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ON THE INTEGRATION OF THE MATRIX MODIFIED KORTEWEG-DE VRIES EQUATION WITH A SELF-CONSISTENT SOURCE

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Abstract. In this work we deduce laws of the evolution of the scattering data for the matrix Zakharov Shabat system with the potential that is the solution of the matrix modified KdV equation with a self consistent source.

1. Introduction

The inverse scattering transform method was first proposed by Gardner, Green, Kruskal and Miura (GGKM) [1] in 1967 for solving the Caushy problem for the classical KdV equation

$$u_t = 6uu_x + u_{xxx}$$

Their approach was based on the connection between the KdV equation and the spectral theory for the Sturm-Liouville operator on the line. Shortly thereafter, Lax [2] pointed out the general character of the inverse scattering method. A few years later, Zakharov Shabat [3] managed to solve another important nonlinear evolution equation, the so-called nonlinear Schrödinger equation, using a nontrivial extension of the methods used in [1], [2]. Thus, a way was found for construction of several other classes of equations that can be solved by similar methods.

It was shown that modified KdV equation can be solved by the inverse scattering method in [4]. The inverse scattering problem for the matrix Schrödinger equation was discussed and the inverse scattering method was generalized to the matrix form in [5].

The direct and inverse scattering theory of the matrix Zakharov-Shabat system was studied in [6]. A detailed exposition of the relations between the inverse problems and nonlinear equations of the mathematical physics is provided, for example, in the monographs [7]-[12].

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There was also given much attention to the soliton equations with self- consistent sources in the recent literature. Physically, the sources appear in solitary waves with non-constant velocity and lead to a variety of dynamics of physical models. For applications, these kinds of systems are usually used to describe interactions between different solitary waves and are relevant in some problems related with, among others, hydrodynamics, solid state physics or plasma physics [13]-[17]. Different techniques have been used to construct their solutions, such as inverse scattering [14, 15, 18, 19, 20].

The multisoliton solutions of the matrix KdV equation was considered by Goncharenko [21] and the matrix KdV equation with self-consistent source was studied in [22]. The matrix modified KdV equation was studied in [23]. In this work we study the matrix modified Korteweg-de Vries equation with self-consistent source and give the representations for the evolution equations of the scattering data.

We consider the integration of the following problem

$$\begin{aligned} &U_t + 3U^2U_x + 3U_xU^2 + U_{xxx} \\ &\quad = 2\sum_{n=1}^{N}\left(\Phi_{1,n}(\lambda_n,x,t)\otimes\Phi_{1,n}^T(-\lambda_n,x,t)+\Phi_{2,n}(\lambda_n,x,t)\otimes\Phi_{2,n}^T(-\lambda_n,x,t)\right), \qquad (1)\\ &\quad -iJ\Phi_n' - V\Phi_n = \lambda_n\Phi_n \qquad n=1,2,\dots,2N, \qquad (2)\end{aligned}$$

under the initial condition

$$U|_{t=0} = U_0(x), \tag{3}$$

with the normalizing conditions

$$\int_{-\infty}^{\infty}\hat{\Phi}_n^T(x,t)\Phi_n(x,t)dx = a_n^2(t) \quad n=1,2,\dots,2N. \tag{4}$$

Here $U=U(x,t)$ is a real symmetric $m\times m$ matrix $U=U^T$; $\Phi_n = \begin{pmatrix}\Phi_{1,n}\\ \Phi_{2,n}\end{pmatrix}$, $\Phi_{1,n}\in R^m$ and $\Phi_{2,n}\in R^m$are column vector functions $\Phi_n=\Phi(\lambda_n,x,t)$; $J=\begin{bmatrix} I_m & 0\\ 0 & -I_m\end{bmatrix}$, I_m is the unit matrix; the potential $V=\begin{bmatrix}0 & U(x)\\ -U(x) & 0\end{bmatrix}$ is $(2m\times 2m)$ matrix ; $\hat{Y}(x,t)=\sigma Y(x,t)$ and $\sigma=\begin{pmatrix}0 & I_m\\ I_m & 0\end{pmatrix}$; $a_n^2(t)$, $n=1,2,\dots,2N$ are nonzero continuous scalar functions.

