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AN INVERSE PROBLEM FOR THE SECOND-ORDER INTEGRO-DIFFERENTIAL PENCIL

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Abstract. We consider the second-order (Sturm-Liouville) integro-differential pencil with polynomial dependence on the spectral parameter in a boundary condition. The inverse problem is solved, which consists in reconstruction of the convolution kernel and one of the polynomials in the boundary condition by using the eigenvalues and the two other polynomials. We prove uniqueness of solution, develop a constructive algorithm for solving the inverse problem, and obtain necessary and sufficient conditions for its solvability.

1. Introduction

This paper concerns the inverse problem theory for integro-differential operators. Inverse problems of spectral analysis consist in reconstruction of operators by their spectral characteristics. The basic results in the theory of inverse problems were obtained for *differential* operators (see the monographs [1, 2, 3, 4] and references therein).

Nowadays nonlocal operators, in particular, *integro-differential* operators attract much attention of mathematicians. On the one hand, such operators are natural for modeling real-world processes in physics, biology, engineering and other applications (see, e.g., [5]). On the other hand, nonlocality of integro-differential operators is an insuperable obstacle for classical methods of inverse problem theory.

First fragmentary results on inverse problems for integro-differential operators appeared in [6, 7, 8]. Later on, a powerful method has been developed for recovering convolution perturbations of various differential operators (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). This method is based on the reduction of an inverse problem to a uniquely solvable nonlinear equation of a special form. Recently the general theory of such nonlinear equations has been constructed in [19]. Some other types of inverse problems for integro-differential operators were investigated in [20, 21, 22, 23, 24, 25, 26].

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This paper deals with a new class of operators, namely, integro-differential pencils with nonlinar dependence on the spectral parameter. We consider the boundary value problem $L = L(M, A_1, A_2, A_3)$ for the Sturm-Liouville equation with an integral delay

$$-y''(x) + \int_0^x M(x-t)y'(t)\,dt = \lambda y(x), \quad x \in (0,\pi), \tag{1}$$

with the boundary conditions, depending polynomially on the spectral parameter:

$$y(0) = 0, \quad A_1(\lambda)y'(\pi) + A_2(\lambda)y(\pi) + A_3(\lambda)y'(0) = 0. \tag{2}$$

Here $M(x)$ is a complex-valued function from the class $L_{2,\pi} := \{f : (\pi - x)f(x) \in L_2(0,\pi)\}$,

$$A_j(\lambda) = \sum_{k=0}^{n} a_{jk}\lambda^k, \quad j = \overline{1,3}, \tag{3}$$

where $\{a_{jk}\}_{j=\overline{1,3},\,k=\overline{0,n}}$ are complex numbers, $a_{1n} \neq 0$. Without loss of generality, we assume that $a_{1n} = 1$. For simplicity, we also assume that $a_{3n} = 0$. The case $a_{3n} \neq 0$ requires some technical modifications.

Investigation of inverse problems for *differential* Sturm-Liouville pencils with boundary conditions, depending polynomially on the spectral parameter, leads to essential difficulties, comparing with standard differential operators (see [27, 28] and references therein). Inverse spectral problems for the integro-differential Strum-Liouville equation (1) without dependence on the spectral parameter in the boundary conditions have been solved in [10, 11, 12]. In this paper, we develop the ideas of [10, 11] for the pencil (1)-(2). We describe the asymptotic behavior of the spectrum, and solve the following inverse problem.

Inverse Problem 1. *Given the spectrum Λ of the boundary value problem L and the polynomials $A_j(\lambda)$, $j = 1,2$, construct the convolution kernel $M(x)$ and the polynomial $A_3(\lambda)$.*

We prove the uniqueness of solution, develop a constructive procedure for solving Inverse Problem 1 and provide necessary and sufficient conditions for its solvability. We proceed with formulations of the main theorems.

