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INTEGRAL TRANSFORMS CONNECTED WITH DIFFERENTIAL SYSTEMS WITH A SINGULARITY

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Abstract. We consider some integral transforms with the kernels expressed in terms of solutions of the system of differential equations $y' = (x^{-1}A + B)y$, where A and B are constant $n \times n$, $n > 2$, matrices. We study analytical and asymptotical properties of such transforms. We also study the transforms as operators acting in some functional spaces.

1. Introduction

Various dilates of classical integral transforms often appear in differential equations theory. In recent paper [1], for instance, the following integral transform:

$$f \to v(f, x, \rho) = \int_0^x f(t) \exp(i\rho t)\, dt$$

was considered in connection with the questions of asymptotical behavior of resolvent and eigenvalues of higher order ordinary differential operators with distributional coefficients. On the other hand, the kernels of such integral transforms themselves are usually connected with certain differential equations. In particular, the kernel $\exp(i\rho t)$ is a solution of the "simplest" differential equation $-iy' = \rho y$. Similarly, the kernels of more general transforms of the form:

$$f \to \int f(t) \exp\big((a-b)\rho(t-s) + (c-b)\rho(x-t)\big)\, dt$$

considered in the same work can be associated in a natural way with the model equation $y^{(n)} = \rho^n y$ or with the system of the form $y' = \rho B y$, where B is a diagonal matrix with (arbitrary) complex entries.

In the present paper, we consider some integral transforms connected with the following system of differential equations:

$$y' = (x^{-1}A + B)y, \tag{1}$$

where A and B are constant $n \times n$, $n > 2$, matrices. Such transforms can be considered as generalizations of the Fourier–Hankel transform, see, for instance, [2], see also [3, 4] for some

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generalizations, in particular, connected with the higher-order operators with singularities. The integral transforms we consider are certain functions with the values in an exterior algebra $\wedge\mathbb{C}^n = \bigoplus \wedge^m\mathbb{C}^n$, their particular construction (which will be written below) is determined by their role in spectral theory of the "perturbed system"

$$y' = (x^{-1}A + q(x))y + \rho B y$$

(where ρ denote a spectral parameter), see, for instance, [5]; see also [6], where such constructions appeared first. One can notice that, unlike the studies mentioned above, the kernels of the considered transforms can not be written explicitly in terms of exponential functions, moreover, they can not be expressed in terms of the “Bessel-type” special functions arising in the case $n = 2$. This makes the analysis more complicated and technical. Contrary to the case $n = 2$ (connected with the classical Hankel transform and its dilates) the general case $n > 2$ has not been studied yet.

2. Assumptions and notations. Formulations of the results

Assumption 1. *Matrix A is off-diagonal. The eigenvalues $\{\mu_j\}_{j=1}^n$ of matrix A are distinct and such that $\mu_j - \mu_k \notin \mathbb{Z}$ for $j \neq k$, moreover, $Re\mu_1 < Re\mu_2 < \cdots < Re\mu_n$, $Re\mu_k \neq 0$, $k = \overline{1,n}$.*

Assumption 2. *$B = diag(b_1,\dots,b_n)$, the entries $b_1,\dots,b_n$ are nonzero distinct noncolinear points on complex plane such that $\sum_{j=1}^n \gamma_j b_j = 0$, where $\gamma_j \in \{-1,0,1\}$, $\sum_{j=1}^n \gamma_j^2 > 0$, is true if and onle if $\gamma_j = 1$, $j = \overline{1,n}$.*

Under Assumption 1 system (1) has the fundamental matrix $c(x) = (c_1(x),\dots,c_n(x))$, where

$$c_k(x) = x^{\mu_k}\hat{c}_k(x),$$

$\det c(x) \equiv 1$ and all $\hat{c}_k(\cdot)$ are entire functions, $\hat{c}_k(0) = \mathfrak{h}_k$, $\mathfrak{h}_k$ is an eigenvector of the matrix A corresponding to the eigenvalue μ_k. We define $C_k(x,\rho) := c_k(\rho x)$, $x \in (0,\infty)$, $\rho \in \mathbb{C}$.

Let Σ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \bigcup_{(k,j):j\neq k} \left\{z : \operatorname{Re}(zb_j) = \operatorname{Re}(zb_k)\right\}.$$

By virtue of Assumption 2 for any $z \in \mathbb{C}\setminus\Sigma$ there exists the ordering $R_1,\dots,R_n$ of the numbers $b_1,\dots,b_n$ such that $\operatorname{Re}(R_1 z) < \operatorname{Re}(R_2 z)\cdots < \operatorname{Re}(R_n z)$. Let $\mathscr{S}$ be a sector $\{z = r\exp(i\gamma), r \in (0,\infty), \gamma \in (\gamma_1,\gamma_2)\}$ lying in $\mathbb{C}\setminus\Sigma$. Then [7], [8] system (1) has the fundamental matrix $e(x) = (e_1(x),\dots,e_n(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}}\setminus\{0\}$ and admits the asymptotics:

$$e_k(x) = \mathrm{e}^{xR_k}(\mathfrak{f}_k + x^{-1}\eta_k(x)),\ \eta_k(x) = O(1),\ x \to \infty,\ x \in \overline{\mathscr{S}},$$

where $(\mathfrak{f}_1,\dots,\mathfrak{f}_n)=\mathfrak{f}$ is a permutation matrix such that $(R_1,\dots,R_n)=(b_1,\dots,b_n)\mathfrak{f}$. We define $E(x,\rho):=e(\rho x)$.

Everywhere below we assume that the following additional condition is satisfied.

Condition 1. For all $k=\overline{2,n}$ the numbers

$$\Delta_k := \det(e_1(x),\dots,e_{k-1}(x),c_k(x),\dots,c_n(x))$$

are not equal to 0.

Under Condition 1 system (1) has the fundamental matrix $\psi(x)=(\psi_1(x),\dots,\psi_n(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}}\setminus\{0\}$ and admits the asymptotics:

$$\psi_k(xt)=\exp(xtR_k)(\mathfrak{f}_k+o(1)),\, t\to\infty, x\in\mathscr{S},\ \psi_k(x)=O(x^{\mu_k}), x\to 0.$$

We define $\Psi(x,\rho):=\psi(\rho x)$.

