)\,d\xi + \int_s^{x-\frac{t-s}{2}} q(\xi)K(\xi,s)\,d\xi\Big)\,ds,$$

$$\mathcal{K}_4(x,\lambda) = \frac{1}{2}\int_0^x \cos\rho(x-t)\,dt \int_0^t M(s)\,ds \int_0^{t-s} \Big(\int_\xi^{\frac{t-s+\xi}{2}} K(\eta,\xi)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} K(\eta,\xi)\,d\eta\Big)\,d\xi.$$

By virtue of these four relations, equality (16) holds for all $\lambda \in \mathbb{C}$ if and only if the function $K(x,t)$ satisfies the integral equation

$$\begin{aligned}
K(x,t) = h + \frac{1}{2}\Big(&\int_0^{x-\frac{t}{2}} q(s)\,ds + \int_0^{\frac{t}{2}} q(s)\,ds + \int_0^t (x-s)M(s)\,ds \\
&+ \int_0^t \Big(\int_s^{\frac{t+s}{2}} q(\xi)K(\xi,s)\,d\xi + \int_s^{x-\frac{t-s}{2}} q(\xi)K(\xi,s)\,d\xi\Big)ds \\
&+ \int_0^t M(s)\,ds \int_0^{t-s} \Big(\int_\xi^{\frac{t-s+\xi}{2}} K(\eta,\xi)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} K(\eta,\xi)\,d\eta\Big)d\xi\Big),\ 0 \le t \le x \le \pi, \qquad (17)
\end{aligned}$$

which, in turn, in accordance with (12) and (13), is equivalent to equation (7) for $\tau = 0$.

The rest properties of $K(x,t)$ immediately follow from the form of equation (17). □

Using (6) and Lemma 2, by simple calculations we arrive at the following lemma, which gives a fundamental representation of the characteristic function.

Lemma 3. *The characteristic function of the problem L has the form*

$$\Delta(\lambda) = -\rho\sin\rho\pi + \omega\pi\cos\rho\pi + \int_0^\pi w(x)\cos\rho x\,dx, \quad w(x) \in L_2(0,\pi). \qquad (18)$$

Here ω is determined by (4) *and*

$$w(\pi - x) = K_1(\pi,x;h,q,M) + K_2(\pi,x;h,q,M) + HK(\pi,x;h,q,M), \quad 0 < x < \pi, \qquad (19)$$

where

$$K_1(x,t;h,q,M) = \frac{\partial}{\partial x}K(x,t;h,q,M), \quad K_2(x,t;h,q,M) = \frac{\partial}{\partial t}K(x,t;h,q,M). \qquad (20)$$

Using representation (18), by the known method (see, e.g., [2] involving Rouché's theorem, the proof of Theorem 1 can be accomplished. Moreover, with the help of (18)) and Hadamard's factorization theorem, by the standard approach (see, e.g., [4]) one can prove the following lemma.

Lemma 4. *The characteristic function is determined uniquely by its zeros. Moreover, the following representation holds:*

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^\infty \frac{\lambda_n - \lambda}{n^2}. \qquad (21)$$

Relation (19) can be considered as a nonlinear equation with respect to the function $M(x)$, which we call the *main nonlinear equation* or shortly *main equation* of the inverse problem.

The central place in solving the inverse problem is occupied by the following theorem, which gives global solvability of the nonlinear equation (19).

Theorem 3. *For any complex-valued functions $q(x), w(x) \in L_2(0,\pi)$ and complex numbers h and H, equation (19) has a unique solution $M(x)$, $(\pi - x)M(x) \in L_2(0,\pi)$.*

In the next section we study dependence of the transformation operator kernel $K(x,t)$ on the function $M(x)$. Based on the obtained properties, in Section 4 we give the proof of Theorem 3.

3. Further properties of the transformation operator kernel

In the present section we reveal some important properties of the kernel $K(x,t)$, which allow us to prove the global solvability of the main nonlinear equation (19). First, we note that for each fixed $\delta \in (0,\pi]$ the linear integral equation (7) can be restricted to the set

$$\mathcal{D}_\delta := \Big\{(x,t,\tau) : 0 \le x \le \pi,\ 0 \le \tau \le t \le \min\{\delta, x\}\Big\}.$$

In other words, for $(x,t,\tau) \in \mathcal{D}_\delta$ the right-hand side of equation (7) depends on values of the unknown function $G(x,t,\tau)$ only on the subset $\mathcal{D}_\delta$. Moreover, it is obvious, that on $\mathcal{D}_\delta$ the solution of the restricted equation coincides with the solution of the initial one. Therefore, the function $G(x,t,\tau)$ for $(x,t,\tau) \in \mathcal{D}_\delta$ depends on values of the function $M(s)$ only on the interval $(0,\delta)$. In particular, due to (12) or (17), we make the following observation.

