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AN INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATORS WITH SINGULAR POTENTIALS ON ARBITRARY COMPACT GRAPHS

S. V. VASILIEV

Abstract. Sturm-Liouville differential operators with singular potentials on arbitrary compact graphs are studied. The uniqueness of recovering operators from Weyl functions is proved and a constructive procedure for the solution of this class of inverse problems is provided.

1. Introduction

The paper is devoted to the theory of inverse spectral problems for differential operators on geometrical graphs. The inverse problem consists in recovering the potential from the given spectral characteristics. Differential operators on graphs are intensively studied by mathematicians in recent years and have applications in different branches of science and engineering. The inverse problem for the classical Sturm-Liouville operator on the interval has been studied comprehensively in the papers [3] - [6]. The case of inverse problem for Sturm-Liouville operators with potentials from the class W_2^{-1}, which we call the singular potentials, on an interval was extensively studied in [7]-[9]. The inverse problems for the classical Strum-Liouville operator on the graphs were investigated in many papers [15]-[21]. The main result for such operators was obtain in [21], where the arbitrary graph has been considered. The case of inverse problem for Sturm-Liouville operators with singular potentials on graphs is more difficult for investigating, and nowadays there are only a few papers in this area. The inverse problem on star-type graph with such type of potentials has been studied in [33]. Also, some specific types of graph has been considered in papers [30]-[32]. The inverse spectral problem for Sturm-Liouville operators with singular potentials on arbitrary graph has not been studied yet. In this paper we consider the solution of the inverse spectral problem for Sturm-Liouville differential operators with singular potentials on compact arbitrary graphs. As the spectral characteristic we consider the Weyl functions, as itîŰÿ done in [33]. A constructive procedure

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for the solution of the inverse problem from the given spectrums are provided. We develop the ideas of the method of spectral mappings [6] for studying this inverse problem.

Let G be a metric graph with a set of vertices $V(G)$ and a set of edges $E(G)$. We assume that all edges are the smooth curves which can intersect only in the vertices. Let l_e be the length of the edge $e \in E(G)$. We consider each edge e as a segment $[0, l_e]$ and parameterize it by the parameter $x_e \in [0, l_e]$. Let μ be the map, which assigns to each edge an order pair of vertices $e^{\pm} \in V(G)$: $\mu(e) = [e^-,\ e^+]$, where e^- and e^+ are *initial* and *terminal* vertices of e respectively. It is convenient for us to choose the orientation such that $x_e = l_e$ corresponds to the vertex e^- and $x_e = 0$ corresponds to the vertex e^+. For every vertex v, we denote by $I(v, G)$ the set of all the edges incidental to v. The number of elements in $I(v, G)$ is called the valency of vertex v and is denoted by $val(v)$. The vertex v is called a *boundary vertex*, if $val(v) = 1$. All other vertices are called *internal*. Let $V^B(G)$ be a set of boundary vertices and $V^I(G)$ be a set of internal vertices. The edge e is called a *boundary edge*, if $e^+ \in V^B(G)$ or $e^- \in V^B(G)$. All other edges are called *internal*. Let $E^B(G)$ be a set of boundary edges and $E^I(G)$ be a set of internal edges.

A chain of edges $\{e_1, \dots, e_n\}$ is called a *cycle* if it forms a closed curve. The edge is called *simple*, if it is not a part of any cycle. The set of simple edges is denoted by $E^S(G)$. Let $E^C(G)$ be the set of edges, which form the set of cycles. For definiteness, we suppose, that set $V^B(G)$ is not empty and contain at least two vertices. Fix some boundary vertex v_0 and call it the *root*. The corresponding edge $r_0 \in I(v_0, G)$ is called the *rooted edge*. We agree, that if $e \in E^S(G)$, than e^- is nearer to the root, than e^+. For each edge $e \in E^B(G)$ we define

$$\mu_B(e, G) := \begin{cases} e^+, & \text{if } e^+ \in V^B(G), \\ e^-, & \text{if } e^- \in V^B(G). \end{cases}$$

If we contract each cycle to a point, then we obtain a new graph G^*, such that $E(G^*) = E^S(G)$. Clearly, G^* is a tree. Fix some $e \in G^*$. The minimal number χ_e of edges on the G^* between rooted edge and edge e, including e, we called the *order* of edge e. The order of the rooted edge r_0 is equal to zero. The number

$$\chi := \max_{e \in E(G^*)} \chi_e$$

is called the *order* of the G^*. The set of edges of order ν is denoted by $E^{(\nu)}$, $\nu = \overline{0, \chi}$.