In the sequel we use the following notations:

- $\mathscr{A}_m$ is the set of all ordered multi-indices $\alpha=(\alpha_1,\dots,\alpha_m)$, $\alpha_1<\alpha_2<\dots<\alpha_m$, $\alpha_j\in\{1,2,\dots,n\}$;
- for a sequence $\{u_j\}$ of vectors and a multi-index $\alpha=(\alpha_1,\dots,\alpha_m)$ we define $u_\alpha:=u_{\alpha_1}\wedge\dots\wedge u_{\alpha_m}$;
- for a numerical sequence $\{a_j\}$ and a multi-index α we define
$$a_\alpha:=\sum_{j\in\alpha}a_j,\ a^\alpha:=\prod_{j\in\alpha}a_j;$$
- for a multi-index α the symbol α' denotes the ordered multi-index that complements α to $(1,2,\dots,n)$;
- for $k=\overline{1,n}$ we denote
$$\overrightarrow{a}_k:=\sum_{j=1}^{k}a_j,\ \overleftarrow{a}_k:=\sum_{j=k}^{n}a_j,\ \overrightarrow{a}^k:=\prod_{j=1}^{k}a_j,\ \overleftarrow{a}^k:=\prod_{j=k}^{n}a_j.$$
We note that Assumptions 1,2 imply, in particular, $\sum\limits_{k=1}^{n}\mu_k=\sum\limits_{k=1}^{n}R_k=0$ and therefore for any multi-index α one has $R_{\alpha'}=-R_\alpha$ and $\mu_{\alpha'}=-\mu_\alpha$.
- the symbol $V^{(m)}$, where V is $n\times n$ matrix, denotes the operator acting in $\wedge^m\mathbb{C}^n$ so that for any vectors $u_1,\dots,u_m$ the following identity holds [6]:
$$V^{(m)}(u_1\wedge u_2\wedge\dots\wedge u_m)=\sum_{j=1}^{m}u_1\wedge u_2\wedge\dots\wedge u_{j-1}\wedge Vu_j\wedge u_{j+1}\wedge\dots\wedge u_m;$$

- if $h \in \wedge^n \mathbb{C}^n$ then $|h|$ is a number such that $h = |h| \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \cdots \wedge \mathfrak{e}_n$, $\{\mathfrak{e}_k\}_{k=1}^n$ is a standard basis in $\mathbb{C}^n$;
- $T_k(x,\rho) := C_k(x,\rho) \wedge \cdots \wedge C_n(x,\rho)$.
- for $h \in \wedge^m \mathbb{C}^n$ we set: $\|h\| := \sum\limits_{\alpha \in \mathscr{A}_m} |h_\alpha|$, where $\{h_\alpha\}$ are the coefficients from the expansion $h = \sum\limits_{\alpha \in \mathscr{A}_m} h_\alpha \mathfrak{e}_\alpha$.

We use the same notation $L_p(a,b)$ for all the spaces of the form $L_p((a,b),\mathscr{E})$, where $\mathscr{E}$ is a finite-dimensional space. The notation $C[a,b]$ for the spaces of continuous functions will be used in a similar way.

Suppose $\mathbf{Y}$ is a Banach space. We denote by $C([0,\infty),\mathbf{Y})$ the set of continuous functions $[0,\infty) \to \mathbf{Y}$ and we denote by $BC([0,\infty),\mathbf{Y})$ the Banach space of bounded continuous functions $[0,\infty) \to \mathbf{Y}$ with the standard sup norm.

Suppose Y is some number set (typically sector or ray in the complex plane) and $\mathbf{Y}$ is some Banach space of functions on Y. Let $\mathbf{Q}$ be some Banach space. For a function $F = F(Q,x,\rho), Q \in \mathbf{Q}, x \in [0,\infty), \rho \in Y$ with values in some finite-dimensional space we write $F \in \mathscr{L}(\mathbf{Q}, BC([0,\infty),\mathbf{Y}))$ iff:

- for any fixed $Q \in \mathbf{Q}, x \in [0,\infty)$ the function $F(Q,x,\cdot)$ belongs to the Banach space $\mathbf{Y}$;
- for any fixed $Q \in \mathbf{Q}$ the function $F(Q,\cdot,\cdot) \in BC([0,\infty),\mathbf{Y})$;
- the map $\mathbf{Q} \ni Q \to F(Q,\cdot,\cdot) \in BC([0,\infty),\mathbf{Y})$ is a linear continuous operator.

We set $X_p := L_1(0,\infty) \cap L_p(0,\infty)$ and denote by $\mathscr{X}_p^m$ the space of functions $Q = Q(t)$, $t \in (0,\infty)$ such that for any t $Q(t)$ is a linear operator acting in $\wedge^m \mathbb{C}^n$, $\left|(Q(\cdot)\mathfrak{e}_\alpha) \wedge \mathfrak{e}_{\beta'}\right| \in X_p$ for all $\alpha, \beta \in \mathscr{A}_m$ and $(Q(t)\mathfrak{e}_\alpha) \wedge \mathfrak{e}_{\alpha'} \equiv 0$ for all $\alpha \in \mathscr{A}_m$.

In the paper, we study the following integral transforms ($k = \overline{1,n}$):

$$\mathscr{T}_k(Q,x,\rho) = \int_0^x \sum_{\alpha \in \mathscr{A}_{n-k+1}} \sigma_\alpha \left| \big(Q(t) T_k(t,\rho)\big) \wedge C_{\alpha'}(t,\rho) \right| C_\alpha(x,\rho)\, dt, \tag{2}$$

where $\sigma_\alpha = |\mathfrak{h}_\alpha \wedge \mathfrak{h}_{\alpha'}|$ and $Q \in \mathscr{X}_p^{n-k+1}$.

Our considerations will be concentrated mostly on properties of $\mathscr{T}_k(Q,x,\rho)$ as functions of (complex) variable ρ. Also we are interested in how these functions depend on the parameters $x \in (0,\infty)$ and $Q \in \mathscr{X}_p^{n-k+1}$. Our guiding line (and one of our key tools) is the following result concerning the classical Laplace transform [9].

Proposition 1. *Suppose*

$$F(\rho)=\int_0^\infty f(t)\exp(-\rho t)\,dt,$$

where $f\in L_p(0,\infty)$. Let l be a ray $\{\rho=z\tau,\tau\in(0,\infty)\}$, $Re z>0$. Then $F\in L_{p'}(l)$, $p'=p/(p-1)$. Moreover, the map $f\to F$ is a linear continuous operator from $L_p(0,\infty)$ to $L_{p'}(l)$.

As for integral transforms (2), it is natural to consider them in certain sectors of the complex ρ-plane rather then half-planes. Everywhere below the symbol $\mathscr{S}$ denotes some (arbitrary) open sector with the vertex at the origin lying in $\mathbb{C}\setminus\Sigma$.