Observation 1. The kernel $K(x,t) = K(x,t;h,q,M)$ on the trapezium

$$D_\delta := \Big\{(x,t) : 0 \le x \le \pi,\ 0 \le t \le \min\{\delta, x\}\Big\}$$

depends on values of $M(s)$ only for $s \in (0,\delta)$.

Denote

$$\|f\|_\delta := \|f\|_{L_2(0,\delta)} = \sqrt{\int_0^\delta |f(x)|^2\,dx}, \quad B_{\delta,r} := \{f \in L_2(0,\delta) : \|f\|_\delta \le r\}. \tag{22}$$

The following lemma gives estimates for the kernel $K(x,t) = K(x,t;h,q,M)$ on the trapezium D_δ for small $\delta > 0$.

Lemma 5. *For any $R > 0$ there exist $\delta_R = \delta_R(h,q) \in (0,\pi]$, $C_R = C_R(h,q) > 0$ and $\hat{C}_R = \hat{C}_R(h,q) > 0$, all depending only on h, $q(x)$ and R, such that for any $\delta \in (0,\delta_R]$ and for all $M(x), \tilde{M}(x) \in B_{\delta,R}$ the following estimates hold:*

$$|K(x,t)| \le C_R, \quad |\hat{K}(x,t)| \le \hat{C}_R\sqrt{\delta}\,\|\hat{M}\|_\delta, \qquad (x,t) \in D_\delta, \tag{23}$$

where $\hat{K}(x,t) = K(x,t;h,q,M) - K(x,t;h,q,\tilde{M})$ and $\hat{M}(x) = M(x) - \tilde{M}(x)$.

Proof. Put

$$C_R := 2|h| + \pi^{\frac{3}{2}} R + 2\int_0^\pi |q(x)|\,dx, \ \ \delta_R := \frac{1}{C_R}, \ \ K := \max_{(x,t)\in D_\delta} |K(x,t)|, \ \ \hat{K} := \max_{(x,t)\in D_\delta} |\hat{K}(x,t)|.$$

Then, by virtue of (17), we have $2K \le C_R + C_R\delta K$, which yields $K \le (2 - C_R\delta)^{-1} C_R \le C_R$ for $\delta \in (0, \delta_R]$. Thus, the first estimate in (23) is established.

Further, subtracting the integral equation for the function $K(x,t;h,q,\tilde{M})$ (i.e. equation (17) with $\tilde{M}(x)$ instead of $M(x)$) termwise from equation (17), we get

$$\begin{aligned}\hat{K}(x,t) = \frac{1}{2}\Big(&\int_0^t (x-s)\hat{M}(s)\,ds + \int_0^t \Big(\int_s^{\frac{t+s}{2}} q(\xi)\hat{K}(\xi,s)\,d\xi + \int_s^{x-\frac{t-s}{2}} q(\xi)\hat{K}(\xi,s)\,d\xi\Big)ds \\ &+ \int_0^t \hat{M}(s)\,ds \int_0^{t-s}\Big(\int_\xi^{\frac{t-s+\xi}{2}} K(\eta,\xi)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} K(\eta,\xi)\,d\eta\Big)\,d\xi\Big) \\ &+ \int_0^t \tilde{M}(s)\,ds \int_0^{t-s}\Big(\int_\xi^{\frac{t-s+\xi}{2}} \hat{K}(\eta,\xi)\,d\eta + \int_\xi^{x-\frac{t+s-\xi}{2}} \hat{K}(\eta,\xi)\,d\eta\Big)\,d\xi\Big),\end{aligned}$$

whence we get $2\hat{K} \le \delta C_R \hat{K} + \hat{C}_R\sqrt{\delta}\|\hat{M}\|_\delta$, where $\hat{C}_R = \pi(1 + \pi C_R)$. This yields $\hat{K} \le (2 - C_R\delta)^{-1}\hat{C}_R\sqrt{\delta}\|\hat{M}\|_\delta \le \hat{C}_R\sqrt{\delta}\|\hat{M}\|_\delta$ for $\delta \in (0, \delta_R]$, and we arrive at the second estimate in (23). $\square$

In what follows, for any fixed $\delta \in (0,\pi]$ we use the following designations

$$M_1(x) = \begin{cases} M(x), & x \in (0,\delta), \\ 0, & x \in (\delta,\sigma), \end{cases} \qquad M_2(x) = \begin{cases} 0, & x \in (0,\delta), \\ M(x), & x \in (\delta,\sigma), \end{cases} \tag{24}$$

where $\sigma = \min\{2\delta, \pi\}$.