A function y on graph G is considered as $y = [y_e(x_e)]_{e \in E(G)}$, $x_e \in [0, l_e]$. Let $q = [q_e(x_e)]_{e \in E(G)}$ be a real-valued function on G such that $q_e \in W_2^{-1}[0, l_e]$, i.e. $q_e(x_e) = \sigma'_e(x_e)$, $\sigma_e(x_e) \in L_2[0, l_e]$, where the derivative is considered in the sense of distributions. Function $\sigma = [\sigma_e(x_e)]_{e \in E(G)}$ we call the *potential*. The Sturm-Liouville differential operator on the edges $e \in E(G)$ is defined by the following expression:

$$\ell_e y_e := -\left(y_e^{[1]}\right)' - \sigma_e(x_e) y_e^{[1]} - \sigma_e^2(x_e) y_e,$$

where $y_e^{[1]} := y_e' - \sigma_e(x_e) y_e$ - is a quasi-derivative, and

$$dom(\ell_e) = \{y_e \mid y_e \in W_2^1[0, l_e],\ y_e^{[1]} \in W_1^1[0, l_e],\ \ell_y e_e \in L_2[0, l_e]\}.$$

We consider the Sturm-Liouville equation on $e \in E(G)$:

$$(\ell_e y_e)(x_e) = \lambda y_e(x_e),\ x_e \in (0, l_e),\ y_e \in dom(\ell_e). \tag{1}$$

At internal vertex v we consider the following matching conditions $MC(v, G)$:

$$y_e|_v = y_r|_v,\ e, r \in I(v, G), \qquad \sum_{e \in I(v,G)} \partial_e y_e|_v = 0, \tag{2}$$

where

$$y_e|_v := \begin{cases} y_e(0), & v = e^+ \\ y_e(l_e), & v = e^- \end{cases}, \quad \partial_e y_e|_v := \begin{cases} y_e^{[1]}(0), & v = e^+ \\ -y_e^{[1]}(l_e), & v = e^- \end{cases}$$

We denote by $MC(G)$ the matching conditions $MC(v, G)$, $v \in V^I(G)$.

Fix some $k \in E^C(G)$. In the case $k^+ \neq k^-$ at internal vertex $u = k^+$ we consider the following matching conditions $MC_k(u, G)$:

$$y_e|_u = y_r|_u, \quad e, r \in I(u, G) \setminus \{k\}, \sum_{e \in I(u,G) \setminus \{k\}} \partial_e y_e|_u = 0. \tag{3}$$

In the case $|I(u, G) \setminus \{k\}| = 1$ this matching condition has the form $\partial_e y_e|_u = 0,\ e \in I(u, G) \setminus \{k\}$.

In the case $k^+ = k^-$ at internal vertex $u = k^+$ we consider the following matching conditions $MC_k(u, G)$:

$$y_k|_{k^-} = y_e|_u,\ y_e|_u = y_r|_u,\ e, r \in I(u, G) \setminus \{k\}, \sum_{e \in I(u,G) \setminus \{k\}} \partial_e y_e|_u + \partial_k y_k|_{k^-} = 0. \tag{4}$$

We denote by $MC_k(G)$ the matching conditions $MC(v, G)$, $v \in V^I(G) \setminus \{k^+\}$ and the matching condition $MC_k(k^+, G)$.

Fix some $r \in E^B(G) \setminus \{r_0\}$. Let us consider the solution φ_{er} of the equation (1) on edge $e \in E(G)$, satisfying $MC(G)$ and the boundary conditions:

$$\partial_e \varphi_{er}|_{\mu_B(e,G)} = \delta_{er}, \quad e \in E^B(G), \tag{5}$$

where δ_{er} is the Kronecker delta. The function $M_r(\lambda, G) := \varphi_{rr}|_{\mu_B(r,G)}$, $r \in E^B(G) \setminus \{r_0\}$ we call the *Weyl function* for (1) with respect to the edge $r \in E^B(G) \setminus \{r_0\}$.

Fix some $k \in E^C(G)$. Let us consider the solution φ_{ek} of the equation (1) on edge $e \in E(G)$, satisfying $MC_k(G)$ and the boundary conditions:

$$\partial_e \varphi_{ek}|_{\mu_B(e,G)} = 0, \quad \partial_k \varphi_{kk}|_{k^+} = 1, \quad e \in E^B(G) \tag{6}$$

The function $M_k(\lambda, G) := \varphi_{kk}|_{k^+}$, $k \in E^C(G)$ we call the *Weyl function* for (1) with respect to the edge $k \in E^C(G)$. Denote $M(\lambda, G) = [M_e(\lambda, G)]_{e \in E^B(G)\setminus\{r_0\}\cup E^C(G)}$. The vector $M(\lambda, G)$ is called the *Weyl vector*. The inverse problem is formulated as follows.

Inverse problem 1. Given the Weyl vector $M(\lambda, G)$, construct the potential σ.

Everywhere below if a symbol α denotes an object, related to σ, then $\tilde{\alpha}$ will denote the analogous object, related to $\tilde{\sigma}$ and $\hat{\alpha} = \alpha - \tilde{\alpha}$. Let us formulate the uniqueness theorem for the solution of Inverse Problem 1.