The matrix function $U_0(x)$ satisfy the following properties:

1. $\int_{-\infty}^{\infty}(1+|x|)\,\|U_0(x)\|\,dx<\infty$, where $\|X\| = \max\limits_j\sum_{k=1}^m|x_{jk}|$, $X=\left(x_{jk}\right)_{j,k=1}^m$;
2. Operator $L(0) = -iJ\frac{d}{dx} - V(x)$ possess exactly $2N$ simple eigenvalues $\lambda_1(0),\lambda_2(0),\dots,\lambda_{2N}(0)$ in the orthogonal direct sum of $2m$ copies of $L^2(R)$ if the potential $U(x)$ have entries in $L^1(R)$.

Our purpose is to find the solution of the problem (1)-(4) which is a collection

$$\{U(x,t), \Phi_1(x,t), \Phi_2(x,t), \dots, \Phi_{2N}(x,t)\},$$

satisfying the problem (1)-(4) and the following conditions:

1. For all $t > 0$,
$$\sum_{r=0}^{3} \int_{-\infty}^{\infty} (1+|x|) \left\| \frac{\partial^r}{\partial x^r} U(x,t) \right\| dx < \infty; \tag{5}$$
2. Operator $L(t) = -iJ\frac{d}{dx} - V(x,t)$ possess exactly $2N$ simple eigenvalues $\lambda_1(t), \lambda_2(t), \dots, \lambda_{2N}(t)$ on the orthogonal direct sum of $2m$ copies of $L^2(\mathrm{R})$ if the potential $U(x,t)$ have entries in $L^1(\mathrm{R})$;
3. The column vector functions $\Phi_n(x,t)$ for $n = 1,2,\dots,2N$ belong to the domain of $L^2(\mathrm{R})$ corresponding to the eigenvalue λ_n.

We assume that the solution $\{U(x,t), \Phi_1(x,t), \Phi_2(x,t), \dots, \Phi_{2N}(x,t)\}$ of the problem (1)-(4) exists in the sense of described above. The main aim of the work is to derive the representations for the evolution equations of the scattering data with which it is available to find the collection of solution of the problem (1)-(4) in the framework of the inverse scattering method for the operator $L(t)$.

2.The auxiliary statements from the direct scattering theory

In this section we give the brief information about the scattering theory for the matrix Zakharov Shabat system on the line $(-\infty < x < \infty)$,

$$LX \equiv -iJX' - VX = \lambda X, \tag{6}$$

and the auxiliary equation

$$-i(Y^T)'J + Y^T V = \mu Y^T, \tag{7}$$

where $X(\lambda,x)$ and $Y(\lambda,x)$ are column vector functions $X = \left(x_j\right)_{j=1}^{2m}$. We assume that the entries of $V(x)$ satisfy the required conditions for $U_0(x)$.

Lemma 1. *Let $X(\lambda,x)$ and $Y(\mu,x)$ be solutions of the equations (6) and (7), respectively, then following relations hold*

$$i(\mu+\lambda)Y^T(\mu,x)X(\lambda,x)) = (Y^T JX)', \tag{8}$$

$$i(\lambda-\mu)\hat{Y}^T(\mu,x)X(\lambda,x)) = (\hat{Y}^T JX)'. \tag{9}$$

In the proof we use $JV + VJ = 0$ and $J^2 = I_{2m}$. It is easy to check that the $\hat{X}(-\lambda, x)$ also satisfy the equation (6).

For $\lambda \in \mathrm{R}$ the Jost matrix $F(\lambda, x)$ and $G(\lambda, x)$ as the $2m \times 2m$ matrix solutions of (6) satisfy the following asymptotic conditions [6]:

$$\begin{aligned} F(\lambda,x) &= [\bar{\psi}(\lambda, x)\ \psi(\lambda, x)] \to e^{i\lambda Jx} I_{2m}, \quad x \to \infty, \\ G(\lambda,x) &= [\phi(\lambda, x)\ \bar{\phi}(\lambda, x)] \to e^{i\lambda Jx} I_{2m}, \quad x \to -\infty. \end{aligned} \tag{10}$$