Lemma 6. *For each $\delta \in (0, \pi/2]$ the following representation holds:*

$$K(x,t;h,q,M) = K(x,t;h,q,M_1) + \int_\delta^t G(x,t,\tau;h,q,M_1)M_2(\tau)\,d\tau, \quad (x,t) \in D_{2\delta}, \tag{25}$$

where the function $G(x,t,\tau) = G(x,t,\tau;h,q,M_1)$ is a solution of equation (7) for $(x,t,\tau) \in \mathscr{D}_{2\delta}$ with $M_1(x)$ instead of $M(x)$ and with the free term $G_0(x,t,\tau) = G_0(x,t,\tau;h,q,M_1)$ of the form

$$G_0(x,t,\tau) = \frac{x-\tau}{2} + \frac{1}{2}\int_0^{t-\tau}\Big(\int_s^{\frac{t-\tau+s}{2}} K(\xi,s;h,q,M_1)\,d\xi + \int_s^{x-\frac{t+\tau-s}{2}} K(\xi,s;h,q,M_1)\,d\xi\Big)ds. \tag{26}$$

Proof. After the direct substitution one can see that the right-hand side of (25) is a solution of integral equation (17) for $(x,t) \in D_{2\delta}$ if and only if for $(x,t) \in D_{2\delta}$ the following relation holds:

$$\int_0^t G(x,t,\tau;h,q,M_1)M_2(\tau)\,d\tau = \int_0^t G_0(x,t,\tau)M_2(\tau)\,d\tau + \frac{1}{2}\sum_{k=1}^2 (\mathscr{Q}_k(x,t) + \mathscr{M}_k(x,t)), \tag{27}$$

where the function $G_0(x,t,\tau)$ is determined by formula (26) and

$$\mathcal{Q}_1(x,t) = \int_0^t ds \int_s^{\frac{t+s}{2}} q(\xi)\,d\xi \int_0^s G(\xi,s,\tau;h,q,M_1)M_2(\tau)\,d\tau$$
$$= \int_0^t M_2(\tau)\,d\tau \int_\tau^t ds \int_s^{\frac{t+s}{2}} q(\xi)G(\xi,s,\tau;h,q,M_1)\,d\xi,$$
$$\mathcal{Q}_2(x,t) = \int_0^t ds \int_s^{x-\frac{t-s}{2}} q(\xi)\,d\xi \int_0^s G(\xi,s,\tau;h,q,M_1)M_2(\tau)\,d\tau$$
$$= \int_0^t M_2(\tau)\,d\tau \int_\tau^t ds \int_s^{x-\frac{t-s}{2}} q(\xi)G(\xi,s,\tau;h,q,M_1)\,d\xi,$$
$$\mathcal{M}_1(x,t) = \int_0^t M(s)\,ds \int_0^{t-s} d\xi \int_\xi^{\frac{t-s+\xi}{2}} d\eta \int_0^\xi G(\eta,\xi,\tau;h,q,M_1)M_2(\tau)\,d\tau, \tag{28}$$
$$\mathcal{M}_2(x,t) = \int_0^t M(s)\,ds \int_0^{t-s} d\xi \int_\xi^{x-\frac{t+s-\xi}{2}} d\eta \int_0^\xi G(\eta,\xi,\tau;h,q,M_1)M_2(\tau)\,d\tau.$$

Changing the order of integration in (28) and taking into account that $M(x) = M_1(x) + M_2(x)$ for $x \in (0,2\delta)$, we get

$$\mathcal{M}_1(x,t) = \int_0^t (M_1(s)+M_2(s))\,ds \int_0^{t-s} M_2(\tau)\,d\tau \int_\tau^{t-s} d\xi \int_\xi^{\frac{t-s+\xi}{2}} G(\eta,\xi,\tau;h,q,M_1)\,d\eta$$
$$= \int_0^t M_2(\tau)\,d\tau \int_0^{t-\tau} (M_1(s)+M_2(s))\,ds \int_\tau^{t-s} d\xi \int_\xi^{\frac{t-s+\xi}{2}} G(\eta,\xi,\tau;h,q,M_1)\,d\eta. \tag{29}$$

Since $M_2(x) = 0$ on $(0,\delta)$ and $t \in [0,2\delta]$, we have

$$\int_0^t M_2(\tau)\,d\tau \int_0^{t-\tau} M_2(s)\,ds \int_\tau^{t-s} d\xi \int_\xi^{\frac{t-s+\xi}{2}} G(\eta,\xi,\tau;h,q,M_1)\,d\eta = 0.$$

Thus, under the second integral in (29) the summand $M_2(s)$ disappears, i.e.

$$\mathcal{M}_1(x,t) = \int_0^t M_2(\tau)\,d\tau \int_0^{t-\tau} M_1(s)\,ds \int_\tau^{t-s} d\xi \int_\xi^{\frac{t-s+\xi}{2}} G(\eta,\xi,\tau;h,q,M_1)\,d\eta.$$

