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THE FORMULA FOR THE REGULARIZED TRACE OF THE STURM-LIOUVILLE OPERATOR WITH A LOGARITHMIC POTENTIAL

KHABIR KABIROVICH ISHKIN AND LEISAN GABDULLOVNA VALIULLINA

Abstract. We have obtained a regularized trace formula for the Sturm-Liouville operator on a semi-axis with a logarithmic potential.

1. Introduction

Let the function q be defined on $(0,+\infty)$, real-valued, summable on any finite interval $(0,b)$, $b>0$, and

$$\lim_{x\to+\infty} q(x)=+\infty.$$

Further, let L be the operator generated in $L^2(,+\infty)$ by the differential expression $ly := -y''+qy$ and the boundary condition $y(0)=0$ [1, Ch. V, § 18]. According to the well-known A. M. Molchanov theorem [2], operator L has a discrete spectrum. In the paper [3] an asymptotic equation for the spectrum of the operator L whose potential can grow arbitrarily slowly was obtained. This equation allows us to calculate the first few (up to a summable remainder) terms of the asymptotic series for the eigenvalues in the case

$$q=\underbrace{\log\ldots\log}_{m} x,\ m\in\mathbb{N},\ a=\text{const}>\begin{cases} e^{m-2}, & m\ge 2,\\ 0, & m=1.\end{cases}$$

In particular, when $q=\log(x+a)$

$$\lambda_k = s_k + O\left(k^{-1}(\log k)^{-3/2}\right), \tag{1}$$

$$s_k = \log(2\sqrt{\pi}k) - k^{-1}\left(\frac{a}{\pi}\sqrt{\log k}+k_0-1/4+\frac{c_0}{\sqrt{\log k}}\right), \tag{2}$$

where $c_0=\dfrac{a}{2\pi}\left(1+\log(2\sqrt{\pi}/a)\right)$. A parameter k_0 is some positive integer (regularization defect) that ensures the convergence of the series

$$\sum_{k=2}^{\infty}(\lambda_k - s_k). \tag{3}$$

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The purpose of this paper is to find the value of k_0 and to calculate the sum of the series (3), wich is called the regularized trace of the operator L_a with the potential $q = \log(x+a)$.

The first L operator with the potential $q(x) = \log^+ x := \max\{\log x, 0\}$ was considered in [4] as an example of Feynman–Kac theory application [5, § X.11] (in combination with Karamata's Tauberian theorem [6, Ch. XII, § 7]) to the problem of finding the asymptotics of the function

$$N(\lambda) := \sum_{\lambda_k<\lambda} 1. \tag{4}$$

This result was summarized by K. Kh. Boymatov [7] on Sturm–Liouville operators with a matrix potential that allows growth of order $\underbrace{\log\ldots\log}_{m} x$ ($m \in \mathbb{N}$).

Regularized traces of the form (3) in the case of power growth potentials are well studied (see [8] and references therein). From the formulas (1) – (2) it is clear that for any n operator L_a^{-n} is not a trace-class operator, therefore the classical method of zeta-functions [9, 10] is not applicable in this situation. On the other hand, due to the exponential growth of the function (4) θ-function of the operator L_a

$$\Theta_a(t) := \sum_{k=1}^{\infty} e^{-t\lambda_k}. \tag{5}$$

is determined only on the half-plane $\Re t > c$ with some positive c. Therefore, the parabolic equation method [11] using asymptotics $\Theta_a(t)$ as $t \to +0$ seems inapplicable in this case. However, formulas (1) – (2) allow us to construct an analytic continuation of the function Θ_a on the half-plane $\Re t > 0$. Using this fact, it is possible to find the sum of a series (3).

2. Asymptotics of the function Θ_a

By virtue of formulas (1) – (2) and (5) the function Θ_a holomorphic in the half-plane $\Re t > 1$.