Here $\bar{\psi}(\lambda, x)$, $\psi(\lambda, x)$, $\phi(\lambda, x)$ and $\bar{\phi}(\lambda, x)$ are the submatrices with m rows and m columns, respectively, which are usually called Jost solutions. It is easy to show that the matrices $F(\lambda, x)$ and $G(\lambda, x)$ at any $\lambda \in \mathrm{R}$ satisfy the integral equations

$$\begin{aligned} G(\lambda,x) &= e^{i\lambda Jx} - iJ\int_{-\infty}^{x} e^{i\lambda J(x-z)} V(z) G(\lambda, z) dz, \\ F(\lambda,x) &= e^{i\lambda Jx} - iJ\int_{x}^{\infty} e^{i\lambda J(x-z)} V(z) F(\lambda, z) dz. \end{aligned} \tag{11}$$

For $\lambda \in \mathrm{R}$ there exists $2m \times 2m$ matrices $A(\lambda)$ and $C(\lambda)$ such that

$$\begin{aligned} G(\lambda,x) &= F(\lambda,x)A(\lambda), \\ F(\lambda,x) &= G(\lambda,x)C(\lambda). \end{aligned} \tag{12}$$

Here $A(\lambda)$ and $C(\lambda)$ consist of block matrices such as $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, X_s, $s = 1,2,3,4$ are $m \times m$ matrices.

Assuming that the potential $U(x)$ have entries in $L^1(\mathrm{R})$, we can say that for each fixed $x \in \mathrm{R}$ the matrix functions $\bar{\psi}(\lambda, x)e^{-i\lambda x}$ and $\bar{\phi}(\lambda, x)e^{i\lambda x}$ ($\psi(\lambda, x)e^{i\lambda x}$ and $\phi(\lambda, x)e^{-i\lambda x}$) can be continuated to the half-plane $Im\lambda > 0$ ($Im\lambda < 0$) and for all $Im\lambda > 0$ ($Im\lambda < 0$) the matrix functions $\bar{\psi}(\lambda, x)e^{-i\lambda x}$ and $\bar{\phi}(\lambda, x)e^{i\lambda x}$ ($\psi(\lambda, x)e^{i\lambda x}$ and $\phi(\lambda, x)e^{-i\lambda x}$) are bounded. Invertible matrix function $A_1(\lambda)$ ($A_4(\lambda)$) can be analytically continuated to the half-plane $Im\lambda > 0$ ($Im\lambda < 0$) and the equation $\det A_1(\lambda) = 0$ ($\det A_4(\lambda) = 0$) has exactly a finite number of simple zeros λ_j $j = 1, 2, \dots N$ ($j = N+1, N+2, \dots 2N$) which correspond to the simple eigenvalues of the operator L with the requirement of the absence of spectral singularities. The matrix function $(A_1(\lambda))^{-1}((A_4(\lambda))^{-1})$ has simple poles in the points λ_j, $j = 1, 2, \dots N$. Let $N_j = \underset{\lambda=\lambda_j}{Res}(A_1(\lambda))^{-1}$, $j = 1,2,\dots,N$. Then there are matrices R_j such that

$$\bar{\phi}(\lambda_j, x)N_j = i\bar{\psi}(\lambda_j, x)R_j, j = 1, 2, \dots N. \tag{13}$$

The following matrix for $\lambda \in \mathrm{R}$

$$R(\lambda) = -A_1^{-1}(\lambda)A_2(\lambda), \tag{14}$$

is called reflection coefficient.

The set $\{R(\lambda),\, \lambda_1,\, \lambda_2, \dots \lambda_{2N},\, R_1, R_2, \dots, R_{2N}\}$ represents the scattering data associated with the equation (6).

The potential $U(x,t)$ can be determined from the scattering data [5].

3. Evolution of the scattering data

It is easy to show that equation (1) can be represented as a Lax operator equality:

$$L_t = [B,L] - \sum_{n=1}^{2N} \left[J, \Phi_n \otimes \hat{\Phi}_n^T\right], \qquad [B,L] = BL - LB, \tag{15}$$

where

$$B = \begin{pmatrix} -3(U^2)_x - 6U^2\frac{d}{dx} - 4I\frac{d^3}{dx^3} & 3iU_{xx} + 6iU_x\frac{d}{dx} \\ \\ 3iU_{xx} + 6iU_x\frac{d}{dx} & -3(U^2)_x - 6U^2\frac{d}{dx} - 4I\frac{d^3}{dx^3} \end{pmatrix}. \tag{16}$$

Here, both sides of the equality (15) turn out to be operators of multiplication by a matrix function.