Let $W_0(\xi)$ be the function defined as follows:

$$W_0(\xi)=(1-|\xi|)\xi+|\xi|^2,\ |\xi|\le 1,\ W_0(\xi):=(W_0\left(\xi^{-1}\right))^{-1},\ |\xi|>1.$$

Notice that $W_0(\xi)$ is continuous in $\xi\in\mathbb{C}$, never vanishes for nonzero ξ and admits the estimate:

$$M_1|\xi|\le|W_0(\xi)|\le M_2|\xi|$$

for all $\xi\in\mathbb{C}$. Moreover, we have $W_0(\xi)=1$ if $|\xi|=1$ and the asymptotics $W_0(\xi)=\xi(1+o(1))$ hold as $\xi\to 0$ and $\xi\to\infty$.

We introduce the following weight functions:

$$W_k(\xi):=\begin{cases}W_0\left(\xi^{\mu_k}\right)\exp(R_k\xi),\ |\xi|\le 1\\ \exp(R_k\xi),\ |\xi|>1.\end{cases}$$

From the definition and the above-mentioned properties of $W_0(\cdot)$ it follows that the weight functions $W_k(\cdot),k=\overline{1,n}$ are all continuous in $\overline{\mathscr{S}}\setminus\{0\}$, never vanish and admit the asymptotics $W_k(\xi)=\xi^{\mu_k}(1+o(1))$ as $\xi\to 0$.

Theorem 1. *$\mathscr{T}_k(Q,x,\rho)$ is analytic with respect to $\rho\in\mathscr{S}$ and the following representation holds:*

$$\mathscr{T}_k(Q,x,\rho)=\overleftarrow{W}^k(\rho x)\omega_k(Q,x,\rho),$$

where $\omega_k\in\mathscr{L}\left(\mathscr{X}_p^{n-k+1},BC\left([0,\infty),C_0(\overline{\mathscr{S}})\right)\right)$. Moreover, for any ray $\{\rho=zt,t\in[0,\infty)\}$, where $z\in\overline{\mathscr{S}}\setminus\{0\}$, the restriction $\omega_k|_l\in\mathscr{L}(\mathscr{X}_p^{n-k+1},BC([0,\infty),\mathscr{H}(l)))$, $\mathscr{H}(l):=C_0(l)\cap L_2(l)$. Here and below the symbol $C_0(\overline{\mathscr{S}})$ denotes the Banach space of continuous vanishing at infinity functions on $\overline{\mathscr{S}}$ with the standard sup norm, the symbol $C_0(l)$ is treated in analogous way.

Theorem 2. *Suppose that $Q(\cdot)$ is absolutely continuous, $Q(0)=0$ and both $Q(\cdot)$, $Q'(\cdot)$ are from $\mathscr{X}_p^{n-k+1}$. Then the following representation holds:*

$$\rho\mathscr{T}_k(Q,x,\rho)=\sum_{\alpha\in\mathscr{A}_{n-k+1}}\chi_\alpha\left|(\hat{Q}(x)T_k(x,\rho))\wedge E_{\alpha'}(x,\rho)\right|E_\alpha(x,\rho)+\overleftarrow{W}^k(\rho x)\tilde{\omega}_k(Q,x,\rho),$$

where $\hat{Q}(x)$ is the operator acting in $\wedge^{n-k+1}\mathbb{C}^n$ which is off-diagonal with respect to the basis $\{\mathfrak{e}_\alpha\}_{\alpha\in\mathscr{A}_{n-k+1}}$ and such that $[B^{(n-k+1)},\hat{Q}(x)] = Q(x)$ for all $x \in [0,\infty)$; $\tilde{\omega}_k(Q,\cdot,\cdot) \in BC([0,\infty), C_0(\overline{\mathscr{S}}))$. Moreover, the following asymptotics holds for any fixed $x > 0$ as $\rho \to \infty$, $\rho \in \overline{\mathscr{S}}$:

$$\mathscr{T}_k(Q,x,\rho) = \rho^{-1} \sum_{\alpha,\beta\in\mathscr{A}_{n-k+1}} T_{k\beta} \exp(\rho x R_\beta)\hat{Q}_{\alpha\beta}(x)\mathfrak{f}_\alpha + o\left(\rho^{-1}\exp(\rho x \overleftarrow{R}_k)\right).$$

Here the constants $\{T_{k\alpha}\}$ are such that:

$$T_k(x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} T_{k\alpha} E_\alpha(x,\rho)$$

and $\hat{Q}_{\alpha\beta}(x) = \chi_\alpha \left|(\hat{Q}(x)\mathfrak{f}_\beta)\wedge \mathfrak{f}_{\alpha'}\right|$, $\chi_\alpha := |\mathfrak{f}_\alpha \wedge \mathfrak{f}_{\alpha'}|$.

3. Auxiliary propositions

In the sequel we write $f \in PC^\pm(\overline{\mathscr{S}})$, where $f = f(x,\rho)$ is a function defined on $[0,\infty)\times\overline{\mathscr{S}}$ iff:

- f is continuous and bounded on the set $\left([0,\infty)\times\overline{\mathscr{S}}\right)\cap\{(x,\rho) : \pm(|\rho|x-1) > 0\}$;
- f can be extended up to the function continuous on the set $\left([0,\infty)\times\overline{\mathscr{S}}\right)\cap\{(x,\rho) : \pm(|\rho|x-1) \geq 0\}$

We write $f \in PC(\overline{\mathscr{S}})$ if $f \in PC^+(\overline{\mathscr{S}})$ and $f \in PC^-(\overline{\mathscr{S}})$ at the same time. We write $f \in PC_0(\overline{\mathscr{S}})$ if, in addition,

$$\lim_{\rho\to\infty} \sup_{x\in[0,\infty)} \|f(x,\rho)\| = 0.$$

Similarly we define the classes $PC^\pm(l)$, $PC(l)$, $PC_0(l)$, where l is a ray $\{\rho = zt, t \in [0,\infty)\}$ We notice that $f \in BC\left([0,\infty)\times\overline{\mathscr{S}}\right)$ together with $f \in PC_0(\overline{\mathscr{S}})$ imply $f \in BC\left([0,\infty), C_0(\overline{\mathscr{S}})\right)$.

If l is a ray $l = \{\rho = zt, t\in[0,\infty)\}$, where $z \in \overline{\mathscr{S}}\setminus\{0\}$ and R is a positive number then we denote $l^+(R) = l \cap \{\rho : |\rho| > R\}$ and $l^-(R) = l\cap\{\rho : |\rho| \leq R\}$.