Theorem 1. *The function Θ_a admits a meromorphic (with a single pole at $t = 1$) continuation to the half-plane $\Re t > 0$ and as $t \to +0$ the following asymptotic decomposition*

$$\Theta_a(t) \sim -\frac{a}{2\sqrt{\pi}} t^{-1/2} - (k_0 + 1/4) + \theta_0 t^{1/2} + \theta_1 t + O(t^{3/2}), \tag{6}$$

where

$$\begin{aligned}\theta_0 &= \frac{a\left(\log a - 1\right)}{2\sqrt{\pi}},\\ \theta_1 &= \lambda_1 + \sigma(2) + \frac{a}{\pi}\int_1^{\infty} \{x\}\left((\log x)^{1/2} x^{-1}\right)' dx\\ &\quad + \left(k_0 - \frac{1}{4}\right)(\gamma - 1) - \left(k_0 + \frac{5}{4}\right)\log 2\sqrt{\pi} + \frac{1}{2}\log 2\pi\end{aligned}$$

$$+\frac{a}{2\pi}\left(1+\log\frac{2\sqrt{\pi}}{a}\right)\int_1^{\infty}\{x\}\left((\log x)^{-1/2}x^{-1}\right)' dx,$$

γ – *Euler's constant.*

Proof. We transform the decomposition (1) – (2) to a form, wich is more convenient for finding the asymptotics of the function Θ_a:

$$\lambda_k = \log k + c_1 + c_2(\log k)^{1/2}k^{-1} + c_3 k^{-1} + c_0(\log k)^{-1/2}k^{-1} + r_k, \tag{7}$$

where $c_1 = \log 2\sqrt{\pi}$, $c_2 = \dfrac{a}{\pi}$, $c_3 = k_0 - \dfrac{1}{4}$, $c_0 = \dfrac{a}{2\pi}\left(1+\log(2\sqrt{\pi}/a)\right)$,

$$r_k = O(k^{-1}(\log k)^{-3/2}), \quad k\to\infty. \tag{8}$$

Then

$$\begin{aligned}\Theta_a(t) &= \sum_{k=1}^{\infty} e^{-t\lambda_k} = e^{-t\lambda_1} + e^{-tc_1}\sum_{k=2}^{\infty} e^{-t(\log k + c_2(\log k)^{1/2} + c_3 k^{-1} + c_0(\log k)^{1/2}k^{-1} + r_k)}\\ &= e^{-t\lambda_1} + e^{-tc_1}\sum_{k=2}^{\infty} k^{-t} e^{-c_2 t(\log k)^{1/2}k^{-1}} e^{-c_3 t k^{-1}} e^{-c_0 t(\log k)^{-1/2}k^{-1}} e^{-tr_k}.\end{aligned} \tag{9}$$

Putting each exponent according to the Taylor formula, we get

$$\begin{aligned} e^{-c_3 t k^{-1}} &= 1 - c_3 t k^{-1} + O(t^2 k^{-2});\\ e^{-c_0 t(\log k)^{-\frac{1}{2}}k^{-1}} &= 1 - c_0 t(\log k)^{-1/2}k^{-1} + O(t^2(\log k)^{-1}k^{-2});\\ e^{-c_2 t(\log k)^{\frac{1}{2}}k^{-1}} &= 1 - c_2 t(\log k)^{1/2}k^{-1} + O(t^2(\log k)k^{-2});\\ e^{-tr_k} &= 1 - tr_k + O(t^2 {r_k}^2).\end{aligned}$$