Theorem 1. *If $M(\lambda, G) = \widetilde{M}(\lambda, G)$, then $\sigma = \tilde{\sigma}$. Thus, the specification of the Weyl vector $M(\lambda, G)$ uniquely determines the potential σ on G.*

The paper is structured as follows. Section 2 contains some auxiliary propositions. Section 3 is devoted to the solution the so-called auxiliary inverse problems. In the section 4 we prove Theorem 1, provide the descent procedure and the solution of the global inverse problem on the graph.

2. Auxiliary propositions

Let us consider the boundary value problem $L_\Omega(G)$, $\Omega \subset E^B(G)$, for equation (1) with the matching conditions $MC(G)$ and boundary conditions

$$\partial_e y_e|_{\mu_B(e,G)} = 0, \quad e \in E^B(G)\setminus\{\Omega\}, \quad y_r|_{\mu_B(r,G)} = 0, \quad r \in \Omega. \tag{7}$$

We define $L(G) := L_\varnothing(G)$. Also we consider the boundary value problem $L_k^\nu(G)$, $\nu = 0, 1$, $k \in E^C(G)$ for equation (1) with the matching conditions $MC_k(G)$ and boundary conditions

$$\partial_e y_e|_{\mu_B(e,G)} = 0, \quad e \in E^B(G), \quad \partial_k^\nu y_k|_{k^+} = 0, \tag{8}$$

where $\partial_k^0 y_k|_{k^+} = y_k|_{k^+}$ and $\partial_k^1 y_k|_{k^+} = \partial_k y_k|_{k^+}$.

Let $C_e(x_e, \lambda)$, $S_e(x_e, \lambda)$, $\psi_e(x_e, \lambda)$ and $\zeta_e(x_e, \lambda)$ be the solutions of equation (1) on the edge $e \in E(G)$ under initial conditions

$$\begin{gathered} C_e(0, \lambda) = S_e^{[1]}(0, \lambda) = 1, \;\; C_e^{[1]}(0, \lambda) = S_e(0, \lambda) = 0, \\ \zeta_e(l_e, \lambda) = -\psi_e^{[1]}(l_e, \lambda) = 1, \quad \psi_e(l_e, \lambda) = \zeta_e^{[1]}(l_e, \lambda) = 0. \end{gathered} \tag{9}$$

For each fixed $x_e \in [0, l_e]$ the functions $C_e(x_e, \lambda)$, $S_e(x_e, \lambda)$, $C_e^{[1]}(x_e, \lambda)$, $S_e^{[1]}(x_e, \lambda)$ and $\zeta_e(x_e, \lambda)$, $\psi_e(x_e, \lambda)$, $\zeta_e^{[1]}(x_e, \lambda)$, $\psi_e^{[1]}(x_e, \lambda)$, $e \in E(G)$ are entire in λ. Moreover,

$$\langle C_e(x_e, \lambda), S_e(x_e, \lambda)\rangle = 1, \quad \langle \zeta_e(x_e, \lambda), \psi_e(x_e, \lambda)\rangle = -1,$$

where $\langle y, z\rangle := yz^{[1]} - y^{[1]}z$ is the Wronskian of y and z. Let $Y = \{y_e(x_e)\}_{e\in E(G)}$ be a solution of equation (1) on graph G. Then

$$y_e(x_e,\lambda) = M_e^0(\lambda)S_e(x_e,\lambda) + M_e^1(\lambda)C_e(x_e,\lambda). \tag{10}$$

Substituting this representation into matching and boundary conditions of the boundary value problem L in fixed order (analogous to [21]), we obtain a linear algebraic system with respect to $a_e(\lambda)$, $b_e(\lambda)$, $e \in E(G)$. The determinant $\Delta(\lambda, L)$ of this system is an entire function. The zeros of $\Delta(\lambda, L)$ coincide with the eigenvalues of L. The function $\Delta(\lambda, L)$ is called *characteristic function* for boundary value problems L. We denote by $\Lambda_\Omega := \{\lambda_{\Omega n}\}_{n\geq 0}$ the zeros of $\Delta(\lambda, L_\Omega(G))$ and define $\lambda_{\Omega n} = \rho^2_{\Omega n}$, $n \geq 0$.

As in the classical case [21] one can show that the functions $M_e(\lambda)$, $e \in E^B(G)\setminus\{r_0\}\cup E^C(G)$ are meromorphic in λ, namely:

$$M_e(\lambda) = -\frac{\Delta(\lambda, L_e(G))}{\Delta(\lambda, L(G))},\ e \in E^B(G)\setminus\{r_0\}, \quad M_r(\lambda) = -\frac{\Delta(\lambda, L_r^0(G))}{\Delta(\lambda, L_r^1(G))},\ r \in E^C(G) \tag{11}$$

Let H_e, $e \in E(G)$, be the classes of functions, which are entire in ρ for all $x \in [0, l_e]$ and fixed potential σ_e, such that for $\eta_e(x_e,\rho,\sigma_e) \in H_e$ following conditions are valid:

1) $\eta_e(x_e,\rho,\sigma_e) = o(exp(x_e|Im\rho|))$ for $\rho \to \infty$ and any fixed $x_e \in [0, l_e]$ and $\sigma_e \in L_2[0, l_e]$.
2) $\eta_e(x_e,\cdot,\sigma_e) \in L_2(\gamma)$ for all $x_e \in [0, l_e]$, real τ and fixed $\sigma_e \in L_2[0, l_e]$, where

$$\gamma = \gamma(\tau) := (-\infty + i\tau, +\infty + i\tau).$$

3) $\eta_e(\cdot,\cdot,\sigma_e) \in L_2[0, l_e] \times \gamma$ and bounded uniformly on $[0, l_e] \times \gamma$ for any fixed real τ and $\sigma_e \in L_2[0, l_e]$.
4) $\eta_e(x_e,\rho,\sigma_e)$ depends continuously on the potential in the following sense: if $\sigma_{en}(x_e) \to \sigma_e(x_e)$ in $L_2[0, l_e]$ as $n \to \infty$, then the corresponding $\eta_e(x_e,\rho,\sigma_{en}) \in H_e$ converges to $\eta_e(x_e,\rho,\sigma_e) \in H_e$ uniformly as $n \to \infty$ on $[0, l_e] \times \gamma$ for all $\tau > \tau_0$ and

$$\max_{x_e\in[0,l_e]} ||\eta_e(x_e,\cdot,\sigma_{en}) - \eta_e(x_e,\cdot,\sigma_e)||_{L_2(\gamma)} \to 0.$$

Obviously, if $\eta_e(x_e,\rho,\sigma_e)$, $\eta_e^*(x_e,\rho,\sigma_e) \in H_e$, than $\eta_e(x_e,\rho,\sigma_e) + \eta_e^*(x_e,\rho,\sigma_e) \in H_e$.

Define $A_\varepsilon(\tau_0) := \{\rho : Im\rho \geq 0, dist(\rho, Z) > \varepsilon\}$, where $Z \subset \{\rho : 0 \leq Im\rho \leq \tau_0\}$ is countable set with constrained number of points in $Re\rho \in [t, t+1], Im\rho \in [0, \tau_0]$. Let K be the class of meromorphic functions, such that for $\kappa(\rho,\sigma) \in K$ following conditions are valid:

1) $\kappa(\rho,\sigma) = o(1)$ for $\rho \to \infty$ and fixed $\sigma \in L_2(G)$, $\rho \in A_\varepsilon(\tau_0)$, where τ_0 depends on κ.
2) $\kappa(\cdot,\sigma) \in L_2(\gamma)$ for all $\tau > \tau_0$ and fixed $\sigma \in L_2(G)$.

3) $\kappa(\rho,\sigma)$ depends continuously on σ, in the following sense: if $\sigma_n(x) \to \sigma(x)$ in $L_2(G)$ as $n\to\infty$, then $\kappa(\rho,\sigma_n)\in K$ converges to $\kappa(\rho,\sigma)\in K$ uniformly as $n\to\infty$ on γ for all $\tau>\tau_0$ and

$$\lim_{n\to\infty} ||\kappa(\cdot,\sigma_n)-\kappa(\cdot,\sigma)||_{L_2(\gamma)} \to 0.$$

Obviously, if $\kappa(\rho,\sigma)$, $\kappa^*(\rho,\sigma)\in K$, than $\kappa(\rho,\sigma)+\kappa^*(\rho,\sigma)\in K$ and $\kappa(\rho,\sigma)\kappa^*(\rho,\sigma)\in K$. Define $[1] := 1+\kappa(\rho)$, $\kappa(\rho)\in K$. We consider the solutions of equation (1):

$$\xi_e(x_e,\lambda) := C_e(x_e,\lambda) - i\rho S_e(x_e,\lambda), \quad E_e(x_e,\lambda) := \zeta_e(x,\lambda) - i\rho\psi_e(x,\lambda), \quad e\in E(G)$$

Let $\lambda=\rho^2$, $Im\rho\ge 0$. Analogous to [33], we obtain

$$\begin{gathered}\xi_e(l_e,\lambda) = e^{-i\rho l_e}[1], \quad E_e(0,\lambda) = e^{-i\rho l_e}[1], \\ \xi_e^{[1]}(l_e,\lambda) = -i\rho e^{-i\rho l_e}[1], \quad E_e^{[1]}(0,\lambda) = i\rho e^{-i\rho l_e}[1],\end{gathered} \tag{12}$$

Using Liouville's formula and (12), we obtain $\langle \xi_e, E_e\rangle = 2i\rho e^{-i\rho l_e}[1]$. Clearly, that $\langle\xi_e,E_e\rangle \not\equiv 0$ and consequently $\{\xi_e(x_e,\lambda), E_e(x_e,\lambda)\}$ is a fundamental system of solutions. We consider the case of some fixed $r\in E^B(G)$. Then

$$\varphi_{er}(x_e,\lambda) = A_{er}(\lambda)\xi_e(x_e,\lambda) + B_{er}(\lambda)E_e(x_e,\lambda), \quad e\in E(G). \tag{13}$$

Substituting (13) into (2) and (5) in fixed order (analogous to [21]), we obtain the system of linear equations with variables $A_{er}(\lambda)$ and $B_{er}(\lambda)$. Determinant of this system we define as $\Delta_E(\lambda,G)$.