Theorem 1. *The spectrum of the problem L is a countable set of eigenvalues, which can be represented in the form $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde\lambda_j\}_{j=1}^n$, where*

$$\lambda_k = \left(k - \tfrac{1}{2} + \kappa_k\right)^2, \quad k \in \mathbb{N}, \qquad \{\kappa_k\} \in l_2. \tag{4}$$

Theorem 2. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde\lambda_j\}_{j=1}^n$ be an arbitrary sequence of complex numbers, satisfying the asymptotic relation (4), and let $A_j(\lambda)$, $j = 1,2$, be polynomials of degree n with arbitrary complex coefficients $\{a_{jk}\}_{j=1,2,\,k=\overline{0,n-1}} \cup \{a_{2n}\}$ and $a_{1n} = 1$. Then there exist the unique complex-valued function $M(x) \in L_{2,\pi}$ and the unique polynomial $A_3(\lambda) = \sum_{k=0}^{n-1} a_{3k}\lambda^k$, such that the spectrum of the boundary value problem $L = L(M, A_1, A_2, A_3)$ coincides with Λ.*

The paper is organized as follows. In Section 2, some preliminaries are provided and Theorem 1 is proved. We obtain a special representation for the characteristic function (see Lemma 1), which allows us to derive asymptotic formulas for the eigenvalues and then is used for solving the inverse problem. In Section 3, the characteristic function is constructed by its zeros as an infinite product and the nonlinear main equation (26) is derived. Finally, we prove Theorem 2 and provide a constructive algorithm for solving Inverse Problem 1. One can apply our approach to other classes of integro-differential pencils with nonlinear dependence on the spectral parameter in boundary conditions or/and in an equation.

2. Preliminaries and Eigenvalue Asymptotics

The goal of this section is to prove Theorem 1. We start with some preliminaries. Let $\lambda = \rho^2$. We will use the following notations:

$$(f * g)(x) = \int_0^x f(x-t)g(t)\,dt,$$
$$f^{*1} := f, \quad f^{*(\nu+1)} := f^{*\nu} * f, \quad \nu \geq 1,$$
$$g_k(x) = \frac{x^k}{k!}, \quad k \geq 0.$$

Clearly, the functions $g_k(x)$ have the following properties:

$$g'_{k+1} = g_k, \quad g_k * g_0 = g_{k+1}, \quad k \geq 0. \tag{5}$$

Let $S(x,\lambda)$ be the solution of equation (1), satisfying the initial conditions $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$. According to the results of [10], the solution $S(x,\lambda)$ can be represented in the form

$$S(x,\lambda) = \frac{\sin \rho x}{\rho} + \int_0^x P(x,t)\frac{\sin \rho(x-t)}{\rho}\,dt, \tag{6}$$

where

$$P(x,t) = \sum_{\nu=1}^{\infty} g_\nu(x-t)N^{*\nu}(t), \tag{7}$$

and $N(t)$, $t \in (0,\pi)$, is the solution of the integral equation

$$M(x) = 2N(x) - \int_0^x N^{*2}(t)\,dt. \tag{8}$$

Note that equation (8) is a special case of (26). By Proposition 4, equation (8) has a unique solution $N(t)$ in $L_2(0,T)$ for every $T \in (0,\pi)$. One can easily show that $N \in L_{2,\pi}$.

The spectrum of L is purely discrete and consists of complex eigenvalues, which coincide with the zeros of the entire characteristic function

$$\Delta(\lambda) := A_1(\lambda)S'(\pi,\lambda) + A_2(\lambda)S(\pi,\lambda) + A_3(\lambda). \tag{9}$$

The relation (6) implies

$$S(\pi,\lambda)=\frac{\sin\rho\pi}{\rho}+\int_0^\pi w(t)\frac{\sin\rho(\pi-t)}{\rho}\,dt, \tag{10}$$

$$S'(\pi,\lambda)=\cos\rho\pi+\int_0^\pi v(t)\cos\rho(\pi-t)\,dt, \tag{11}$$

where $w(t):=P(\pi,t)$, $v(t):=P(\pi,t)+\int_0^t P_x(\pi,s)ds$. Using (7), we derive the relations

$$w(t)=\sum_{\nu=1}^\infty g_\nu(\pi-t)N^{*\nu}(t),\quad v(t)=\sum_{\nu=1}^\infty g_\nu(\pi-t)N^{*\nu}(t)+\sum_{\nu=1}^\infty\int_0^t g_{\nu-1}(\pi-s)N^{*\nu}(s)\,ds. \tag{12}$$