Substituting these expansions into (9), we have

$$\begin{aligned}\Theta_a(t) = e^{-t\lambda_1} + e^{-tc_1}\{\zeta(t) - 1 - t\,[c_2\varphi_1(t) + c_3(\zeta(t+1)-1)\\ + c_0\varphi_2(t) + \varphi_3(t)]\} + \varphi_4(t),\end{aligned} \tag{10}$$

where

$$\varphi_1(t) = \sum_{k=2}^{\infty}(\log k)^{1/2}k^{-1-t}, \qquad \varphi_2(t) = \sum_{k=2}^{\infty}(\log k)^{-1/2}k^{-1-t},$$

$$\varphi_3(t) = \sum_{k=2}^{\infty}k^{-t}r_k, \qquad \varphi_4(t) = e^{-tc_1}\sum_{k=2}^{\infty}k^{-t}R_k(t),$$

$$R_k(t) = O\left(t^2(\log k)k^{-2}\right).$$

Therefore, the function Θ_a admits a meromorphic extension to the half-plane $\Re t > 0$ with a single pole at $t = 1$ (because of the term $\zeta(t)$). Find the asymptotics $\Theta_a(t)$ as $t \to +0$.

Let $\rho(x) = x - \{x\}$, where $\{x\}$ is fractional part of the number x. Then $\quad\forall \quad 0 < \varepsilon < 1$

$$\varphi_1(t) = \int_{1+\varepsilon}^{\infty} (\log x)^{1/2} x^{-1-t} d\rho(x) = \int_{1+\varepsilon}^{\infty} (\log x)^{1/2} x^{-1-t} dx$$
$$+\varepsilon \left(\log(1+\varepsilon)\right)^{1/2} (1+\varepsilon)^{-1-t} + \int_{1+\varepsilon}^{\infty} \{x\} \left[(\log x)^{1/2} x^{-1-t}\right]' dx,$$

whence as $\varepsilon \to +0$ we get

$$\varphi_1(t) = \int_1^{\infty} (\log x)^{1/2} x^{-1-t} dt + \int_1^{\infty} \{x\} \left((\log x)^{1/2} x^{-1-t}\right)' dx =: \varphi_{11}(t) + \varphi_{12}(t).$$

Direct calculation gives

$$\varphi_{11}(t) = \frac{\sqrt{\pi}}{2} t^{-3/2}.$$

Further,

$$\varphi_{12}(t) = \int_1^{\infty} \{x\} \left[-(1+t)(\log x)^{1/2} + \frac{1}{2}(\log x)^{-1/2}\right] x^{-2-t} dx.$$

Denote the integrand by $f(x,t)$. Then $\quad \forall k$

$$\frac{\partial^k}{\partial t^k} f(x,t) = O\left((\log x)^{-1/2+k} x^{-2}\right), \quad x \geq 1,$$

uniformly in $\Re\, t > 0$, therefore φ_{12} is holomorphic at zero, so

$$\varphi_{1,2}(t) = \varphi_{1,2}(0) + O(t), \quad t \to 0.$$

Therefore,

$$\varphi_1(t) = \frac{\sqrt{\pi}}{2} t^{-3/2} + \int_1^{\infty} \{x\} \left((\log x)^{1/2} x^{-1}\right)' dx + O(t), \quad t \to +0. \tag{11}$$

Similarly, it is proved that

$$\varphi_2(t) = \sqrt{\pi}\, t^{-1/2} + \int_1^{\infty} \{x\} \left((\log x)^{-1/2} x^{-1}\right)' dx. \tag{12}$$

Let us introduce the function

$$\sigma(x) = \sum_{k \geq x} r_k.$$

From (1) – (2) it follows that

$$\sigma(x) = O\left((\log x)^{-1/2}\right), \quad x \to \infty.$$

Then

$$\varphi_3(t) = -\int_2^{\infty} x^{-t} d\sigma(x) = \frac{\sigma(2)}{2^t} - t\int_2^{\infty} x^{-t-1} \sigma(x) dx = \sigma(2) + O(t^{1/2}),\ t \to +0. \tag{13}$$

Further, due to the well-known properties of ζ-functions (see, for example, [12, Ch. II, 1^0])

$$\zeta(t) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)\,t + O(t^2), \tag{14}$$

$$\zeta(t+1) = \frac{1}{t} + \gamma + O(t). \tag{15}$$

Now substituting (11) – (15) into (10), after simple calculations we get (6). The theorem is proved.