Lemma 1. *Define* $\Theta := \{\sum_{e\in E(G)} \theta_e l_e,\ \theta_e\in\{0,1,2\}\}$. *The following representation is valid:*

$$\Delta_E(\lambda,G) = (i\rho)^n \sum_{l\in\Theta} A_l(G) e^{-i\rho l}[1], \quad A_{|G|}\neq 0, \tag{14}$$

where $n := |V^I(G)|+|V^B(G)|$, $|G| := 2\sum_{e\in E(G)} l_e$.

Proof. In the each internal vertex $u\in V^I(G)$ we define a variable α_u and we define the indicator function

$$J_s^{\pm}(u) := \begin{cases} 1, & s^{\pm}=u, \\ 0, & s^{\pm}\neq u. \end{cases} \tag{15}$$

Substitute (13) into (2) and (5), we obtain

$$\begin{gathered}\xi_e(l_e,\lambda)A_{er}(\lambda) + B_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^-(v)\alpha_v = 0, \\ A_{er}(\lambda) + E_e(0,\lambda)B_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^+(v)\alpha_v = 0, \\ \sum_{e\in E(G)}\Big\{J_e^+(u)\Big[E_e^{[1]}(0,\lambda)B_{er}(\lambda) - i\rho A_{er}(\lambda)\Big] - J_e^-(u)\Big[\xi_e^{[1]}(l_e,\lambda)A_{er}(\lambda) + i\rho B_{er}(\lambda)\Big]\Big\} = 0,\end{gathered} \tag{16}$$

for $e \in E(G)$ and $u \in V^I(G)$ and boundary conditions for $k \in E^B(G)$

$$\begin{aligned} -i\rho A_{kr}(\lambda) + E_k^{[1]}(0,\lambda)B_{kr}(\lambda) = \delta_{kr}, \quad \text{if } k^+ \in V^B(G), \\ \xi_k^{[1]}(l_k,\lambda)A_{kr}(\lambda) + i\rho B_{kr}(\lambda) = \delta_{kr}, \quad \text{if } k^- \in V^B(G), \end{aligned} \tag{17}$$

Using (16), (17) and (12), we obtain

$$\Delta_E(\lambda, G) = (i\rho)^n \sum_{l\in\Theta} A_l(G) e^{-i\rho l}[1],$$

where $A_{|G|}(G)$ is the determinant of system

$$A_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^-(v)\alpha_v = 0, \quad e \in E(G), \tag{18}$$

$$B_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^+(v)\alpha_v = 0, \quad e \in E(G), \tag{19}$$

$$\sum_{e\in E(G)} \Big(A_{er}(\lambda)J_e^-(u) + B_{er}(\lambda)J_e^+(u)\Big) = 0, \quad u \in V^I(G), \tag{20}$$

We denote by P the matrix of this system (18)-(20) and by P_e^1, P_e^2 and P_u the rows of the matrix P, which correspond to the equation (18), (19) and (20) respectively for $e \in E(G)$ and $u \in V^I(G)$. Transform the matrix P by following equation

$$P_u = P_u - \sum_{e\in E(G)} J_e^-(u)P_e^1 - \sum_{e\in E(G)} J_e^+(u)P_e^2, \quad u \in V^I(G),$$

we obtain, that $A_{|G|}(G)$ is the determinant of system

$$\begin{aligned} A_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^-(v)\alpha_v = 0, \\ B_{er}(\lambda) - \sum_{v\in V^I(G)} J_e^+(v)\alpha_v = 0, \\ \sum_{e\in E(G)} \Big(J_e^-(u) + J_e^+(u)\Big)\alpha_u = 0, \ u \in V^I(G) \end{aligned} \tag{21}$$

In each vertex $u \in V^I(G)$ we denote

$$N^\pm(u) := \sum_{e\in E(G)} J_e^\pm(u).$$

Graph G is connected, thus $N^+(u) + N^-(u) \neq 0$ for each $u \in V^I(G)$. Consequently, using (21), we obtain

$$A_{|G|} = \prod_{u\in V^I(G)} (N^+(u) + N^-(u)) \neq 0,$$

□

Using standard methods [27], we obtain

Lemma 2. *For sufficiently large* $|\rho|$, *such that* $\rho \in A_\varepsilon(\tau_0)$, τ_0 *is some fix number, following inequality is valid*

$$C_1|\rho|^n e^{|G|Im\rho} < |\Delta_E(\lambda, G)| < C_2|\rho|^n e^{|G|Im\rho}. \tag{22}$$