Everywhere below the symbols $\theta^\pm(\cdot)$ denote the Heaviside step functions:

$$\theta^+(\xi) = \begin{cases} 0, \xi \leq 0 \\ 1, \xi > 0, \end{cases}$$

$$\theta^-(\xi) = \begin{cases} 1, \xi \leq 0 \\ 0, \xi > 0 \end{cases} = 1 - \theta^+(\xi).$$

Lemma 3.1. *Suppose that a function $F(x,\rho)$, $(x,\rho)\in[0,\infty)\times l$ is such that:*

1. $F\in PC(l)$;
2. *for each fixed* $x\in[0,\infty)$ $F(x,\cdot)\in L_2(l)$ *and* $\sup\limits_{x\in[0,\infty)}\|F(x,\cdot)\|_{L_2(l)}<\infty$;
3. *for any* $T\in(0,\infty)$ $\sup\limits_{x\in[0,T]}\|F(x,\cdot)\|_{L_2(l^+(R))}\to 0$ *as* $R\to\infty$.

Then $F\in BC([0,\infty),L_2(l))$.

Proof. Take an arbitrary $x_0>0$ and suppose that $x_1\to x_0$. Denote as $\rho_\pm$ the points from l such that $|\rho_-|=\min\{x_0^{-1},x_1^{-1}\}$, $|\rho_+|=\max\{x_0^{-1},x_1^{-1}\}$.

First, given $\varepsilon>0$, we find $R>0$ such that $\sup\limits_{x\in[0,T]}\|F(x,\cdot)\|_{L_2(l^+(R))}<\varepsilon/4$. For definiteness we assume $|\rho_+|<R$. Continuity properties of F guarantee that

$$\|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l^-(R)\cap l^+(|\rho_+|))}<\frac{\varepsilon}{4},\ \|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l^-(|\rho_-|))}<\frac{\varepsilon}{4}$$

for all x_1 sufficiently close to x_0. Finally, since $|\rho_+-\rho_-|\to 0$ as $x_1\to x_0$ and F is bounded we can assert that the estimate

$$\|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l^-(|\rho_+|)\cap l^+(|\rho_-|))}<\frac{\varepsilon}{4}$$

is also true for all x_1 sufficiently close to x_0.

Now consider the case $x_0=0$. We can take the same R as above and (assuming that $x_1<R^{-1}$) split as follows:

$$\|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l)}\le\|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l^+(R))}+\|F(x_1,\cdot)-F(x_0,\cdot)\|_{L_2(l^-(R))}.$$

The first term is less than $\varepsilon/4$ because of our choice of R. The second term becomes less that $3\varepsilon/4$ for all x_1 sufficiently close to x_0 because of continuity properties of F. □

The following extension is obvious.

Lemma 3.2. *Suppose that $F=F(f,x,\rho)$ is such that $F(f,\cdot,\cdot)\in BC([0,\infty),L_2(l))$ for any fixed $f\in X_p$ and $F\in\mathscr{L}(BC([0,\infty),L_2(l)))$. Suppose also that $g=g(x,\rho)$ is a function from $PC(l)$. Set $\mathscr{F}(f,x,\rho):=g(x,\rho)F(f,x,\rho)$. Then for any fixed $f\in X_p$ $\mathscr{F}(f,\cdot,\cdot)\in BC([0,\infty),L_2(l))$; moreover, $\mathscr{F}\in\mathscr{L}(X_p,BC([0,\infty),L_2(l)))$.*

Lemma 3.3. *Consider the integral transform:*

$$F(f,x,\rho)=\int_0^x f(t)\exp\big((\lambda_1(x-t)+\lambda_2 t)\rho\big)\,dt,$$

where $\lambda_2\neq\lambda_1$ are such that $Re(\lambda_1\rho)\le 0$, $Re(\lambda_2\rho)\le 0$ for all $\rho\in\overline{\mathscr{S}}$. Then:

1. *for any fixed* $f \in X_p$ $F(f,x,\rho)$ *is continuous and bounded w.r.t.* $(x,\rho) \in [0,\infty) \times \overline{\mathscr{S}}$*;*
2. $\sup_{x\in[0,\infty)} \|F(f,x,\rho)\| \to 0$ *as* $\rho \to \infty$, $\rho \in \overline{\mathscr{S}}$*;*
3. *for any ray* $l = [0, z\cdot\infty)$, $z \in \overline{\mathscr{S}} \setminus \{0\}$ $F(f,\cdot,\cdot) \in BC([0,\infty), L_2(l))$*;*
4. *for any ray* $l = [0, z\cdot\infty)$, $z \in \overline{\mathscr{S}} \setminus \{0\}$ *the map* $X_p \ni f \to F(f,\cdot,\cdot) \in BC([0,\infty), L_2(l) \cap C_0(l))$ *is continuous.*

Proof.

1. is obvious.

2. is obvious for smooth compactly supported f, in general case we use the standard arguments based on the passing to the limit procedure.

3. and 4. For any fixed ray l one can write $\lambda_k \rho = |\rho|(is_k - c_k)$ with some fixed c_k, s_k, where $c_k \geq 0$. We are to consider two different cases: $c_1 \neq c_2$ (actually one can assume $c_2 > c_1$) and $c_1 = c_2$, $s_1 \neq s_2$.

1) Case $c_2 > c_1$.

$$|F(f,x,\rho)| \leq \int_{\tau_1 x}^{\tau_2 x} |f(\xi - \tau_1 x)| \exp(-c|\rho|\xi)\, d\xi = \mathscr{F}_L(f_x, c|\rho|),$$

where $c = c_2 - c_1$, $\tau_k = c_k / c$,

$$f_x(\xi) := \theta^+((\xi - \tau_1 x)(\tau_2 x - \xi))|f(\xi - \tau_1 x)|,$$

and $\mathscr{F}_L$ denote the classical Laplace transform. From Proposition 1 it follows that:

$$\|F(f,x,\cdot)\|_{L_2(l)} \leq M\|f_x\|_{L_2(0,\infty)} \leq M\|f\|_{L_2(0,\infty)}.$$

Furthermore, the Hölder inequality yields:

$$|F(f,x,\rho)| \leq \left\{\int_{\tau_1 x}^{\tau_2 x} |f(\xi - \tau_1 x)|^p\, d\xi\right\}^{1/p} \left\{\int_{\tau_1 x}^{\tau_2 x} \exp(-cp'|\rho|\xi)\, d\xi\right\}^{1-1/p} \leq M|\rho|^{-1+1/p},$$

where M do not depend on x. Therefore we have

$$\sup_{x\in[0,\infty)} \|F(f,x,\cdot)\|_{L_2(l^+(R))} \leq MR^{-1}.$$

2) Case $c_1 = c_2$, $s_2 > s_1$.