3. Operator e^{-tL_a} and its trace

3.1. Estimation of the resolvent kernel for the operator L_a

We introduce the notation. Let be $\Omega_a = \mathbb{C}\setminus[\log a, +\infty)$, $\Omega_{r\sigma} = \{\sigma \le \arg(\lambda - \log a - r) \le 2\pi - \sigma\}$, $r > 0, 0 < \sigma < \pi$, $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ — solutions of the equation

$$-y'' + q_a y = \lambda y \tag{16}$$

that satisfy the following conditions:

$$\varphi(0,\lambda) = 0,\ \frac{\partial\varphi}{\partial x}(0,\lambda) = 1,$$
$$\psi(x,\lambda) \sim (q_a(x)-\lambda)^{-1/4}\exp\left(-\int_0^x (q_a(t)-\lambda)^{1/2}dt\right),\ x \to +\infty. \tag{17}$$

Hereinafter, the expression $z^{1/n}$ will mean that branch of the root $\sqrt[n]{z}$, which is positive for positive z. Since for each fixed $\lambda \notin [q_a(b), +\infty)$ the function

$$\alpha(x,\lambda) = \frac{1}{8}\frac{q_a''(x)}{(q_a(x)-\lambda)^{3/2}} - \frac{5}{32}\frac{q_a'^2(x)}{(q_a(x)-\lambda)^{5/2}}, \tag{18}$$

is summable on $(b,+\infty)$, there exists a unique solution ψ, satisfying (17),[13, Ch. II, § 6].

The resolvent kernel for the operator L_a has the form

$$G(x,y,\lambda) = \frac{1}{\psi(0,\lambda)}\begin{cases}\varphi(x,\lambda)\psi(t,\lambda),\ x<t<\infty,\\ \psi(x,\lambda)\varphi(t,\lambda),\ 0<t<x.\end{cases} \tag{19}$$

Therefore,

$$G(x,x,\lambda) = \frac{1}{\psi(0,\lambda)}\varphi(x,\lambda)\psi(x,\lambda). \tag{20}$$

Lemma 1. *For each $\varepsilon > 0$ there is a constant $r_\varepsilon > 0$ such that if $\varepsilon \le \sigma < \pi, r \ge r_\varepsilon$, then for all $x \ge 0$ and $\lambda \in \Omega_{r\sigma}$ function $\psi(x,\lambda)$ permits the representation*

$$\psi(x,\lambda) \sim (q_a(x)-\lambda)^{-1/4}\exp\left(-\int_0^x (q_a(t)-\lambda)^{1/2}dt\right)\times$$

$$\times\left[1+\int_x^{+\infty}\left(\alpha+\frac{\alpha'}{2\sqrt{q-\lambda}}+R\right)dt\right], \tag{21}$$

where α is defined by the formula (18) *and*

$$\sup_{x\ge 0,\lambda\in\Omega_{r\sigma}}\left|R(x,\lambda)(x+a)^3(q-\lambda)^{5/2}\right|<\infty. \tag{22}$$

Proof. Substituting

$$\psi(x,\lambda)=(q_a(x)-\lambda)^{-1/4}\exp\left(-\int_0^x(q_a(t)-\lambda)^{1/2}dt+\int_{x_0}^x(-\alpha+\beta)dt\right), \tag{23}$$

we get the equation for β

$$\beta'-2(\beta_0+\beta_1+\alpha)\beta+2\beta_1\alpha+\alpha^2-\alpha'+\beta^2=0,$$

where

$$\beta_0=\sqrt{q_a-\lambda},\ \beta_1=\frac{1}{4}\frac{q_a'}{q_a-\lambda}.$$

The method of variation of constants leads to the equation

$$\beta=\gamma+\int_x^{+\infty}e^{-2\int_x^t(\beta_0+\alpha)dt}(q_a(x)-\lambda)^{1/2}(q_a(t)-\lambda)^{-1/2}\beta^2dt, \tag{24}$$

where

$$\gamma=\int_x^{+\infty}e^{-2\int_x^t(\beta_0+\alpha)dt}(q_a(x)-\lambda)^{1/2}(q_a(t)-\lambda)^{-1/2}(2\beta_1\alpha-\alpha'+\alpha^2)dt.$$