Consequently, we obtain for fixed $r \in E^B(G)$:

Lemma 3. *For each fixed* $x_e \in [0, l_e]$ *and for* $\rho \in A_\varepsilon(\tau_0)$, τ_0 *is some fix number,* $\rho \to \infty$, *following representations are valid*

$$\begin{gathered} \varphi_{er}(x_e, \lambda) = O\Big(\frac{1}{\rho} e^{-x_e Im\rho}\Big), \quad \varphi_{er}^{[1]}(x_e, \lambda) = O\Big(e^{-x_e Im\rho}\Big), \\ \hat{\varphi}_{er}(x_e, \lambda) = \frac{1}{\rho} e^{i\rho x_e} \hat{\kappa}(\rho), \quad \kappa(\rho), \widetilde{\kappa}(\rho) \in K. \end{gathered} \tag{23}$$

Proof. Analogous to [21], using (22) and Cramer's rule, we obtain

$$\begin{gathered} A_{er} = O\Big(\frac{1}{\rho} e^{-2l_e Im\rho}\Big), \quad B_{er} = O\Big(\frac{1}{\rho} e^{-l_e Im\rho}\Big)., \\ \hat{A}_e(\lambda) = \frac{1}{\rho} e^{2i\rho l_e} \hat{\kappa}(\rho),\ \hat{B}_e(\lambda) = \frac{1}{\rho} e^{i\rho l_e} \hat{\kappa}(\rho), \quad \kappa(\rho), \widetilde{\kappa}(\rho) \in K. \end{gathered} \tag{24}$$

Together with (12) and (13), this yields for each fixed $x_e \in [0, l_e]$ to (23). □

Using the same arguments, one can prove, that Lemma 3 is valid also for some fixed $r \in E^C(G)$.

3. Auxiliary inverse problem

Fix some edge $e \in E^B(G) \backslash \{r_0\}$ and consider the following auxiliary inverse problem on the edge e, which is called $IP(e, G)$:

Auxiliary inverse problem $IP(e, G)$. Given $M_e(\lambda, G)$, construct $\sigma_e(x_e)$, $x_e \in [0, l_e]$.

Using properties of functions from class K, analogous to [33] one can prove following theorem:

Theorem 2. *If* $M_e(\lambda, G) \equiv \widetilde{M}_e(\lambda, G)$, *then* $\sigma_e(x_e) \equiv \widetilde{\sigma}_e(x_e)$ *almost everywhere on* $[0, l_e]$.

In the ρ-plane consider the contour $\gamma = \gamma(\tau)$, where $\tau > 0$ is such that $\inf\{\Lambda_e \cup \widetilde{\Lambda}_e\} > -\tau^2$. Let Γ be the contour in the λ-plane which is the image of γ under the mapping $\lambda = \rho^2$. Denote by D^+ the image of the half-plane $\{Im\rho > \tau\}$ and $D^- := C \backslash D^+$. We define the sequence of real numbers δ_n, $n \in \mathbb{N}$, such that $\forall n, k \in \mathbb{N}$ $\delta_n \neq |\rho_{ek}|$, $\delta_n \neq |\widetilde{\rho}_{ek}|$, $\delta_k < \delta_{k+1}$, and $\delta_n \to \infty$ as $n \to \infty$. Let $C_N := \{|\lambda| = \delta_N^2\}$, $\gamma_N = \gamma \cap \{\rho : |\rho|^2 = \delta_N^2\}$, where $C_N^- := C_N \cap D^-$ be the contours with clockwise orientation. Denote $\Gamma_N = \Gamma \cap intC_N$, $\quad \Gamma_N^- = \Gamma_N \cup C_N^-$. Denote $\theta^2 = \mu$. Define the functions

$$D_e(x_e, \lambda, \mu) := \frac{\langle C_e(x_e, \lambda), C_e(x_e, \mu)\rangle}{\lambda - \mu} = \int_0^{x_e} C_e(t, \lambda) C_e(t, \mu)\, dt,$$

$$\widetilde{D}_e(x_e,\lambda,\mu) := \frac{\langle \widetilde{C}_e(x_e,\lambda), \widetilde{C}_e(x_e,\mu)\rangle}{\lambda-\mu} = \int_0^{x_e} \widetilde{C}_e(t,\lambda)\widetilde{C}_e(t,\mu)\,dt,$$
$$r_e(x_e,\rho,\theta) := D_e(x_e,\lambda,\mu)\theta\hat{M}_e(\mu), \quad \widetilde{r}_e(x_e,\rho,\theta) := \widetilde{D}_e(x_e,\lambda,\mu)\theta\hat{M}_e(\mu).$$