Analogously we obtain

$$\mathcal{M}_2(x,t) = \int_0^t M_2(\tau)\,d\tau \int_0^{t-\tau} M_1(s)\,ds \int_\tau^{t-s} d\xi \int_\xi^{x-\frac{t+s-\xi}{2}} G(\eta,\xi,\tau;h,q,M_1)\,d\eta.$$

Thus, if the function $G(x,t,\tau;h,q,M_1)$ obeys the conditions of the lemma, then equality (27) is fulfilled. Hence, both the sides of (25) satisfy one and the same equation (17) for $(x,t) \in D_{2\delta}$, having a unique solution, which finishes the proof. $\square$

4. Solution of the main equation. Proof of Theorem 3

Let us represent the main equation (19) in the form

$$2w(\pi - x) = (\pi - x)M(x) + \mathscr{P}M(x), \quad 0 < x < \pi, \tag{30}$$

where

$$\mathscr{P} = \sum_{\nu=1}^{3} \mathscr{P}_\nu, \quad \left.\begin{array}{l} \mathscr{P}_1 M(x) = 2K_1(\pi, x; h, q, M), \\ \mathscr{P}_2 M(x) = 2K_2(\pi, x; h, q, M) - (\pi - x)M(x), \\ \mathscr{P}_3 M(x) = 2HK(\pi, x; h, q, M). \end{array}\right\} \tag{31}$$

Our plan is to use Theorem 1 from [33], which states, in particular, that for any function $f(x) \in L_2(0, b)$ the equation

$$f(x) = u(x) + \mathscr{D}u(x), \quad 0 < x < b,$$

has a unique solution $u(x) \in L_2(0, b)$, if $\mathscr{D}$ is an operator of the class $\mathscr{E}_{b,1}$. For convenience of the reader, we provide here the definition of $\mathscr{E}_{b,1}$.

Definition 1. The operator $\mathscr{D} : L_2(0, b) \to L_2(0, b)$ belongs to the class $\mathscr{E}_{b,1}$, if the following four conditions are fulfilled:

(i) For each $u(x) \in L_2(0, b)$ and for each number $\gamma \in (0, b)$ the image function $\mathscr{D}u(x)$ on the interval $(0, \gamma)$ does not depend on values of $u(x)$ on (γ, b);

(ii) For all $R > 0$ and $r > 0$ there exists $\delta \in (0, b]$ such that $\mathscr{D} : B_{\delta,R} \to B_{\delta,r}$;

(iii) For all $R > 0$ and $\alpha > 0$ there exists $\delta \in (0, b]$ such that

$$\|\mathscr{D}u - \mathscr{D}\tilde{u}\|_\delta \le \alpha \|u - \tilde{u}\|_\delta$$

for any functions $u(x), \tilde{u}(x) \in B_{\delta,R}$;

(iv) For all $\delta \in (0, b/2]$ and $u(x) \in L_2(0, 2\delta)$ the following representation holds:

$$\mathscr{D}u(x) = \mathscr{D}u_1(x) + \int_\delta^x Q_\delta(x, t; u_1)u_2(t)\,dt, \quad 0 < x < 2\delta, \tag{32}$$

where the kernel $Q_\delta(x, t; u_1)$ belongs to $L_2((\delta, 2\delta)^2)$ and does not depend on $u_2(x)$, while

$$u_1(x) = \begin{cases} u(x), & x \in (0, \delta), \\ 0, & x \in (\delta, 2\delta), \end{cases} \qquad u_2(x) = \begin{cases} 0, & x \in (0, \delta), \\ u(x), & x \in (\delta, 2\delta). \end{cases} \tag{33}$$

We note that in conditions (ii)–(iv), one and the same symbol $\mathscr{D}$ denotes natural extensions of the operator $\mathscr{D}$ to the spaces $L_2(0, \gamma)$, $\gamma \in (0, b)$, which, by virtue of (i), are determined uniquely.

Let us show that the operator $\mathscr{P}$ determined in (31) belongs to $\mathscr{E}_{\pi,1}$. Since the class $\mathscr{E}_{\pi,1}$ is, obviously, closed with respect to the sum of operators, it is sufficient to prove that $\mathscr{P}_\nu \in \mathscr{E}_{\pi,1}$, $\nu = \overline{1,3}$. For this purpose, using (17), (20) and (31), we calculate:

$$\begin{aligned}\mathscr{P}_1 M(x) &= q\Big(\pi - \frac{x}{2}\Big) + \int_0^x M(s)\,ds + \int_0^x q\Big(\frac{s-x}{2}+\pi\Big)K\Big(\frac{s-x}{2}+\pi, s\Big)\,ds\\ &\quad + \int_0^x M(s)\,ds \int_0^{x-s} K\Big(\frac{\xi - x - s}{2} + \pi, \xi\Big)\,d\xi, \qquad 0<x<\pi,\\ \mathscr{P}_2 M(x) &= \frac{1}{2}\Big(q\Big(\frac{x}{2}\Big) - q\Big(\pi - \frac{x}{2}\Big)\Big) + \int_x^\pi q(s)K(s,x)\,ds + \frac{1}{2}\int_0^x q\Big(\frac{x+s}{2}\Big)K\Big(\frac{x+s}{2}, s\Big)\,ds\\ &\quad - \frac{1}{2}\int_0^x q\Big(\frac{s-x}{2}+\pi\Big)K\Big(\frac{s-x}{2}+\pi, s\Big)\,ds + \int_0^x M(s)\Big(\int_{x-s}^{\pi-s} K(\xi, x-s)\,d\xi\\ &\quad + \frac{1}{2}\int_0^{x-s} K\Big(\frac{\xi+x-s}{2}, \xi\Big)\,d\xi - \frac{1}{2}\int_0^{x-s} K\Big(\frac{\xi - x - s}{2}+\pi, \xi\Big)\,d\xi\Big)\,ds, \qquad 0<x<\pi,\end{aligned}$$

where $K(x,t) = K(x,t;h,q,M)$ is the solution of equation (17). Using these representations and (17) along with Observation 1, Lemma 5 and Lemma 6, one can check that for the operators $\mathscr{P}_\nu$, $\nu = \overline{1,3}$, all conditions (i)–(iv) of Definition 1 for $b = \pi$ are fulfilled. Note that condition (iv) for the operators $\mathscr{P}_1$ and $\mathscr{P}_2$ can be checked easier by using representations (34) and (35) below. Thus, $\mathscr{P} \in \mathscr{E}_{\pi,1}$ and, consequently, $(\pi - x)^{-1}\mathscr{P} \in \mathscr{E}_{b,1}$ for all $b \in (0,\pi)$. Hence, equation (30) has a unique solution $M(x)$, which belongs to $L_2(0,b)$ for any $b \in (0,\pi)$. It remains to prove that $(\pi - x)M(x) \in L_2(0,\pi)$.

Fix $\delta \in (0, \pi/2]$. By virtue of (25) and (7) along with (26), for $x \in (\delta, 2\delta)$ we have

$$K_1(\pi, x; h, q, M) = K_1(\pi, x; h, q, M_1) + \int_\delta^x G_1(\pi, x, t; h, q, M_1)M_2(t)\,dt, \tag{34}$$

$$K_2(\pi, x; h, q, M) = K_1(\pi, x; h, q, M_1) + \frac{\pi - x}{2}M_2(x) + \int_\delta^x G_2(\pi, x, t; h, q, M_1)M_2(t)\,dt, \tag{35}$$

where

$$G_1(x,t,\tau;h,q,M_1) = \frac{\partial}{\partial x}G(x,t,\tau;h,q,M_1), \quad G_2(x,t,\tau;h,q,M_1) = \frac{\partial}{\partial t}G(x,t,\tau;h,q,M_1). \tag{36}$$

Using (25), (34) and (35) for $\delta = \pi/2$, one can restrict the main equation (19) to the subinterval $(\pi/2, \pi)$ in the form

$$g(x) = (\pi - x)M_2(x) + \int_{\frac{\pi}{2}}^x R(x,t)M_2(t)\,dt, \quad \frac{\pi}{2} < x < \pi, \tag{37}$$

where

$$\begin{aligned} g(x) &= 2\Big(w(\pi - x) - K_1(\pi, x; h, q, M_1) - K_2(\pi, x; h, q, M_1) - HK(\pi, x; h, q, M_1)\Big),\\ R(x,t) &= 2\Big(G_1(\pi, x, t; h, q, M_1) + G_2(\pi, x, t; h, q, M_1) + HG(\pi, x, t; h, q, M_1)\Big). \end{aligned} \tag{38}$$

Further, putting

$$N(x) := (\pi - x)M_2(x), \quad Q(x,t) := \frac{R(x,t) - 1}{\pi - t}, \quad \phi(x) := \int_{\frac{\pi}{2}}^x Q(x,t)N(t)\,dt, \tag{39}$$

we rewrite (37) in the form

$$g(x) = N(x) + \int_{\frac{\pi}{2}}^{x} \frac{N(t)\,dt}{\pi - t} + \phi(x), \quad \frac{\pi}{2} < x < \pi. \tag{40}$$