$$F(f,x,\rho) = \exp(-c_1|\rho|x)F_0(f,x,\rho),$$

$$F_0(f,x,\rho) = \int_0^x f(t)\exp(i|\rho|(s_1(x-t) + s_2 t))\, dt = \mathscr{F}(f_x, s|\rho|),$$

where $\mathscr{F}$ denote the classical Fourier transform and

$$f_x(\xi) := \theta^+((\xi - \tau_1 x)(\tau_2 x - \xi)) f(\xi - \tau_1 x),$$

$\tau_k := s_k / s$, $s = s_2 - s_1$. We note that the map $x \to f_x$ is continuous from $[0,\infty)$ to $L_2(-\infty,\infty)$, moreover, $\|f_x\|_{L_2(-\infty,\infty)} \leq \|f\|_{L_2(0,\infty)}$. □

Lemma 3.4. *Suppose that $\mathscr{G}(x,t,\rho)$ is some linear operator acting (for any fixed x,t,ρ) from $\mathscr{L}(\wedge^m\mathbb{C}^n)$ to $\wedge^m\mathbb{C}^n$ such that:*

1. *$\mathscr{G}(x,x\tau,\rho)$ is continuous in $[0,\infty)\times[0,1]\times\overline{\mathscr{S}}$;*
2. *$\|\mathscr{G}(x,x\tau,\rho)\|\le M$, where the constant M does not depend on $(x,\tau,\rho)\in[0,\infty)\times[0,1]\times\overline{\mathscr{S}}$.*

Then for each of the functions:

$$F^-(Q,x,\rho)=\theta^-(|\rho x|-1)\int_0^x \mathscr{G}(x,t,\rho)Q(t)\,dt,$$
$$F^+(Q,x,\rho)=\theta^+(|\rho x|-1)\int_0^{|\rho|^{-1}} \mathscr{G}(x,t,\rho)Q(t)\,dt,$$

the following assertions are true:

1. *for any fixed $Q\in\mathscr{X}_p^m$ $F^\pm(Q,\cdot,\cdot)\in PC_0(\overline{\mathscr{S}})$;*
2. *for any fixed $Q\in\mathscr{X}_p^m$ $F^\pm(Q,\cdot,\cdot)\in BC([0,\infty),L_2(l))$, where l is a ray $l=\{\rho=zt, t\in[0,\infty)\}$ with arbitrary fixed $z\in\overline{\mathscr{S}}\setminus\{0\}$;*
3. *$F^\pm\in\mathscr{L}(\mathscr{X}_p^m, BC([0,\infty),L_2(l)))$, i.e., the map $\mathscr{X}_p^m\ni Q\to F^\pm(Q,\cdot,\cdot)\in BC([0,\infty),L_2(l))$ is a linear continuous operator.*

Proof. 1) One has:

$$\|F^-(Q,x,\rho)\|\le M\theta^-(|\rho x|-1)\int_0^x\|Q(t)\|\,dt\le M\int_0^{|\rho|^{-1}}\|Q(t)\|\,dt,$$

in particular one has $F^-(Q,0,\rho)\equiv 0$ and

$$\lim_{\rho\to\infty}\sup_{x\in(0,\infty)}\|F^-(Q,x,\rho)\|=0.$$

Furthermore, since for $x>0$

$$\int_0^\infty \theta^-(|\rho x|-1)d|\rho|=x^{-1}$$

the Cauchy-Bunyakowsky-Schwartz inequality yields:

$$\|F^-(Q,x,\cdot)\|_{L_2(l)}\le Mx^{-1/2}\int_0^x\|Q(t)\|\,dt,$$

that implies for $Q\in\mathscr{X}_p^m$:

$$\|F^-(Q,x,\cdot)\|_{L_2(l)}\le M\left(\|Q\|_{L_2(0,1)}+\|Q\|_{L_1(1,\infty)}\right).$$

Furthermore, using again the Cauchy-Bunyakowsky-Schwartz and the Hölder inequality we obtain:

$$\|F^-(Q,x,\cdot)\|_{L_2(l^+(R))}\le M\theta^-(Rx-1)(1-Rx)^{1/2}x^{-1/2}\int_0^x\|Q(t)\|dt\le$$

$$M\theta^-(Rx-1)(1-Rx)^{1/2}x^{1/2-1/p}\|Q\|_{L_p(0,\infty)}.$$

This yields:

$$\sup_{x\in[0,T]}\|F^-(Q,x,\cdot)\|_{L_2(l^+(R))}\le M\sup_{x\in[0,R^{-1}]}(1-Rx)^{1/2}x^{1/2-1/p}=MR^{1/p-1/2}\sup_{t\in[0,1]}(1-t)^{1/2}t^{1/2-1/p}.$$

Since $p>2$ we see that

$$\lim_{R\to\infty}\sup_{x\in[0,T]}\|F^-(Q,x,\cdot)\|_{L_2(l^+(R))}=0.$$

By virtue of Lemma 3.1 this completes the proof in what concerns F^-.

2) Proceeding as above we obtain $F^+(Q,0,\rho)\equiv 0$ and:

$$\|F^+(Q,x,\rho)\|\le M\theta^+(|\rho x|-1)\int_0^{|\rho|^{-1}}\|Q(t)\|\,dt,$$

that yields

$$\lim_{\rho\to\infty}\sup_{x\in(0,\infty)}\|F^+(Q,x,\rho)\|=0.$$

Then, for $x>0$ one has:

$$\begin{aligned}\|F^+(Q,x,\cdot)\|_{L_2(l)} &\le M\left\{\int_0^\infty\theta^+(|\rho x|-1)\left(\int_0^{|\rho|^{-1}}\|Q(t)\|\,dt\right)^2 d|\rho|\right\}^{1/2}\\ &\le M\int_0^\infty\|Q(t)\|\left\{\int_0^{t^{-1}}d|\rho|\right\}^{1/2}=M\int_0^\infty t^{-1/2}\|Q(t)\|\,dt\\ &\le M(\|Q\|_{L_p(0,1)}+\|Q\|_{L_1(1,\infty)}).\end{aligned}$$

Furthermore, one has:

$$\begin{aligned}\|F^+(Q,x,\cdot)\|_{L_2(l^+(R))} &\le M\left\{\int_R^\infty\theta^+(rx-1)\left(\int_0^\infty\theta^-(rt-1)\|Q(t)\|\,dt\right)^2 dr\right\}^{1/2}\le\\ &\le M\int_0^{R_+^{-1}}\|Q(t)\|\left\{\int_{R_+}^{t^{-1}}dr\right\}^{1/2}dt\\ &\le M\int_0^{R_+^{-1}}\|Q(t)\|(t^{-1}-R_+)^{1/2}\,dt\\ &\le M\|Q\|_{L_p(0,R_+^{-1})}\left\{\int_0^{R_+^{-1}}(t^{-1}-R_+)^{p'/2}\,dt\right\}^{1/p'}\\ &=M\|Q\|_{L_p(0,R_+^{-1})}R_+^{1/p-1/2}\left\{\int_0^1(\tau^{-1}-1)^{p'/2}\,d\tau\right\}^{1/p'}\\ &\le MR_+^{1/p-1/2}\|Q\|_{L_p(0,\infty)},\end{aligned}$$