The eigenvector function $\Phi_n(x,t)$corresponds to the eigenvalue λ_n, then $\hat{\Phi}_n = \begin{pmatrix} \Phi_{2,n} \\ \Phi_{1,n} \end{pmatrix}$ corresponds to the eigenvalue $-\lambda_n$.

Let us consider the following system of equations

$$LF_0 = \lambda F_0, \lambda \in \mathrm{R} \tag{17}$$

$$\frac{\partial F_n}{\partial x} = i\hat{\Phi}_n{}^T F_0, \quad n = 1,2,\dots,2N. \tag{18}$$

Then, the matrix functions

$$H_0 = \dot{F}_0 - BF_0 + \sum_{n=1}^{2N} \Phi_n \otimes F_n \tag{19}$$

$$H_n = \hat{\Phi}_n{}^T JF_0 - (\lambda - \lambda_n) F_n, \quad n = 1,2,\dots,2N. \tag{20}$$

satisfy the equality

$$LH_0 - \lambda H_0 = \sum_{n=1}^{2N} \Phi_n \otimes H_n.$$

In fact, if we take the derivative from equation (17) with respect to t

$$\dot{L}F_0 + L\dot{F}_0 = \lambda \dot{F}_0,$$

here, we find $L\dot{F}_0$

$$L\dot{F}_0 = \lambda \dot{F}_0 - \dot{L}F_0 = \lambda \dot{F}_0 - BLF_0 + LBF_0 - \sum_{n=1}^{2N} \left[J, \Phi_n \otimes \hat{\Phi}_n{}^T\right] F_0.$$

Using this equality now we calculate LH_0

$$LH_0 = \lambda \dot{F}_0 - \lambda B F_0 - \sum_{n=1}^{2N} \left[J, \Phi_n \otimes \hat{\Phi}_n{}^T\right] F_0 + \sum_{n=1}^{2N} L(\Phi_n \otimes F_n),$$

$$LH_0 - \lambda H_0 = -\lambda \sum_{n=1}^{2N} \Phi_n \otimes F_n - \sum_{n=1}^{2N} \left[J, \Phi_n \otimes \hat{\Phi}_n{}^T\right] F_0 + \sum_{n=1}^{2N} L(\Phi_n \otimes F_n),$$

$$LH_0 - \lambda H_0 = -\lambda \sum_{n=1}^{2N} \Phi_n \otimes F_n - \sum_{n=1}^{2N} J\Phi_n \otimes \hat{\Phi}_n{}^T F_0 + \sum_{n=1}^{2N} \Phi_n \otimes \hat{\Phi}_n{}^T J F_0$$
$$-i \sum_{n=1}^{2N} J\Phi'_n \otimes F_n - i \sum_{n=1}^{2N} J\Phi_n \otimes F'_n - \sum_{n=1}^{2N} V\Phi_n \otimes F_n\,,$$

$$LH_0 - \lambda H_0 = -\lambda \sum_{n=1}^{2N} \Phi_n \otimes F_n - \sum_{n=1}^{2N} J\Phi_n \otimes \hat{\Phi}_n^T F_0 + \sum_{n=1}^{2N} \Phi_n \otimes \hat{\Phi}_n{}^T J F_0$$
$$+ \sum_{n=1}^{2N} \lambda_n \Phi_n \otimes F_n + \sum_{n=1}^{2N} J\Phi_n \otimes \hat{\Phi}^T{}_n F_0\,,$$

$$LH_0 - \lambda H_0 = -\sum_{n=1}^{2N} (\lambda - \lambda_n)\, \Phi_n \otimes F_n + \sum_{n=1}^{2N} \Phi_n \otimes \hat{\Phi}_n{}^T J F_0.$$

According to the Lemma 1, it is easy to see that

$$\frac{\partial H_n}{\partial x} = i\,(\lambda - \lambda_n)\, \hat{\Phi}_n{}^T F_0 - (\lambda - \lambda_n)\, \frac{\partial F_n}{\partial x} = 0 \tag{21}$$

Thus, H_n does not depend on x.