By direct calculations, it is easy to verify that for every $r>0,\ 0<\sigma<\pi$

$$|q_a(t)-\lambda|>\sin\sigma|q_a(x)-\lambda|\ \forall\ t\ge x,\lambda\in\Omega_{r\sigma}.$$

Integrating in parts and taking into account the last inequality, we get

$$\gamma=-\frac{\alpha'}{2\sqrt{q_a-\lambda}}+O\left((x+a)^{-3}(q_a(x)-\lambda)^{-5/2}\right)$$

uniformly in $x\ge 0$ and $\lambda\in\Omega_{r\sigma}$ for all $r>0, 0<\sigma<\pi$.

Replacing

$$R=(x+a)^3(q_a(x)-\lambda)^{5/2}\left(\beta+\frac{\alpha'}{2\sqrt{q_a-\lambda}}\right) \tag{25}$$

converts the equation (24) to the form

$$R(x,\lambda)=\widetilde{\gamma}(x,\lambda)+\int_x^{+\infty}K(x,t,\lambda)R^2(t,\lambda)dt, \tag{26}$$

where functions $\widetilde{\gamma}(x,\lambda)$ and $K(x,t,\lambda)$ are continuous on $[0,+\infty)\times\Omega_a$ and $[0,+\infty)^2\times\Omega_a$ respectively. Moreover,

$$\widetilde{\gamma}(x,\lambda)=O(1),\ K(x,t,\lambda)=O\left((x+a)^{-3}(q_a(x)-\lambda)^{-5/2}\right)$$

uniformly in $x\geq 0$ and $\lambda\in\Omega_{r\sigma}$ (for all $r>0, 0<\sigma<\pi$). According to the second estimate for any $\varepsilon>0$ there is a sufficiently large $r_\varepsilon>0$ such that for all $r\geq r_\varepsilon$ and $\varepsilon\leq\sigma<\pi$ the integral operator on the right side of (26) is a contraction in a Banach space $C([0,+\infty)\times\Omega_{r\sigma})$.The method of successive approximations shows that for all $r\geq r_\varepsilon$ and $\varepsilon\leq\sigma<\pi$, the equation (26) has a unique solution $R\in C([0,+\infty)\times\Omega_{r\sigma})$. Hence, according to the equalities (25) and (23), for $x_0=+\infty$, the representation (21) with the estimate (22) follows. The lemma is proved.

According to the proof of the lemma it follows that if $\varepsilon\leq\sigma<\pi, r\geq r_\varepsilon$ then the function $\psi(x,\lambda)$ for every $\lambda\in\Omega_{r\sigma}$ does not have zeros on $[0,+\infty)$. Therefore, the solution $\varphi(x,\lambda)$ can be accepted in the form

$$\varphi(x,\lambda)=\psi(0,\lambda)\psi(x,\lambda)\int_0^x\psi^{-2}(t,\lambda)\,dt. \tag{27}$$

Substituting this expression into (20), we get

$$G(x,x,\lambda)=\psi^2(x,\lambda)\int_0^x\psi^{-2}(t,\lambda)\,dt. \tag{28}$$

We set

$$\Psi(x,\lambda)=\int_0^x\psi^{-2}(t,\lambda)\,dt, x>0,\ \lambda\in\Omega_{r\sigma},\ \varepsilon\leq\sigma<\pi, r\geq r_\varepsilon. \tag{29}$$