Using (7), (26), (36), (38) and (39), we obtain

$$\begin{aligned}
Q(x,t) = {} & \frac{1}{\pi - t}\Big(\int_0^{\pi - x} K(x - t + s, x - t)\,ds + \frac{1}{2}\int_0^{x-t} \Big(K\Big(\frac{x - t + s}{2}, s\Big) + K\Big(\pi + \frac{s - x - t}{2}, s\Big)\Big)\,ds \\
& + \int_x^{\pi} q(s)G(s,x,t)\,ds + \int_{\frac{x+t}{2}}^{x} q(s)G(s, 2s - x, t)\,ds + \int_{\pi - \frac{x-t}{2}}^{\pi} q(s)G(s, 2(s-\pi) + x, t)\,ds \\
& + \int_0^{x-t} M_1(s)\Big(\int_x^{\pi} G(\xi - s, x - s, t)\,d\xi + \frac{1}{2}\int_t^{x-s}\Big(G\Big(\frac{x - s + \xi}{2}, \xi, t\Big) \\
& + G\Big(\pi + \frac{\xi - x - s}{2}, \xi, t\Big)\Big)\,d\xi\Big) + H + \frac{H}{\pi - t}\Big(\int_0^{x-t}\Big(\int_s^{\frac{x-t+s}{2}} K(\xi,s)d\xi + \int_s^{\pi - \frac{x+t-s}{2}} K(\xi,s)d\xi\Big)ds \\
& + \int_t^{x}\Big(\int_s^{\pi - \frac{x-s}{2}} q(\xi)G(\xi,s,t)\,d\xi + \int_s^{\frac{x+s}{2}} q(\xi)G(\xi,s,t)\,d\xi\Big)\,ds \\
& + \int_0^{x-t} M_1(s)\int_t^{x-s}\Big(\int_\xi^{\frac{x-s+\xi}{2}} G(\eta,\xi,t)\,d\eta + \int_\xi^{\pi - \frac{x+s-\xi}{2}} G(\eta,\xi,t)\,d\eta\Big)d\xi\Big),
\end{aligned}$$

where $K(x,t) = K(x,t;h,q,M_1)$ and $G(x,t,\tau) = G(x,t,\tau;h,q,M_1)$. By virtue of boundedness of these two functions, we get

$$|Q(x,t)| \le C\Big(1 + \frac{1}{\pi - t}\int_t^{\pi} |q(s)|\,ds\Big) =: f(t) \in L_2\Big(\frac{\pi}{2}, \pi\Big), \quad \frac{\pi}{2} < t < x < \pi, \tag{41}$$

(see, e.g., Lemma 2.1 in [35]). This, in particular, implies $Q(x,t) \in L_2((\pi/2,\pi)^2)$ and hence, according to the analogue of Lemma 2.5 in [14] (or Lemma 5.2 in [26]) for $\eta < 0$, we get $(\pi - x)N(x) \in L_2(\pi/2,\pi)$. Let us show that $N(x) \in L_2(\pi/2,\pi)$. For this purpose, we use the following assertion, which follows from (39) and (41).

Lemma 7. *Fix $\theta \in [1/3, 1]$. If $(\pi - x)^{\theta} N(x) \in L_2(\pi/2,\pi)$, then $(\pi - x)^{\theta - 1/3}\phi(x) \in L_2(\pi/2,\pi)$.*

Applying Lemma 7 along with Lemma 2.4 in [14] (or Lemma 5.1 in [26]) subsequently three times, we finally refine that $N(x) \in L_2(\pi/2,\pi)$, which finishes the proof of Theorem 3.

6. Solution of the inverse problem

Before proceeding directly to the proof of Theorem 2, we provide one more auxiliary assertion.

Lemma 8. *Let arbitrary complex numbers λ_n, $n \ge 0$, of the form (3) be given. Then the function $\Delta(\lambda)$, determined by formula (21), has the form (18) with some function $w(x) \in L_2(0,\pi)$.*

Proof. Let us obtain the assertion of the lemma as a corollary from Lemma 3.3 in [14], which implies that the function $\Delta_1(\lambda) := (\lambda_0 - \lambda)^{-1}\Delta(\lambda)$ has the form

$$\Delta_1(\lambda) = \frac{\sin\rho\pi}{\rho} - \omega\pi\frac{\cos\rho\pi}{\rho^2} + \int_0^\pi w_1(x)\frac{\cos\rho x}{\rho^2}\,dx, \quad w_1(x) \in L_2(0,\pi), \quad \int_0^\pi w_1(x)\,dx = \omega\pi.$$

Thus, we have

$$\Delta(\lambda) = -\rho\sin\rho\pi + \omega\pi\cos\rho\pi - \int_0^\pi w_1(x)\cos\rho x\,dx + \lambda_0\Big(\frac{\sin\rho\pi}{\rho} - \omega\pi\frac{\cos\rho\pi}{\rho^2} + \int_0^\pi w_1(x)\frac{\cos\rho x}{\rho^2}\,dx\Big).$$

Since

$$\frac{\sin\rho x}{\rho} = \int_0^x \cos\rho t\,dt, \quad \frac{\cos\rho x}{\rho^2} = \frac{1}{\rho^2} - \int_0^x (x-t)\cos\rho t\,dt,$$

we arrive at representation (18), where

$$w(x) = -w_1(x) + \lambda_0\Big(1 + (\pi - x)\omega\pi - \int_x^\pi (t-x)w_1(t)\,dt\Big) \in L_2(0,\pi),$$

which finishes the proof. □

Now we are in position to give the proof of Theorem 2.