Lemma 2. *Let $F(\lambda, x, t)$ and $G(\lambda, x, t)$ be the Jost solutions of equation (17) for $x \to \infty$ and $x \to -\infty$, respectively. Then the vector functions*

$$\begin{aligned} F_n{}^- &= i \int_{-\infty}^{x} \hat{\Phi}_n^T(x,t) G(\lambda, x, t)\, dx, \\ F_n{}^+ &= -i \int_{x}^{\infty} \hat{\Phi}_n^T(x,t) F(\lambda, x, t)\, dx. \end{aligned} \tag{22}$$

satisfy (18), therefore the matrices

$$\begin{aligned} H_0^+ &= \dot{F} - BF + \sum_{n=1}^{2N} \Phi_n \otimes F_n^+\,, \\ H_0^- &= \dot{G} - BG + \sum_{n=1}^{2N} \Phi_n \otimes F_n^- \end{aligned}\,, \tag{23}$$

are solutions of equation (17).

Proof. Substituting (10) and (22) into (19) and (20), we define H_0^-, H_0^+, H_n^-, H_n^+. According to (21), it is easy to show for $n = 1, 2, \ldots, 2N$ that $H_n^- = H_n^+ = 0$. Therefore, we can conclude that the matrix functions H_0^+ and H_0^- are solutions of equation (17) as $x \to \infty$ and $x \to -\infty$, respectively, that $LH_0^- - \lambda H_0^- = 0$ and $LH_0^+ - \lambda H_0^+ = 0$.

Remark 1. According to (5), (16) and considering the asymptotics (10) and the integral equation (11) for the solutions in (23), we obtain

$$H_0{}^{+} \to -4i\lambda^3 \begin{pmatrix} e^{i\lambda x}I & 0 \\ 0 & e^{-i\lambda x}I \end{pmatrix} J, \quad \text{as } x \to \infty$$

$$H_0{}^{-} \to -4i\lambda^3 \begin{pmatrix} e^{i\lambda x}I & 0 \\ 0 & e^{-i\lambda x}I \end{pmatrix} J, \quad \text{as } x \to -\infty.$$

By the uniqueness of the Jost solutions we get

$$\begin{cases} H_0^{+} = -4i\lambda^3 FJ\ , \\ H_0^{-} = -4i\lambda^3 GJ\ . \end{cases} \tag{24}$$

Lemma 3. *For all* $\lambda \in \mathrm{R}$ *the following equality is valid*

$$\dot{R}(\lambda) = 8i\lambda^3 R(\lambda). \tag{25}$$

Proof. We introduce the following function

$$H = H_0^{-} - H_0^{+} A(\lambda) \tag{26}$$

Substituting (24) into (26), we receive

$$H = -4i\lambda^3 FA(\lambda)J + 4i\lambda^3 FJA(\lambda) = 4i\lambda^3 F\,[J, A(\lambda)] \tag{27}$$

Using the representations (23) in (26), we have

$$H = F\dot{A(\lambda)} + \sum_{n=1}^{2N} \Phi_n \otimes (F_n^{-} - F_n^{+} A(\lambda)).$$

Here,

$$F_n^{-} - F_n^{+} A(\lambda) = i \int_{-\infty}^{+\infty} \hat{\Phi}_n{}^{T} G dx.$$

Since, $\Phi_n(x,t)$ belong to the $L_2(\mathrm{R})$ and by virtue of Lemma 1, we find that

$$\int_{-\infty}^{+\infty} \hat{\Phi}_n{}^{T} G dx = \frac{(\hat{\Phi}_n^{T} JG)}{i(\lambda - \lambda_n)}\Bigg|_{-\infty}^{\infty} = 0.$$

So, we get

$$H = F\dot{A(\lambda)}. \tag{28}$$

Comparing (27) and (28) we find

$$4i\lambda^3\,[J, A(\lambda)] = \dot{A(\lambda)}. \tag{29}$$

Particularly,

$$\dot{A}_1(\lambda) = 0, \quad \dot{A}_2(\lambda) = 8i\lambda^3 A_2(\lambda), \tag{30}$$

$$\dot{A}_3(\lambda) = -8i\lambda^3 A_3(\lambda), \quad \dot{A}_4(\lambda) = 0. \tag{31}$$

According to $R(\lambda) = -A_1^{-1}(\lambda)A_2(\lambda)$ and the assumptions on the block matrices of $A_1(\lambda)$ $(A_4(\lambda))$ we can find

$$A_1(\lambda)\cdot R(\lambda) = -A_2(\lambda)$$

Taking the derivative respect to t we obtain

$$A_1(\lambda)\dot{R}(\lambda) = -8i\lambda^3 A_2(\lambda)$$

and we find that $\dot{R}(\lambda) = -8i\lambda^3 A_1^{-1}(\lambda)A_2(\lambda)$, which is (25).