Since $N\in L_{2,\pi}$, we have $w,v\in L_2(0,\pi)$. Applying integration by parts, we transform (10) into the relation

$$S(\pi,\lambda)=\int_0^\pi (g_0+w*g_0)(t)\cos\rho(\pi-t)\,dt. \tag{13}$$

The following Lemma 1 provides an important represenation for the characteristic function $\Delta(\lambda)$, which will be used for solving the inverse problem.

Lemma 1. *The following relation holds*

$$\Delta(\lambda)=\lambda^n\left(\cos\rho\pi+\sum_{j=1}^n\frac{c_{n-j}}{\lambda^j}+\int_0^\pi r(t)\cos\rho(\pi-t)\,dt\right), \tag{14}$$

where

$$c_j=\sum_{k=0}^j(-1)^k\big(a_{1,j-k}(g_{2k}+v*g_{2k})+a_{2,j-k}(g_{2k+1}+w*g_{2k+1})\big)(\pi)+a_{3j},\quad j=\overline{0,n-1}, \tag{15}$$

$$\begin{aligned} r(t)&=v(t)+a_{2n}(g_0+w*g_0)(t)\\ &\quad+\sum_{k=1}^n(-1)^k(a_{1,n-k}(g_{2k-1}+v*g_{2k-1})+a_{2,n-k}(g_{2k}+w*g_{2k}))(t),\quad r\in L_2(0,\pi). \end{aligned}\tag{16}$$

Proof. Substituting (11), (13) and (3) into (9), we derive

$$\begin{aligned} \Delta(\lambda)&=A_1(\lambda)\cos\rho\pi+\int_0^\pi(A_1(\lambda)v(t)+A_2(\lambda)(g_0+w*g_0)(t))\cos\rho(\pi-t)\,dt+A_3(\lambda)\\ &=\lambda^n\Big(\cos\rho\pi+\sum_{k=1}^n\frac{a_{1,n-k}\cos\rho\pi}{\lambda^k}+\sum_{k=1}^n\frac{a_{3,n-k}}{\lambda^k}\\ &\quad+\sum_{k=0}^n\frac{1}{\lambda^k}\int_0^\pi(a_{1,n-k}v+a_{2,n-k}(g_0+w*g_0))(t)\cos\rho(\pi-t)\,dt\Big). \end{aligned}\tag{17}$$

Using the relations (5), integration by parts and induction, we prove that

$$\frac{\cos\rho\pi}{\lambda^k}=\sum_{j=1}^k\frac{(-1)^{k-j}g_{2(k-j)}(\pi)}{\lambda^j}+(-1)^k\int_0^\pi g_{2k-1}(t)\cos\rho(\pi-t)dt, \tag{18}$$

$$\frac{1}{\lambda^k}\int_0^\pi u(t)\cos\rho(\pi-t)dt=\sum_{j=1}^k\frac{(-1)^{k-j}(u*g_{2(k-j)})(\pi)}{\lambda^j}+(-1)^k\int_0^\pi(u*g_{2k-1})(t)\cos\rho(\pi-t)dt, \tag{19}$$

for any function $u \in L_2(0,\pi)$ and $k \in \mathbb{N}$.