where $R_+ = R_+(R,x) := \max\{R, x^{-1}\}$ (we take into account that $p > 2 > p'$ and therefore the integrals converge; we use the same symbol M for denoting possibly different constants that do not depend on x, R, Q). Since $\sup_{x\in[0,\infty)} (R_+(R,x))^{1/p-1/2} = R^{1/p-1/2} \to 0$ as $R \to \infty$ we obtain

$$\lim_{R\to\infty} \sup_{x\in[0,\infty)} \|F^+(Q,x,\cdot)\|_{L_2(l^+(R))} = 0$$

for any $T \in (0,\infty)$. Applying Lemma 3.1 completes the proof. $\square$

Lemma 3.5. *Suppose that $\mathscr{G}(x,t,\rho)$ is some linear operator acting (for any fixed x,t,ρ) from $\mathscr{L}(\wedge^m\mathbb{C}^n)$ to $\wedge^m\mathbb{C}^n$ such that:*

1. *$\mathscr{G}(x,t,\rho)$ is continuous in $\{(x,t,\rho) : \rho \in \overline{\mathscr{S}}, |\rho|^{-1} \le t \le x < \infty\}$;*
2. *$\|\mathscr{G}(x,t,\rho)\| \le M$, where the constant M does not depend on (x,t,ρ).*

Then for the function:

$$F(Q,x,\rho) = \theta^+(|\rho x| - 1)\int_{|\rho|^{-1}}^{x} (\rho t)^{-1}\mathscr{G}(x,t,\rho)Q(t)\,dt$$

the assertions of Lemma 3.4 are true.

Proof. Proceeding as above we note first:

$$\begin{aligned}\|F(Q,x,\rho)\| &\le M\theta^+(|\rho x| - 1)\int_{|\rho|^{-1}}^{x} |\rho t|^{-1}\|Q(t)\|\,dt\\ &\le M|\rho|^{-1/2}\int_0^\infty t^{-1/2}\|Q(t)\|\,dt\\ &\le M|\rho|^{-1/2}(\|Q\|_{L_p(0,1)} + \|Q\|_{L_1(1,\infty)}).\end{aligned}$$

In particular, this implies:

$$\lim_{\rho\to\infty} \sup_{x\in(0,\infty)} \|F(Q,x,\rho)\| = 0.$$

Furthermore, one has:

$$\begin{aligned}\|F(Q,x,\cdot)\|_{L_2(l)} &\le M\left\{\int_{x^{-1}}^{\infty}\left(\int_{|\rho|^{-1}}^{x} |\rho t|^{-1}\|Q(t)\|dt\right)^2 d|\rho|\right\}^{1/2}\\ &\le M\int_0^x \left\{\int_{t^{-1}}^{\infty} |\rho t|^{-2}\|Q(t)\|^2\,d|\rho|\right\}^{1/2} dt = M\int_0^x t^{-1/2}\|Q(t)\|\,dt\\ &\le M(\|Q\|_{L_p(0,1)} + \|Q\|_{L_1(1,\infty)})\end{aligned}$$

and, moreover, for arbitrary $R > 0$:

$$\|F(Q,x,\cdot)\|_{L_2(l^+(R))} \le M\left\{\int_R^\infty \theta^+(rx-1)\left(\int_0^x \theta^+(rt-1)(rt)^{-1}\|Q(t)\|\,dt\right)^2 dr\right\}^{1/2}$$

$$\begin{aligned}&\le M\int_0^x t^{-1}\|Q(t)\|\left\{\int_R^\infty r^{-2}\theta^+(rx-1)\theta^+(rt-1)\,dr\right\}^{1/2}dt\\&= M\int_0^x t^{-1}(R_+(R,t))^{-1/2}\|Q(t)\|\,dt,\end{aligned}$$

where (as in proof of the previous lemma) $R_+(R,t)=\max\{R,t^{-1}\}$. Thus, we obtain:

$$\begin{aligned}\|F(Q,x,\cdot)\|_{L_2(l^+(R))} &\le M\int_0^{R^{-1}} t^{-1/2}\|Q(t)\|\,dt + MR^{-1/2}\int_{R^{-1}}^x t^{-1}\|Q(t)\|\,dt\\&\le M\int_0^{R^{-1}} t^{-1/2}\|Q(t)\|\,dt + MR^{-1/2}\|Q\|_{L_p(0,\infty)}\left\{\int_{R^{-1}}^\infty t^{-p'}\,dt\right\}^{1/p'}\\&\le M\int_0^{R^{-1}} t^{-1/2}\|Q(t)\|\,dt + MR^{1/p-1/2}\|Q\|_{L_p(0,\infty)}.\end{aligned}$$

Since $p>2$ and (as a sequence) $t^{-1/2}\|Q(t)\|\in L_1(0,\infty)$ we can conclude now that

$$\lim_{R\to\infty}\sup_{x\in[0,\infty)}\|F(Q,x,\cdot)\|_{L_2(l^+(R))}=0.$$

In view of Lemma 3.1 this completes the proof. □

4. Proofs of the theorems

Proof of Theorem 1. 1) First we observe that

$$\begin{aligned}\hat{\mathcal{T}}_k(Q,x,\rho) &:= (\rho x)^{-\overleftarrow{\mu}_k}\mathcal{T}_k(Q,x,\rho)\\&= \int_0^1 \sum_{\alpha\in\mathcal{A}_{n-k+1}} \sigma_\alpha \left|\left(Q(x\tau)\hat{T}_k(\rho x\tau)\right)\wedge \hat{c}_{\alpha'}(\rho x\tau)\right| \hat{c}_\alpha(\rho x)\, x\tau^{\overleftarrow{\mu}_k-\mu_\alpha}\,d\tau,\end{aligned}$$

where $\hat{T}_k(x):=\hat{c}_k(x)\wedge\cdots\wedge\hat{c}_n(x)$. Since all $\hat{c}_k(\cdot)$ are entire functions and $\mathrm{Re}(\overleftarrow{\mu}_k-\mu_\alpha)\ge 0$ for any $\alpha\in\mathcal{A}_{n-k+1}$ we can conclude that $\hat{\mathcal{T}}_k(Q,\cdot,\cdot)\in C\left([0,\infty)\times\overline{\mathcal{S}}\right)$. Moreover, the arguments show that $\mathcal{T}_k(Q,x,\cdot)$ is analytic in $\mathcal{S}$.