Lemma 2. *Let $\varepsilon\leq\sigma<\pi, r\geq r_\varepsilon$. Then*

$$\Psi(x,\lambda)\sim\frac{1}{2}\exp\left(2\int_0^x(q_a(t)-\lambda)^{1/2}\,dt\right)\times\left[1-2\int_x^{+\infty}\left(\alpha+\frac{\alpha'}{2\sqrt{q-\lambda}}\right)dt+Q(x,\lambda)\right], \tag{30}$$

where

$$\sup_{x\geq 0,\lambda\in\Omega_{r\sigma}}\left|Q(x,\lambda)(x+a)^2(q-\lambda)^{5/2}\right|<\infty. \tag{31}$$

Proof. Fix $\varepsilon>0$ and everywhere until the end of the proof of the lemma we assume that $\varepsilon<\sigma<\pi,\ \lambda\in\Omega_{r\sigma}, x>0$. The function Ψ can be represented as

$$\Psi(x,\lambda)=\int_0^{x/2}\psi^{-2}(t,\lambda)\,dt+\int_{x/2}^x\psi^{-2}(t,\lambda)\,dt=:\Psi_1(x,\lambda)+\Psi_2(x,\lambda). \tag{32}$$

Since for all $\sigma-\pi<\arg q_a(x)-\lambda<\pi-\sigma$ then $\forall\ 0\leq t<x/2$

$$\Re\left(\int_t^x\sqrt{q_a-\lambda}\,d\tau\right)>\Re\left(\int_{x/2}^x\sqrt{q_a-\lambda}\,d\tau\right)$$

$$> \sin(\sigma/2) \min\left\{\sqrt{q_a(x)-\lambda}, \sqrt{q_a(x/2)-\lambda}\right\}. \tag{33}$$

Futher, since $\forall\ t \in [x/2, x]\quad |q_a(x) - q_a(t)| < \log 2$ and $|q_a(x) - \lambda| > r \sin\sigma$ then for all $r > 2\log 2/\sin\sigma$

$$1/2 < \left|\frac{q_a(x)-\lambda}{q_a(t)-\lambda}\right| < 3/2 \quad \forall\ t \in [x/2, x], \lambda \in \Omega_{r\sigma}. \tag{34}$$

Hence, taking into account (33), we conclude

$$\Re\left(\int_t^x \sqrt{q_a - \lambda}\, d\tau\right) \geq C|q_a(x) - \lambda|^{1/2} x.$$

Therefore, for all $x \geq 0, \lambda \in \Omega_{r\sigma}$, where $r > 2\log 2/\sin\varepsilon$,

$$|\Psi_1(x,\lambda)| \leq C(r)\left|e^{2\int_0^x (q_a(t)-\lambda)^{1/2} dt}(x+a)^{-2}(q(x)-\lambda)^{-5/2}\right|, \tag{35}$$

$C(\varepsilon) > 0$ depends only on ε.

Next, we substitute the expression (21) into the formula for Ψ_2 and integrate the latter in parts. Cosequently, according to (34), we obtain (30), (31).

Lemma 3. *Let $0 < \varepsilon \leq \sigma < \pi$. Then there exists $r_\varepsilon > 0$ so that the representation is true*

$$G(x,x,\lambda) = \frac{1}{2}(q_a(x)-\lambda)^{-1/2}\left(1 - e^{-2\int_0^x \sqrt{q_a(t)-\lambda}\, dt} + g(x,\lambda)\right), \tag{36}$$

$$|g(x,\lambda| \leq C(\varepsilon)(x+a)^{-2}|q(x)-\lambda|^{-5/2} \quad \forall\ x \geq 0, \lambda \in \Omega_{r\sigma}, \tag{37}$$

where $r > r_\varepsilon, C(\varepsilon) > 0$ depends only on ε.

Proof. It follows from the formula (28) and Lemmas 2, 3.