Everywhere below we chose contour $\gamma(\tau)$ such that $\theta\hat{M}_e(\mu) \in L_2(\gamma)$. It is always possible to choose such contour because of the properties of functions from class K and (23). Analogous to [33], one can obtain the main equation

$$\Psi_e(x_e,\rho) = \widetilde{U}_e(x_e)\Psi_e(x_e,\rho) + \widetilde{F}_e(x_e,\rho), \tag{25}$$

where $\Psi_e(x_e,\rho) := C_e(x_e,\lambda) - \widetilde{C}_e(x_e,\lambda)$,

$$\widetilde{F}_e(x_e,\rho) := \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\Gamma_N} \widetilde{D}_e(x_e,\lambda,\mu)\hat{M}_e(\mu)\widetilde{C}_e(x_e,\mu)\,d\mu, \quad \lambda=\rho^2, \tag{26}$$

and for all fixed $x_e \in [0,l_e]$

$$\widetilde{U}_e(x_e)f(\rho) := \frac{1}{\pi i}\int_\gamma \widetilde{r}_e(x_e,\rho,\theta)f(\theta)\,d\theta, \quad U_e(x_e)f(\rho) := \frac{1}{\pi i}\int_\gamma r_e(x_e,\rho,\theta)f(\theta)\,d\theta$$

Operator $\widetilde{U}_e(x_e)$ is a Hilbert-Schmidt operator in $L_2(\gamma)$. Also from [33] we obtain the validity of the following theorem:

Theorem 3. *For each fixed $x_e \in [0,l_e]$ equation (25) is uniquely solvable in $L_2(\gamma)$.*

Using the solution $\Psi_e(x_e,\rho)$ of the main equation (25), one can calculate the function $C_e(x_e,\lambda)$ and then construct $\sigma_e(x_e)$ according to the next theorem [33].

Theorem 4. *The solution $\sigma_e(x_e)$ of the $IP(e,G)$ can be found by the formula*

$$\sigma_e(x) = -\frac{1}{\pi i}\int_\Gamma \widetilde{C}_e(x_e,\mu)\hat{C}_e(x_e,\mu)\hat{M}_e(\mu)\,d\mu + \frac{1}{\pi i}l.i.m._{N\to\infty}\int_{\gamma_N} \rho\cos 2\rho x_e \hat{M}_e(\rho^2)\,d\rho \tag{27}$$

Thus, the solution of the auxiliary inverse problem $IP(e,G)$ can be constructed by the following algorithm.

Algorithm 1. Given $M_e(\lambda)$.

1) Take $\widetilde{\sigma} = 0$ and calculate $\widetilde{C}_e(x_e,\lambda)$, $\widetilde{M}_e(\lambda)$, $\widetilde{D}_e(x_e,\lambda,\mu)$ and $\widetilde{r}_e(x_e,\rho,\theta)$.
2) Construct $\widetilde{F}_e(x_e,\rho))$ by (26).
3) Find $\Psi_e(x_e,\rho)$ by solving the main equation (25) for each $x \in [0,l_e]$.
4) Construct $\sigma_e(x_e)$ using (27), where $\hat{C}_e(x_e,\lambda) = \Psi_e(x_e,\rho)$.

Fix some edge $k \in E^C(G)$ and consider the following auxiliary inverse problem on the edge k, which is called $IP(k,G)$:

Auxiliary inverse problem $IP(k,G)$. Given $M_k(\lambda,G)$, construct $\sigma_k(x_k)$, $x_k \in [0,l_k]$.

Fix some $k \in E^C(G)$. If we unlink the vertex $k^+ = v \in V^I(G)$ and move it slightly to new vertex $v_1 \notin V(G)$ without moving other edges and without changing the length of k, we tear apart the cycle, which contains the edge k, and obtain a new graph G_k with edge k_1 instead of k and with the new vertex v_1, such that $v_1 \neq v$ and $k_1^+ = v_1$. Moreover, $k_1 \notin I(v,G_k)$ and the edge k_1 is a boundary edge for G_k and $v_1 \in V^B(G_k)$. For example, if $E(G) = \{k, r_0, r_1\}$, $E^C(G) = \{k, r_1\}$, $k^- \neq k^+$, $k^+ = v \in V^I(G)$ and $I(v,G) = \{k, r_1\}$, then after this procedure we obtain graph G_k with edge k_1 instead of k and with the new vertex v_1, such that $k_1^+ = v_1$, $v_1 \neq v$, $I(v_1,G_k) = \{k_1\}$, $I(v,G_k) = \{r_1\}$. Clearly, that in this case $r_1 \in E^I(G)$ and $r_1 \in E^B(G_k)$.

Clearly, inverse problem $IP(k,G)$, $k \in E^C(G)$, is equivalent to the inverse problem $IP(k_1,G_k)$. Therefore, $IP(k,G)$, $k \in E^C(G)$, is solved by the same arguments as $IP(r,G)$, $r \in E^B(G)\setminus\{r_0\}$.