Using (18), we obtain

$$\sum_{k=1}^{n} \frac{a_{1,n-k}\cos\rho\pi}{\lambda^k} = \sum_{j=1}^{n} \frac{1}{\lambda^j} \sum_{k=0}^{n-j} a_{1,n-j-k}(-1)^k g_{2k}(\pi) + \sum_{k=1}^{n} a_{1,n-k}(-1)^k \int_0^\pi g_{2k-1}(t)\cos\rho(\pi-t)\,dt. \tag{20}$$

Similarly using (19) and (5), we derive

$$\sum_{k=1}^{n} \frac{1}{\lambda^k} \int_0^\pi (a_{1,n-k}v(t) + a_{2,n-k}(g_0 + w * g_0)(t))\cos\rho(\pi-t)\,dt$$
$$= \sum_{j=1}^{n} \frac{1}{\lambda^j} \sum_{k=0}^{n-j} (-1)^k (a_{1,n-j-k} v * g_{2k} + a_{2,n-j-k}(g_{2k+1} + w * g_{2k+1}))(\pi)$$
$$+ \sum_{k=1}^{n} (-1)^k \int_0^\pi (a_{1,n-k} v * g_{2k-1} + a_{2,n-k}(g_{2k} + w * g_{2k}))(t)\cos\rho(\pi-t)\,dt. \tag{21}$$

Substituting (20) and (21) into (17), we arrive at the relation (14) with the coefficients $\{c_j\}_{j=0}^{n-1}$ and $r(t)$, defined by (15) and (16), respectively. Since v and w belong to $L_2(0,\pi)$, the relation (16) implies $r \in L_2(0,\pi)$. □

Proof of Theorem 1. In view of (14), the function $\Delta(\lambda)$ is asymptotically close to the entire function $\Delta_0(\lambda) = \lambda^n \cos\rho\pi$ as $|\lambda| \to \infty$. Using the standard technique (see, for example, [4, Theorem 1.1.3]), based on Rouché's theorem, we show that the spectrum of L has the form $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}\{\tilde{\lambda}_j\}_{j=1}^n$, where the numbers $\{\lambda_k\}$ satisfy (4). □

3. Solution of Inverse Problem

This section is devoted to the proof of Theorem 2 and constructive solution of Inverse Problem 1. First we need two auxiliary lemmas.

Lemma 2. *The specification of the spectrum $\{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde{\lambda}_j\}_{j=1}^n$ uniquely determines the characteristic function $\Delta(\lambda)$ by the formula*

$$\Delta(\lambda) = R_n(\lambda) \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{\left(k-\frac{1}{2}\right)^2}, \quad R_n(\lambda) := \prod_{j=1}^{n} (\lambda - \tilde{\lambda}_j). \tag{22}$$

Proof. Similar to the proof of [4, Theorem 1.1.4]. □

In view of the relation (14), we have $c_j = \frac{\Delta^{(j)}(0)}{j!}$ for $j = \overline{0, n-1}$, and, consequently, the function

$$g(\lambda) := \lambda^{-n}\left(\Delta(\lambda) - \sum_{j=0}^{n-1} \frac{\Delta^{(j)}(0)}{j!}\lambda^j\right) \tag{23}$$

is entire in the λ-plane and has the form

$$g(\lambda) = \cos\rho\pi + \int_0^\pi r(t)\cos\rho(\pi - t)\,dt, \tag{24}$$

where the function $r \in L_2(0,\pi)$ was defined by (16).

Lemma 3. *Let $\{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde\lambda_j\}_{j=1}^n$ be arbitrary complex numbers, satisfying (4), and let $\Delta(\lambda)$ be the function, constructed by these numbers via (22). Then $\Delta(\lambda)$ has the form (14) with some complex numbers $\{c_j\}_{j=0}^{n-1}$ and some function $v \in L_2(0,\pi)$.*

Proof. One can prove similarly to [10, Lemma 3.3], that any function $d(\lambda)$ in the form

$$d(\lambda) = \prod_{k=1}^\infty \frac{\lambda_k - \lambda}{\left(k - \frac{1}{2}\right)^2}$$

can be represented as follows:

$$d(\lambda) = \cos\rho\pi + \int_0^\pi q(t)\cos\rho(\pi - t)\,dt,$$

where q is some function from $L_2(0,\pi)$. Consequently, the relations (22) and (23) yield

$$g(\lambda) = \cos\rho\pi + \int_0^\pi q(t)\cos\rho(\pi-t)\,dt + O\left(|\lambda|^{-1}\exp(|\mathrm{Im}\,\rho|\pi)\right), \quad |\lambda| \to \infty.$$