3.2. Formula for the kernel of the operator e^{-tL_a}

Let $0 < \varepsilon \leq \sigma < \pi, r > r_\varepsilon$, where r_ε is the constant appearing in the formulation of the Lemma 3, $\gamma_{r\sigma}$ is the boundary of the region $\Omega_{r\sigma}$ with the counterclockwise direction[1].

Lemma 4. *For any $t > 0$ the representation*

$$e^{-tL_a} = \frac{1}{2\pi i}\int_{\gamma_{r\sigma}} e^{-t\lambda}(\lambda - L_a)^{-1}\, d\lambda \tag{38}$$

holds. Here the limit is understood in the sense of convergence in the uniform operator topology.

[1] Since $\lambda_1 > \log a$, then bypassing $\gamma_{r\sigma}$ the entire spectrum of L_a remains on the left.

Proof. Since e^{-tL_a} is a bounded operator, then

$$e^{-tL_a} = \lim_{n\to\infty} e^{-tL_a} P_n,$$

where P_n is orthogonal projector on the linear span of the first n eigenvectors of the operator L_a and the limit is understood in the sense of a convergence in the norm. We have (see, for example, [14, § XII.2])

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\lambda - L_a)^{-1} d\lambda,$$

where γ_n is any contour covering the first n eigenvalues of the operator L_a, the integral is taken in the counterclockwise direction. Then

$$e^{-tL_a} P_n = \frac{1}{2\pi i}\int_{\gamma_n} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda.$$

As γ_n we take a contour composed of l_n – parts of $\gamma_{r\sigma}$ lying in the half-plane $\Re\lambda < (\lambda_n + \lambda_{n+1})/2$, and the segment l_n connecting the ends of γ_n.

Let

$$R_n = \int_{l_n} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda.$$

Let us prove that for any $\Re t > 1$

$$\|R_n\| \to 0, \qquad n \to \infty. \tag{39}$$

We have

$$\|(\lambda - L_a)^{-1}\| \le \frac{1}{\operatorname{dist}(\lambda, \sigma(L_a))}. \tag{40}$$

Hence, for all λ from $[N_-, N_+]$, where $N_\pm = (\lambda_n + \lambda_{n+1})/2 \pm i$, we will have $\|(\lambda - L_a)^{-1}\| \le 2/(\lambda_{n+1} - \lambda_n)$. According to (1) – (2) $\lambda_{n+1} - \lambda_n \sim n^{-1}, \quad n \to \infty$. Hence, for the operator

$$r_n = \int_{[N_-, N_+]} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda$$

we get

$$\|r_n\| = O\left(n^{-t+1}\right), \quad n \to \infty. \tag{41}$$

Further, since according to (40) $\|(\lambda - L_a)^{-1}\| < 1$ on $l_n \backslash [N_-, N_+]$, then

$$\|R_n - r_n\| = O\left(e^{-tn} n\right),\ n \to \infty. \tag{42}$$

Now (39) directly follows from (41) and (42).

3.3. Asymptotics of $\operatorname{tr}\left(e^{-tL_a}\right)$

Lemma 5. *For* $\Re t > 1$

$$\operatorname{tr}\left(e^{-tL_a}\right) = -\frac{1}{2\pi i}\int_0^\infty dx \int_{\gamma_{r\sigma}} e^{-t\lambda} G(x,x,\lambda)\,d\lambda.$$

Proof. By Lemma 4

$$(e^{-tL_a} f)(x) = -\frac{1}{2\pi i}\int_\Gamma d\lambda e^{-t\lambda}\int_0^\infty G(x,y,\lambda) f(y)\,dy. \tag{43}$$