Proof of Theorem 2. Let a complex-valued function $q(x) \in L_2(0,\pi)$ and a complex number h be given along with some sequence of complex numbers $\{\lambda_n\}_{n\ge0}$ of the form (3). Then we find the number H from the relation (4), where ω is determined by the formula

$$\omega = \frac{1}{2}\lim_{n\to\infty}(\lambda_n - n^2). \tag{42}$$

According to Lemma 8, the function $\Delta(\lambda)$, constructed by formula (21), has the form (18) with some function $w(x) \in L_2(0,\pi)$. By virtue of Theorem 3, the main equation (19) with these $w(x)$, $q(x)$, h and H has a unique solution $M(x)$, $(\pi - x)M(x) \in L_2(0,\pi)$. Consider the corresponding boundary value problem $L = L(q, M, h, H)$. Let $\tilde\Delta(\lambda)$ be its characteristic function. Then, by virtue of Lemma 3, it has the form

$$\tilde\Delta(\lambda) = -\rho\sin\rho\pi + \omega\pi\cos\rho\pi + \int_0^\pi \tilde w(x)\cos\rho x\,dx, \quad \tilde w(x) \in L_2(0,\pi), \tag{43}$$

where

$$\tilde w(\pi - x) = K_1(\pi, x; h, q, M) + K_2(\pi, x; h, q, M) + HK(\pi, x; h, q, M), \quad 0 < x < \pi. \tag{44}$$

Comparing (44) with (19), we get $\tilde w(x) = w(x)$ a.e. on $(0,\pi)$ and, hence, $\tilde\Delta(\lambda) \equiv \Delta(\lambda)$. Thus, the spectrum of the constructed boundary value problem L coincides with the sequence $\{\lambda_n\}_{n\ge0}$.

Uniqueness of $M(x)$ follows from uniqueness of solution of the main equation (19). □

This proof is constructive and gives the following algorithm for solving Inverse Problem 1.

Algorithm 1. Let the spectrum $\{\lambda_n\}_{n\ge0}$ of some boundary value problem $L(q, M, h, H)$ along with the function $q(x)$ and the number h be given.

(i) Find the number ω by (42) and the number H by the formula

$$H = \omega\pi - h - \frac{1}{2}\int_0^\pi q(x)\,dx;$$

(ii) In accordance with (18), calculate the function $w(x)$ by the formula

$$w(x) = \frac{2}{\pi}\sum_{n=0}^{\infty}(\Delta(n^2) - (-1)^n\omega\pi)\cos nx,$$

where the function $\Delta(\lambda)$ is determined by (21);

(iii) Find the function $M(x)$ from the main equation (19).

Acknowledgement

This work was supported by Grant 17-11-01193 of the Russian Science Foundation.

References

[1] G. Borg, *Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe*, Acta Math., **78** (1946), 1–96.

[2] V. A. Marchenko, Sturm–Liouville Operators and Their Applications, Naukova Dumka, Kiev, 1977; English transl., Birkhäuser, 1986.

[3] B. M. Levitan, Inverse Sturm–Liouville Problems, Nauka, Moscow, 1984 (Russian); English transl., VNU Sci.Press, Utrecht, 1987.

[4] G. Freiling and V. A. Yurko, Inverse Sturm–Liouville Problems and Their Applications, NOVA Science Publishers, New York, 2001.

[5] V. A. Yurko, Introduction to the Theory of Inverse Spectral Problems, Moscow, Fizmatlit, 2007.

[6] V. A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.

[7] V. A. Yurko, Inverse Spectral Problems for Differential Operators and Their Applications, Gordon and Breach Science Publishers, Amsterdam, 2000.

[8] R. Beals, P. Deift and C. Tomei, Direct and Inverse Scattering on the Line, Mathematical Surveys and Monographs, 28. AMS, Providence, RI, 1988.

[9] V. Lakshmikantham and Rama Mohana Rao M., Theory of integro-differential equations, Stability and Control: Theory, Methods and Applications, v.1, Gordon and Breach Science Publishers, Singapore, 1995.

[10] M. M. Malamud, *On some inverse problems*, Boundary Value Problems of Mathematical Physics, Kiev, 1979, 116–124.

[11] V. A. Yurko, *Inverse problem for integro-differential operators of the first order*, Functional Analysis, Ul'janovsk, 1984, 144–151.

[12] M. S. Eremin, *An inverse problem for a second-order integro-differential equation with a singularity*, Diff. Uravn., **24** (1988), no.2, 350–351.

[13] V. A. Yurko, *An inverse problem for integro-differential operators*, Mat. Zametki, **50** (1991), no.5, 134–146 (Russian); English transl. in Math. Notes, **50** (1991), no. 5–6, 1188–1197.