Clearly, the function $(g(\rho^2) - \cos\rho\pi)$ is entire and even with respect to ρ, it satisfies the estimate $g(\rho^2) = O(\exp(|\mathrm{Im}\,\rho|\pi))$ and is square integrable on the real axis. Therefore, by Paley-Wiener theorem, there exists a function $r \in L_2(0,\pi)$, such that (24) holds. The relations (23) and (24) imply (14) with $c_j = \frac{\Delta^{(j)}(0)}{j!}$, $j = \overline{0, n-1}$. □

Further, using (12) and (16), we will derive the main equation with respect to the function $N(t)$. Define

$$p(t) = r(t) - a_{2n} + \sum_{k=1}^n (-1)^{k+1}(a_{1,n-k}g_{2k-1}(t) + a_{2,n-k}g_{2k}(t)). \tag{25}$$

Then the relation (16) yields

$$p(t) = v(t) + a_{2n}(w * g_0)(t) + \sum_{k=1}^n (-1)^k (a_{1,n-k}v * g_{2k-1} + a_{2,n-k}w * g_{2k})(t).$$

Substituting (12) into the latter formula, we arrive at the nonlinear integral equation

$$f(t) = \sum_{\nu=1}^\infty \left(\psi_\nu(t)N^{*\nu}(t) + \int_0^t \Psi_\nu(t,s)N^{*\nu}(s)\,ds\right), \tag{26}$$

where

$$f(t) = \frac{p(t)}{\pi - t}, \quad \psi_\nu(t) = \tfrac{1}{\nu} g_{\nu-1}(\pi - t), \tag{27}$$

$$\Psi_\nu(t,s) = \frac{1}{\pi - t}\Big(g_{\nu-1}(\pi - s) + a_{2n} g_\nu(\pi - s) + \sum_{k=1}^{n} (-1)^k (a_{1,n-k} g_{2k-1}(t-s) g_\nu(\pi - s)$$
$$+ a_{1,n-k} g_{2k}(t-s) g_{\nu-1}(\pi - s) + a_{2,n-k} g_{2k}(t-s) g_\nu(\pi - s)\Big), \quad \nu \in \mathbb{N}. \tag{28}$$

We call the relation (26) *the main equation* of Inverse Problem 1. The main equation plays a crucial role in the proof of Theorem 2 and in the constructive solution of the inverse problem. Theorem 3 establishes the unique solvability of (26).

Theorem 3. *For any function $f \in L_{2,\pi}$ and the functions $\psi_\nu(t)$, $\Psi_\nu(t,s)$, $\nu \in \mathbb{N}$, defined by* (27) *and* (28)*, the main equation* (26) *has a unique solution $N \in L_{2,\pi}$.*

The proof of Theorem 3 is based on the following proposition, which is a special case of [19, Theorem 3] (see also [9, Theorem 4]).

Proposition 4. *Let f be any function from $L_2(0,T)$, and let $\psi_\nu(t)$ and $\Psi_\nu(t,s)$ for $\nu \in \mathbb{N}$ be square integrable functions on $(0,T)$ and $\mathscr{S} := \{(t,s)\colon 0 < s < t < T\}$, respectively, such that $\psi_1(t) \equiv 1$ and*

$$\|\psi_\nu\|_{L_2(0,T)} \le C^\nu, \quad \|\Psi_\nu\|_{L_2(\mathscr{S})} \le C^\nu, \quad \nu \in \mathbb{N},$$

where $C > 0$ is a constant. Then equation (26) *has a unique solution $N(t)$ in $L_2(0,T)$.*

Proof of Theorem 3. Obviously, the functions $\psi_\nu(t)$ and $\Psi_\nu(t,s)$, defined by (27) and (28), satisfy the conditions of Proposition 4 for every $T \in (0,\pi)$. Thus, the main equation (26) has a unique solution $N \in L_2(0,T)$ for every $T \in (0,\pi)$. One can prove that $N \in L_{2,\pi}$, using the approach of [10, 9]. □