According to formulas (19), (27) and Lemmas 1 and 2

$$|G(x,y,\lambda)| \le C|(q_a(x)-\lambda)^{-1/4}|q_a(y)-\lambda|^{-1/4}\exp\left(-\left|\Re\int_x^y \sqrt{q_a(t)-\lambda}\,dt\right|\right), \tag{44}$$

where $C = \mathrm{const} > 0$. Further, for any fix $x > 0$ and all $f \in L^2(0,+\infty)$

$$\begin{aligned}|\left(e^{-tL_a} f\right)(x)| &\le \int_\Gamma |d\lambda||e^{-t\lambda}|\int_0^\infty |G(x,y,\lambda) f|\,dy\\ &\le \frac{1}{2\pi}\int_\Gamma |d\lambda||e^{-t\lambda}|\sqrt{\int_0^\infty |G(x,y,\lambda)|^2 dy}\cdot\|f\|\end{aligned}$$

Hence, taking into account the estimate (44), we will have

$$|e^{-tL_a} f(x)| \le C\int_{\gamma_{r\sigma}} |e^{-t\lambda}||q_a(x)-\lambda|^{-\frac{1}{4}}|d\lambda| < \infty.$$

Therefore, we can apply the Fubini theorem to the right side of (43). As a result, we get

$$e^{-tL_a} f = -\int_0^\infty \left(\frac{1}{2\pi i}\int_{\gamma_{r\sigma}} d\lambda e^{-t\lambda} G(x,y,\lambda)\right) f\,dy.$$

Consequently, the kernel of the operator e^{-tL_a} has the form

$$H(x,y,t) = -\frac{1}{2\pi i}\int_{\gamma_{r\sigma}} d\lambda e^{-t\lambda} G(x,y,\lambda).$$

According to formulas (1) – (2) for $\Re t > 1$ the operator e^{-tL_a} is a trace-class one, therefore

$$\operatorname{tr}\left(e^{-tL_a}\right) = \int_0^\infty H(x,x,t)\,dx = -\frac{1}{2\pi i}\int_0^\infty dx\int_{\gamma_{r\sigma}} d\lambda e^{-t\lambda} G(x,x,\lambda).$$

Theorem 2. *For the θ-functions of the operator L_a as $t \to +0$ the following decomposition is true*

$$\Theta_a(t) \sim -\frac{a}{2\sqrt{\pi}} t^{-1/2} - 1/4 + \theta_0 t^{1/2} + \frac{\log a}{4} t + O\left(t^{3/2}\right). \tag{45}$$

Here θ_0 is defined in the same way as in (6).

Proof. The statement of the theorem follows directly from the Lemmas 3 and 5.

Corollary 3. *The constant k_0 in formulas (1) – (2) and (6) is equal to 0.*

4. Regularized trace of the operator L_a

The sum

$$\sigma = \lambda_1 + \sum_{k=2}^{\infty}\left[\lambda_k - \log(2\sqrt{\pi}k) - k^{-1}\left(\frac{a}{\pi}(\log k)^{1/2} - 1/4 + c_0(\log k)^{-1/2}\right)\right], \tag{46}$$

where $c_0 = \frac{a}{2\pi}\left(1+\log(2\sqrt{\pi}/a)\right)$, we call the *regularized trace of the operator L_a*.

Theorem 4. *The following formula is true*

$$\sigma = \frac{1}{4}\left(\log\left(2\sqrt{\pi}\right)+\gamma-1+\log a\right) - \frac{a}{\pi}\int_1^{\infty}\{x\}\left((\log x)^{1/2}x^{-1}\right)' dx$$
$$-\frac{a}{2\pi}\left(1+\log\frac{2\sqrt{\pi}}{a}\right)\int_1^{\infty}\{x\}\left((\log x)^{-1/2}x^{-1}\right)' dx. \tag{47}$$

Proof. The formula (47) follows from the formulas (6) and (45).

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UDepartment of Mathematics and Information Technology, Bashkir State University, Zaki Validi street, 83, Ufa, 450076, Russia.

E-mail: Ishkin62@mail.ru

Department of Mathematics and Information Technology, Bashkir State University, Zaki Validi street, 83, Ufa, 450076, Russia.

E-mail: l.matem2012@yandex.ru