[14] S. A. Buterin, *On an inverse spectral problem for a convolution integro-differential operator*, Results Math., **50** (2007), no.3-4, 173–181.

[15] Ju. V. Kuryshova, *Inverse spectral problem for integro-differential operators*, Mat. Zametki, **81** (2007), no.6, 855–866; English transl. in Math. Notes, **81** (2007), no.6, 767–777.

[16] S. A. Buterin, *On the reconstruction of a convolution perturbation of the Sturm–Liouville operator from the spectrum*, Diff. Uravn., **46** (2010), 146–149 (Russian); English transl. in Diff. Eqns., **46** (2010), 150–154.

[17] Yu. V. Kuryshova and C.-T. Shieh, *An inverse nodal problem for integro-differential operators*, J. Inverse and Ill-Posed Problems, **18** (2010), no.4, 357–369.

[18] Y. Wang and G. Wei, *The uniqueness for Sturm–Liouville problems with aftereffect*, Acta Math Sci., **32A** (2012), no.6, 1171–1178.

[19] V. A. Yurko, *An inverse spectral problem for integro-differential operators*, Far East J. Math. Sci., **92** (2014), no.2, 247–261.

[20] S. A. Buterin and A. E. Choque Rivero, *On inverse problem for a convolution integro-differential operator with Robin boundary conditions*, Appl. Math. Lett., **48** (2015), 150–155.

[21] S. A. Buterin and M. Sat, *On the half inverse spectral problem for an integro-differential operator*, Inverse Problems in Science and Engineering, **25** (2017), no.10, 1508–1518.

[22] N. Bondarenko and S. Buterin, *On recovering the Dirac operator with an integral delay from the spectrum*, Results Math., **71** (2017), no.3-4, 1521–1529.

[23] V. A. Yurko, *Inverse spectral problems for first order integro-differential operators*, Boundary Value Problems (2017) 2017:98 https://doi.org/10.1186/s13661-017-0831-8.

[24] N. Bondarenko and S. Buterin, *An inverse spectral problem for integro-differential Dirac operators with general convolution kernels*, Applicable Analysis (2018), 17pp. https://doi.org/10.1080/00036811.2018.1508653.

[25] S. A. Buterin, *On inverse spectral problems for first-order integro-differential operators with discontinuities*, Appl. Math. Lett., **78** (2018), 65–71.

[26] S. A. Buterin, *Inverse spectral problem for Sturm–Liouville integro-differential operators with discontinuity conditions*, Sovr. Mat. Fundam. Napravl., **64** (2018), no.3, 427–458; Engl. transl. in J. Math. Sci. (To appear)

[27] N. P. Bondarenko, *An inverse problem for an integro-differential operator on a star-shaped graph*, Math. Meth. Appl. Sci., **41** (2018), no.4, 1697–1702.

[28] S. A. Buterin and S. V. Vasiliev, *On uniqueness of recovering the convolution integro-differential operator from the spectrum of its non-smooth one-dimensional perturbation*, Boundary Value Problems (2018) 2018:55, https://doi.org/10.1186/s13661-018-0974-2.

[29] V. A. Zolotarev, *Inverse spectral problem for the operators with non-local potential*, Mathematische Nachrichten (2018), 1–21, DOI: https://doi.org/10.1002/mana.201700029.

[30] M. Ignatyev, *On an inverse spectral problem for the convolution integro-differential operator of fractional order*, Results Math., (2018), 73:34. https://doi.org/10.1007/s00025-018-0800-2.

[31] M. Ignatiev, *On an inverse spectral problem for one integro-differential operator of fractional order*, J. Inverse and Ill-posed Probl., **27** (2019), no.1, 17–23.

[32] N. P. Bondarenko, *An inverse problem for the integro-differential Dirac system with partial information given on the convolution kernel*, J. Inverse Ill-Posed Probl., **27** (2019), no.2, 151–157.

[33] S. Buterin and M. Malyugina, *On global solvability and uniform stability of one nonlinear integral equation*, Results Math., (2018), 73:117, 1–19. https://doi.org/10.1007/s00025-018-0879-5.

[34] S. A. Buterin and M. A. Kuznetsova, *On Borg's method for non-selfadjoint Sturm-Liouville operators*, Anal. Math. Phys., (2019), 1–18. https://doi.org/10.1007/s13324-019-00307-9.

[35] S. A. Buterin, *Inverse spectral reconstruction problem for the convolution operator perturbed by a one-dimensional operator*, Matem. Zametki, **80** (2006), no.5, 668–682 (Russian); English transl. in Math. Notes, **80** (2006), no.5, 631–644.

Department of Mathematics, Saratov State University, Astrahanskaya st., 83, Saratov 410026, Russia.

E-mail: buterinsa@info.sgu.ru