Proof of Theorem 2. Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde{\lambda}_j\}_{j=1}^n$ be complex numbers, such that (4) holds, and let $A_j(\lambda)$, $j = 1,2$, be polynomials in the form (3) with $a_{1n} = 1$. Define the function $\Delta(\lambda)$ by the formula (22). By virtue of Lemma 3, the function $\Delta(\lambda)$ can be represented in the form (14) with some (uniquely determined) coefficients $\{c_j\}_{j=0}^{n-1}$ and some function $r \in L_2(0,\pi)$. Define the function $p(t)$ by (25), and then $f(t)$, $\psi_\nu(t)$, $\Psi_\nu(t,s)$, $\nu \in \mathbb{N}$, by (27), (28). Since $r \in L_2(0,\pi)$, we have $p \in L_2(0,\pi)$ and $f \in L_{2,\pi}$. According to Theorem 3, the main equation (26) has a unique solution $N \in L_{2,\pi}$. Let $M(x)$ be the function, defined by (8). Clearly, $M \in L_{2,\pi}$. We also find the coefficients $\{a_{3j}\}_{j=0}^{n-1}$ of the polynomial $A_3(\lambda)$ from (15):

$$a_{3j} = c_j - \sum_{k=0}^{j} (-1)^k \left(a_{1,j-k}(g_{2k} + v * g_{2k}) + a_{2,j-k}(g_{2k+1} + w * g_{2k+1})\right)(\pi), \quad j = \overline{0, n-1}. \tag{29}$$

Consider the boundary value problem $L = L(M, A_1, A_2, A_3)$, where $M(x)$ and $A_3(\lambda)$ were defined in this proof and $A_1(\lambda)$, $A_2(\lambda)$ are the initially given polynomials. Let us prove that the spectrum of L coincides with Λ. Indeed, by virtue of Lemma 1, the characteristic function of L has the form (14) with the coefficients $\{c_j\}_{j=0}^{n-1}$ and the function $r \in L_2(0,\pi)$, defined by (15) and (16), respectively. Consequently, this characteristic function coincides with the defined above $\Delta(\lambda)$, having the zeros $\{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde{\lambda}_j\}_{j=1}^{n}$. In view of the uniqueness of the main equation solution and the uniqueness at the other steps of this proof, the problem L is uniquely specified by its spectrum Λ and the polynomials $A_j(\lambda)$, $j = 1, 2$. □

The proof of Theorem 2 leads to the following algorithm for solving Inverse Problem 1.

Algorithm 1. Let the complex numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \cup \{\tilde{\lambda}_j\}_{j=1}^{n}$, satisfying (4), and the polynomials $A_j(\lambda)$, $j = 1, 2$, in the form (3) with $a_{1n} = 1$, be given.

1. Construct the characteristic function $\Delta(\lambda)$ by its zeros, using (22).
2. Find the coefficients $c_j = \frac{\Delta^{(j)}(0)}{j!}$, $j = \overline{0, n-1}$.
3. Construct the function $g(\lambda)$ by (23).
4. Find $r(t)$, inverting the Fourier transform (24):
$$r(t) = \frac{1}{\pi}(g(0) - 1) + \frac{2}{\pi}\Big((-1)^k g(k) - 1\Big)\cos kt, \quad t \in (0,\pi).$$
5. Construct $p(t)$ by (25), then construct $f(t)$, $\psi_\nu(t)$, $\Psi_\nu(t,s)$, $\nu \in \mathbb{N}$, $0 < s < t < \pi$, by (27) and (28).
6. Solving the main equation (26), obtain $N(t)$, $t \in (0,\pi)$.
7. Find $M(x)$, $x \in (0,\pi)$, using (8).
8. Find $\{a_{3j}\}_{j=0}^{n-1}$ by (29).

Thus, the kernel $M(x)$ and the coefficients of $A_3(\lambda)$ are determined.

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