Further, we have

$$\omega_k(Q,x,\rho)=(\rho x)^{\overleftarrow{\mu}_k}\left(\overleftarrow{W}^k(\rho x)\right)^{-1}\hat{\mathcal{T}}_k(Q,x,\rho).$$

By virtue of the properties of the weight functions $W_k(\cdot)$ the fraction $\xi^{\overleftarrow{\mu}_k}(\overleftarrow{W}^k(\xi))^{-1}$ is continuous in $\overline{\mathcal{S}}$. Therefore $\omega_k(Q,\cdot,\cdot)\in C\left([0,\infty)\times\overline{\mathcal{S}}\right)$.

Now we note that

$$\sum_{\alpha\in\mathcal{A}_m}\sigma_\alpha\left|f\wedge C_{\alpha'}(t,\rho)\right|C_\alpha(x,\rho)=:G_m(x,t,\rho)f$$

define some linear operator acting in $\wedge^m\mathbb{C}^n$ and the operator admits also the following representations:

$$G_m(x,t,\rho)f = \sum_{\alpha\in\mathscr{A}_m} \chi_\alpha \left| f \wedge E_{\alpha'}(t,\rho)\right| E_\alpha(x,\rho). \tag{3}$$

Using the fundamental matrix $\Psi(x,\rho)$ one can obtain another representation for the operator $G_m(x,t,\rho)$:

$$G_m(x,t,\rho)f = \sum_{\alpha\in\mathscr{A}_m} \chi_\alpha \left| f \wedge \Psi_{\alpha'}(t,\rho)\right| \Psi_\alpha(x,\rho). \tag{4}$$

In the sequel for a matrix function $V = V(x,\rho)$ we denote $\tilde{V}(x,\rho) := V(x,\rho)(W(\rho x))^{-1}$, where $W(\xi) := diag(W_1(\xi),\dots,W_n(\xi))$. For instance, we set $\tilde{\Psi}(x,\rho) := \Psi(x,\rho)(W(\rho x))^{-1}$ that means, in particular, $\Psi_k(x,\rho) = W_k(\rho x)\tilde{\Psi}_k(x,\rho)$ and $\Psi_\alpha(x,\rho) = W^\alpha(\rho x)\tilde{\Psi}_\alpha(x,\rho)$ for any arbitrary multi-index α. Using (4) we can write:

$$\omega_k(Q,x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} \chi_\alpha \int_0^x \left| \frac{\overleftarrow{W}^k(\rho t)}{\overleftarrow{W}^k(\rho x)} \cdot W^\alpha(\rho x) W^{\alpha'}(\rho t)\cdot \big(Q(t)\tilde{T}_k(t,\rho)\big)\wedge \tilde{\Psi}_{\alpha'}(t,\rho)\right| \tilde{\Psi}_\alpha(x,\rho)\,dt.$$

Since $\tilde{\Psi}(x,\rho)$ and $\tilde{T}_k(x,\rho)$ are continuous and bounded on $[0,\infty)\times\overline{\mathscr{S}}$ this yields:

$$\|\omega_k(Q,x,\rho)\| \le M\int_0^x \|Q(t)\|\,dt \tag{5}$$

with some absolute constant M. Therefore (in particular) $\omega_k \in \mathscr{L}\left(\mathscr{X}_p^{n-k+1}, BC([0,\infty)\times\overline{\mathscr{S}})\right)$.

2) We split as follows:

$$\omega_k(Q,x,\rho) = \omega_k^{(0)}(Q,x,\rho) + \omega_k^{(1)}(Q,x,\rho) + \omega_k^{(2)}(Q,x,\rho),$$

where:

$$\omega_k^{(1)}(Q,x,\rho) = \theta^-(|\rho x|-1)\int_0^x \mathscr{G}_{n-k+1}(x,t,\rho)\big(Q(t)\tilde{T}_k(t,\rho)\big)\,dt,$$

$$\omega_k^{(2)}(Q,x,\rho) = \theta^+(|\rho x|-1)\int_0^{|\rho|^{-1}} \mathscr{G}_{n-k+1}(x,t,\rho)\big(Q(t)\tilde{T}_k(t,\rho)\big)\,dt,$$

$$\mathscr{G}_{n-k+1}(x,t,\rho) := \frac{\overleftarrow{W}^k(\rho t)}{\overleftarrow{W}^k(\rho x)} G_{n-k+1}(x,t,\rho),$$

and

$$\omega_k^{(0)}(Q,x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} \theta^+(|\rho x|-1)\chi_\alpha \exp(-\rho x \overleftarrow{R}_k)\int_{|\rho|^{-1}}^x \left|\big(Q(t)T_k(t,\rho)\big)\wedge E_{\alpha'}(t,\rho)\right| E_\alpha(x,\rho)\,dt.$$

Using representation (4) we obtain the estimate

$$\|\mathscr{G}_{n-k+1}(x,x\tau,\rho)\| \le M$$

for all $(x,\tau,\rho)\in[0,\infty)\times[0,1]\times\overline{\mathscr{S}}$. Thus, from Lemma 3.4 it follows that $\omega_k^{(\nu)}(Q,\cdot,\cdot)\in PC_0(\overline{\mathscr{S}})$ and $\omega_k^{(\nu)} \in \mathscr{L}\left(\mathscr{X}_p^{n-k+1}, BC([0,\infty), L_2(l))\right)$, $\nu = 1,2$.

Now we consider in details $\omega_k^{(0)}$. Using the expansion

$$T_k(t,\rho) = \sum_{\beta\in\mathscr{A}_{n-k+1}} T_{k\beta} E_\beta(t,\rho)$$

we obtain:

$$\omega_k^{(0)}(Q,x,\rho) = \sum_{\alpha,\beta\in\mathscr{A}_{n-k+1}} \Gamma_{\alpha\beta} h_{\alpha\beta}(Q,x,\rho),$$

where $\Gamma_{\alpha\beta}$ are also some absolute constants and:

$$h_{\alpha\beta}(Q,x,\rho) = \theta^+(|\rho x|-1)\exp(-\rho x \overleftarrow{R}_k)\int_{|\rho|^{-1}}^x \left|\left(Q(t)E_\beta(t,\rho)\right)\wedge E_{\alpha'}(t,\rho)\right| E_\alpha(x,\rho)\,dt.$$