Corollary 1. *Since, $A_1(\lambda)$ and $A_4(\lambda)$ do not depend on t, their determinants* $\det A_1(\lambda)$ *and* $\det A_4(\lambda)$ *, its zeros λ_j, $j = 1, 2, \dots 2N$ also do not depend on t.*

Lemma 4. *The matrix functions $R_j(t)$, $j = 1, 2, \dots 2N$ satisfy the following equations*

$$\frac{dR_j(t)}{dt} = (8i\lambda_j^3 + a_j^2(t))R_j(t). \tag{32}$$

Proof. For λ_j $(Im\lambda_j > 0)$, $j = 1, 2, \dots N$ we denote

$$\begin{aligned} h_0^-(\lambda_j, x, t) &= \dot{\bar{\phi}} - B\bar{\phi} + \sum_{n=1}^{2N} \Phi_n \otimes f_n^-, \\ h_0^+(\lambda_j, x, t) &= \dot{\bar{\psi}} - B\bar{\psi} + \sum_{n=1}^{2N} \Phi_n \otimes f_n^+, \end{aligned} \tag{33}$$

where

$$\begin{aligned} f_n^-(\lambda_j, x, t) &= i\int_{-\infty}^{x} \hat{\Phi}_n^T(x,t)\bar{\phi}(\lambda_j, x, t)\,dx, \\ f_n^+(\lambda_j, x, t) &= -i\int_{-\infty}^{x} \hat{\Phi}_n^T(x,t)\bar{\psi}(\lambda_j, x, t)\,dx. \end{aligned} \tag{34}$$

We now introduce the following matrix function

$$h_j = h_0^-(\lambda_j, x, t)N_j - i h_0^+(\lambda_j, x, t)R_j(t), \quad j = 1, 2, \dots, N. \tag{35}$$

And we know that

$$\Phi_j(x,t) = \bar{\psi}(\lambda_j, x, t)c_j(t), \quad j = 1, 2, \dots, N. \tag{36}$$

Here, $c_j(t)$ are non-zero column vectors.

Using (33) we can rewrite (35) as

$$h_j = \dot{\bar{\phi}}(\lambda_j, x, t)N_j - B\bar{\phi}(\lambda_j, x, t)N_j - i\dot{\bar{\psi}}(\lambda_j, x, t)R_j(t)$$

$$+iB\bar{\psi}(\lambda_j,x,t)R_j(t)+\sum_{n=1}^{2N}\Phi_n\otimes(f_n^- N_j - i f_n^+ R_j(t)), \quad j=1,2,\dots,N. \tag{37}$$

Differentiating (13) with respect to t and taking account of the independence of N_j from t we obtain

$$\dot{\bar{\phi}}(\lambda_j,x,t)N_j = i\dot{\bar{\psi}}(\lambda_j,x,t)R_j(t)+i\bar{\psi}(\lambda_j,x,t)\dot{R}_j(t), \quad j=1,2,\dots,N. \tag{38}$$

Substituting (34) and (38) into (37) we get

$$h_j = i\bar{\psi}(\lambda_j,x,t)\dot{R}_j(t) - i\sum_{n=1}^{2N}\Phi_n\otimes\int_{-\infty}^{\infty}\hat{\Phi}_n^T(x,t)\bar{\psi}(\lambda_j,x,t)\,dx\,R_j(t), \quad j=1,2,\dots,N. \tag{39}$$

If $n\neq j$, according to Lemma 1 we get

$$\int_{-\infty}^{\infty}\hat{\Phi}_n(x,t)\bar{\psi}(\lambda_j,x,t)\,dx=0.$$

In the case of $n=j$, we receive

$$h_j = i\bar{\psi}(\lambda_j,x,t)\dot{R}_j(t) - i\Phi_j\otimes\int_{-\infty}^{\infty}\hat{\Phi}_j^T(x,t)\bar{\psi}(\lambda_j,x,t)\,dx\,R_j(t), \quad j=1,2,\dots,N. \tag{40}$$