4. Descent procedure. Solution of the inverse problem 1

Consider the solution of the equation (1) on the edges $e \in E(G)$, represented by (10). Let us construct graphs T and Q. Fix the edge $r \in E^S(G) \cap E^I(G) \cup \{r_0\}$. Denote $v := r^+$, $v \in V(G)$. The vertex v divide graph G on two parts: $G = Q \cup T$, where $V(Q) \cap V(T) = v$, $E(T) \cap I(v,G) = r$, $r \in E^B(T)$ and $r \notin E(Q) \cap I(v,G)$. Moreover, the rooted edge $r_0 \in E(T)$.

Consider the boundary value problem $L_\Omega(Q,v)$ for equation (1) with the matching conditions $MC(u,Q)$, $u \in V^I(Q)\setminus\{v\}$ and boundary conditions

$$\partial_e y_e|_{\mu_B(e,G)} = 0, \quad e \in E^B(G)\setminus\{\Omega\}, \quad y_r|_{\mu_B(r,G)} = 0, \quad r \in \Omega, \quad \partial_r y_r|_v = 0, \quad r \in I(v,Q).$$

We define $L_\varnothing(Q,v) = L(Q,v)$. Using the Laplace expansion by the columns, corresponding to $M_e^0(\lambda)$, $M_e^1(\lambda)$, $e \in E(T)$, we obtain two possible cases:

1. The vertex $v \in V^B(Q)$, $I(v,Q) =: e$. Clearly, for $k \in E^B(Q) \cap E^B(G)$

$$\begin{pmatrix} \Delta(\lambda, L_e(Q)) & \Delta(\lambda, L(Q)) \\ \Delta(\lambda, L_{ek}(Q)) & \Delta(\lambda, L_k(Q)) \end{pmatrix} \begin{pmatrix} \Delta(\lambda, L(T)) \\ \Delta(\lambda, L_r(T)) \end{pmatrix} = \begin{pmatrix} \Delta(\lambda, L(G)) \\ \Delta(\lambda, L_k(G)) \end{pmatrix}$$

Using Cramer's rule and (11), we obtain

$$M_r(\lambda,T) = \frac{M_k(\lambda,G)\Delta(\lambda,L_e(Q)) + \Delta(\lambda,L_{ek}(Q))}{\Delta(\lambda,L_k(Q)) + \Delta(\lambda,Q)M_k(\lambda,G)}. \tag{28}$$

2. The vertex $v \in V^I(Q)$. Clearly, for $k \in E^B(Q) \cap E^B(G)$

$$\begin{pmatrix} \Delta(\lambda, L(Q)) & \Delta(\lambda, L(Q,v)) \\ \Delta(\lambda, L_k(Q)) & \Delta(\lambda, L_k(Q,v)) \end{pmatrix} \begin{pmatrix} \Delta(\lambda, L(T)) \\ \Delta(\lambda, L_r(T)) \end{pmatrix} = \begin{pmatrix} \Delta(\lambda, L(G)) \\ \Delta(\lambda, L_k(G)) \end{pmatrix}$$

Analogous to the first case, using Cramer's rule and (11), we obtain

$$M_r(\lambda, T) = \frac{M_k(\lambda, G)\Delta(\lambda, L(Q)) + \Delta(\lambda, L_k(Q))}{\Delta(\lambda, L_k(Q, v)) + \Delta(\lambda, L(Q, v))M_k(\lambda, G)}. \tag{29}$$

Descent procedure. Fix edge $r \in E^S(G) \cap E^I(G) \cup \{r_0\}$ and suppose, that $r \in E^{(\nu)}$ be a fixed simple edge of order ν. Denote $v = r^-$. The vertex v divide graph G on two parts $G = Q \cup T$, where $V(Q) \cap V(T) = v$, $E(T) \cap I(v, G) = r$, $r \in E^B(T)$ and $r \notin E(Q) \cap I(v, G)$. We consider the potential σ on the graph Q are known. Fix $e \in E^B(G) \cap E(Q)$. We consider the $M_e(\lambda, G)$ is known.

1. Weyl functions $M_r(\lambda, T)$ find from (28) or (29).
2. Solving the inverse problem $IP(r, T)$, we construct σ_r on r.

We suppose, that $M_e(\lambda, G)$, $e \in E^C(G) \cup E^B(G) \backslash \{r_0\}$ are known. The solution of the inverse problem can be found by the following algorithm

Algorithm 2.

1. For each fixed edge $k \in E^C(G)$ we solve $IP(k, G)$ by the algorithm 1 and find σ_k on the edge k.
2. For each fixed edge $r \in E^B(G) \backslash \{r_0\}$ we solve $IP(r, G)$ by the algorithm 1 and find σ_r on the edge r.
3. For $\nu = \chi - 1, \ldots, 0$ we perform the following operations: for each fixed edge $r \in E^{(\nu)}(G) \cap \left(E^S(G) \cap E^I(G) \cup \{r_0\}\right)$ using the descent procedure, we find σ_r.

The considerations above show that the solution of Inverse Problem 1 is uniquely determined, and Theorem 1 is proved.

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Department of Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: vasilievsv@info.sgu.ru