Then we write:

$$h_{\alpha\beta}(Q,x,\rho) = h_{\alpha\beta}^{(0)}(Q,x,\rho) + h_{\alpha\beta}^{(1)}(Q,x,\rho),$$

where:

$$\begin{aligned}
h_{\alpha\beta}^{(1)}(Q,x,\rho) &= \theta^+(|\rho x|-1)\int_{|\rho|^{-1}}^x (\rho t)^{-1} H_{\alpha\beta}(x,t,\rho)Q(t)\,dt,\\
H_{\alpha\beta}(x,t,\rho)Q(t) &= \exp(\rho x(R_\alpha - \overleftarrow{R}_k)\\
&\qquad + \rho t(R_\beta - R_\alpha))\rho t\cdot\left|\left(Q(t)\tilde{E}_\beta(t,\rho)\right)\wedge \tilde{E}_{\alpha'}(t,\rho) - \left(Q(t)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right|\tilde{E}_\alpha(x,\rho),\\
h_{\alpha\beta}^{(0)}(Q,x,\rho) &= \theta^+(|\rho x|-1)\int_{|\rho|^{-1}}^x \exp(\rho x(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - R_\alpha))\left|\left(Q(t)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right|\tilde{E}_\alpha(x,\rho)\,dt\\
&= (\tilde{h}_{\alpha\beta}^{(0)}(Q,x,\rho) + \hat{h}_{\alpha\beta}^{(0)}(Q,x,\rho))\theta^+(|\rho x|-1)\tilde{E}_\alpha(x,\rho).
\end{aligned}$$

Here

$$\begin{aligned}
\tilde{h}_{\alpha\beta}^{(0)}(Q,x,\rho) &= \int_0^x \exp(\rho x(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - R_\alpha))\left|\left(Q(t)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right|\,dt,\\
\hat{h}_{\alpha\beta}^{(0)}(Q,x,\rho) &= \int_0^{|\rho|^{-1}} \exp(\rho x(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - R_\alpha))\left|\left(Q(t)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right|\,dt
\end{aligned}$$

and we take into account that $\tilde{E}_\alpha(x,\rho) = \exp(-\rho x R_\alpha)E_\alpha(x,\rho)$ if $|\rho|x > 1$.

From the asymptotics of $E_k(x,\rho)$ it follows the estimate:

$$\left\|\left(Q(t)\tilde{E}_\beta(t,\rho)\right)\wedge\tilde{E}_{\alpha'}(t,\rho) - \left(Q(t)\mathfrak{f}_\beta\right)\wedge\mathfrak{f}_{\alpha'}\right\| \le \frac{M}{|\rho|t}\|Q(t)\| \tag{6}$$

with some absolute constant M.

Consider the expression $\rho x(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - R_\alpha)$. We rewrite it as $\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - \overleftarrow{R}_k)$ and notice that $\mathrm{Re}\left(\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho t(R_\beta - \overleftarrow{R}_k)\right) \le 0$ for all (x,t,ρ) such that $t\in[0,x]$, $\rho\in\overline{\mathscr{S}}$. By virtue of Lemma 3.5 and estimate (6) we have $h_{\alpha\beta}^{(1)}(Q,\cdot,\cdot)\in PC_0(\overline{\mathscr{S}})$, $h_{\alpha\beta}^{(1)}\in\mathscr{L}\left(\mathscr{X}_p^{n-k+1}, BC([0,\infty),L_2(l))\right)$, while Lemma 3.4 guarantees that the same is true for

$\hat{h}^{(0)}_{\alpha\beta}$. Moreover, we see that if $\alpha \neq \beta$ then $R_\alpha - \overleftarrow{R}_k \neq R_\beta - \overleftarrow{R}_k$. Therefore, one can apply Lemma 3.3 and conclude that $\tilde{h}^{(0)}_{\alpha\beta}(Q,\cdot,\cdot) \in PC_0(\overline{\mathscr{S}})$, $\tilde{h}^{(0)}_{\alpha\beta} \in \mathscr{L}\left(\mathscr{X}^{n-k+1}_p, BC([0,\infty), L_2(l))\right)$ provided that $\alpha \neq \beta$. Now we recall that $(Q\mathfrak{f}_\alpha) \wedge \mathfrak{f}_{\alpha'} = 0$ for any $\alpha \in \mathscr{A}_{n-k+1}$ and therefore $\tilde{h}^{(0)}_{\alpha\beta} = 0$ if $\alpha = \beta$. Thus, we conclude that $h_{\alpha\beta}(Q,\cdot,\cdot) \in PC_0(\overline{\mathscr{S}})$, $h_{\alpha\beta} \in \mathscr{L}\left(\mathscr{X}^{n-k+1}_p, BC([0,\infty), L_2(l))\right)$ for any pair of multi-indices α, β from $\mathscr{A}_{n-k+1}$, therefore (in view of Lemma 3.2) the same is true for $\omega^{(0)}_k$ and for ω_k. From (5) it follows now that $\omega_k \in \mathscr{L}\left(\mathscr{X}^{n-k+1}_p, BC([0,\infty), C_0(\overline{\mathscr{S}}))\right)$. Together with other results of the first part this completes the proof. □

Proof of Theorem 2. From the identity:

$$\rho(Q(t)T_k(t,\rho)) \wedge E_{\alpha'}(t,\rho) = \frac{d}{dt}\left((\hat{Q}(t)T_k(t,\rho)) \wedge E_{\alpha'}(t,\rho)\right) - \rho(\tilde{Q}(t)T_k(t,\rho)) \wedge E_{\alpha'}(t,\rho),$$

where $\alpha \in \mathscr{A}_{n-k+1}$ is arbitrary and

$$\tilde{Q}(t) := \hat{Q}'(t) + t^{-1}[\hat{Q}(t), A^{(n-k+1)}]$$

it follows the relation:

$$\rho\int_{x_0}^{x} G_{n-k+1}(x,t,\rho)\left(Q(t)T_k(t,\rho)\right)dt =$$

$$G_{n-k+1}(x,t,\rho)\left(\hat{Q}(t)T_k(t,\rho)\right)\Big|_{x_0}^{x} - \int_{x_0}^{x} G_{n-k+1}(x,t,\rho)\left(\tilde{Q}(t)T_k(t,\rho)\right)dt.$$

Under the conditions of the theorem $G_{n-k+1}(x,t,\rho)\left(Q(t)T_k(t,\rho)\right)$, $G_{n-k+1}(x,t,\rho)\left(\tilde{Q}(t)T_k(t,\rho)\right)$ remain bounded as $t \to 0$ (while x, ρ are fixed) and $\hat{Q}(0) = 0$. Therefore, by passing to the limit as $x_0 \to 0$ we obtain:

$$\rho\mathscr{T}_k(Q,x,\rho) = G_{n-k+1}(x,x,\rho)\left(\hat{Q}(x)T_k(x,\rho)\right) - \mathscr{T}_k(\tilde{Q},x,\rho).$$

In order to observe the desired representation it is sufficient to use the representations for $G_{n-k+1}(x,t,\rho)$, $T_k(t,\rho)$ and Theorem 1. The asymptotics of $E_\alpha(x,\rho)$ yield the asymptotics for $\mathscr{T}_k(Q,x,\rho)$. □

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