In the second term of the (40) using (36) we have

$$\begin{aligned}\Phi_j\otimes\int_{-\infty}^{\infty}\hat{\Phi}_j^T(x,t)\bar{\psi}(\lambda_j,x,t)\,dx &= \bar{\psi}(\lambda_j,x,t)c_j(t)\otimes c_j^T(t)\int_{-\infty}^{\infty}\bar{\psi}^T(\lambda_j,x,t)\sigma\bar{\psi}(\lambda_j,x,t)\,dx\\ &= \bar{\psi}(\lambda_j,x,t)P_j(t)\end{aligned}$$

Denoting $P_j(t)$

$$P_j(t)=c_j(t)\otimes c_j^T(t)\int_{-\infty}^{\infty}\bar{\psi}^T(\lambda_j,x,t)\sigma\bar{\psi}(\lambda_j,x,t)\,dx, \tag{41}$$

we write (40) in the following form

$$h_j = i\bar{\psi}(\lambda_j,x,t)\dot{R}_j(t) - i\bar{\psi}(\lambda_j,x,t)P_j(t)R_j(t),\ \ j=1,2,\dots,N. \tag{42}$$

Using the normalization condition we do the calculation $P_j^2(t)$

$$\begin{aligned}P_j^2(t) &= c_j(t)\otimes(c_j^T(t)\int_{-\infty}^{\infty}\bar{\psi}^T(\lambda_j,x,t)\sigma\bar{\psi}(\lambda_j,x,t)\,dx\,c_j(t))\\ &\quad\otimes c_j^T(t)\int_{-\infty}^{\infty}\bar{\psi}^T(\lambda_j,x,t)\sigma\bar{\psi}(\lambda_j,x,t)\,dx\\ P_j^2(t) &= a_j^2(t)P_j(t)\end{aligned} \tag{43}$$

Via the work of [24] the following equality can be taken

$$R_j(t)=P_j(t)(P_j(t)\int_{-\infty}^{\infty}\bar{\psi}^T(\lambda_j,x,t)\sigma\bar{\psi}(\lambda_j,x,t)\,dx\,P_j(t)+(I-P_j(t)))^{-1} \tag{44}$$

By (43) and (44) we have

$$P_j(t)R_j(t) = a_j^2(t)R_j(t) \tag{45}$$

Substituting (45) into (42) we obtain

$$h_j = i\bar{\psi}(\lambda_j, x, t)\dot{R}_j(t) - i\bar{\psi}(\lambda_j, x, t)a_j^2(t)R_j(t), \ \ j = 1, 2, \ldots, N. \tag{46}$$

According (13) and (24) we get

$$h_j = -8\lambda_j^3\bar{\psi}(\lambda_j, x, t)R_j(t) \tag{47}$$

Comparing (46) and (47) we arrive at (32) for $\lambda_j \ (Im\lambda_j > 0), \ \ j = 1, 2, \ldots, N$. Analogically, if we do this process for $\lambda_j \ (Im\lambda_j < 0), \ \ j = N+1, N+2, \ldots, 2N$ we obtain (32).

Thus, we have proved the following theorem.

Theorem 2. *If the matrix function $U(x,t)$, $\Phi_n = \Phi(\lambda_n, x, t)$, $n = 1, 2, \ldots, 2N$ form a solution of the problem* (1)-(4)*, then the scattering data for the operator*

$$L(t) = -iJ\frac{d}{dx} - V(x,t),$$

acting on the orthogonal direct sum of $2m$ copies of $L^2(\mathrm{R})$ satisfy the relations

$$\frac{d\lambda_j}{dt} = 0, j = 1, \ 2, \ \ldots \ 2N,$$

$$\dot{R}(\lambda) = 8i\lambda^3 R(\lambda), \lambda \in \mathrm{R},$$

$$\frac{dR_j(t)}{dt} = (8i\lambda_j^3 + a_j^2(t))R_j(t), j = 1, \ 2, \ \ldots \ 2N.$$

The obtained relations completely specify the evolution of the scattering data for $L(t)$ and this allows using the inverse scattering method to find solutions of the problem (1)-(4).

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