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THE PARTIAL INVERSE NODAL PROBLEM FOR DIFFERENTIAL PENCILS ON A FINITE INTERVAL

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Abstract. In this paper, the partial inverse nodal problem for differential pencils with real-valued coefficients on a finite interval $[0,1]$ was studied. The authors showed that the coefficients $(q_0(x), q_1(x), h, H_0)$ of the differential pencil L_0 can be uniquely determined by partial nodal data on the right(or, left) arbitrary subinterval $[a,b]$ of $[0,1]$. Finally, an example was given to verify the validity of the reconstruction algorithm for this inverse nodal problem.

1. Introduction

The differential pencil $L_\xi = L(q_0, q_1, h, H_\xi)$:

$$\begin{cases} l y := -y'' + (q_0(x) + 2\lambda q_1(x))y = \lambda^2 y, \quad x \in (0,1), & (1.1) \\ U(y) := y'(0) - hy(0) = 0, & (1.2) \\ V_\xi(y) := y'(1) + H_\xi y(1) = 0 & (1.3) \end{cases}$$

is considered, where $h, H_\xi \in \mathbb{R}$, $H_0 \neq H_1$, $q_\xi(x)$ is a real-valued function, $q_\xi \in W_2^\xi[0,1]$, $\xi = 0,1$. We assume that $q_1(x) \neq$ const and $Q_1(1) = 0$, where

$$Q_1(x) := \int_0^x q_1(t)\mathrm{d}t.$$

Denote $\tau = |\mathrm{Im}\lambda|$, $\varphi(x,\lambda)$ and $\psi_\xi(x,\lambda)$ the solutions of (1.1) associated with initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h, \quad \psi_\xi(1,\lambda) = 1, \quad \psi'_\xi(1,\lambda) = -H_\xi.$$

According to [6, 8], we have the following asymptotic formulae

$$\begin{cases} \varphi(x,\lambda) = \cos(\lambda x - Q_1(x)) + O\left(\frac{e^{\tau x}}{\lambda}\right), \\ \varphi'(x,\lambda) = -\lambda \sin(\lambda x - Q_1(x)) + O(e^{\tau x}), \end{cases} \tag{1.4}$$

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and

$$\begin{cases} \psi_\xi(x,\lambda) = \cos(\lambda(1-x)+Q_1(x)) + O\left(\frac{e^{\tau(1-x)}}{\lambda}\right), \\ \psi'_\xi(x,\lambda) = \lambda\sin(\lambda(1-x)+Q_1(x)) + O(e^{\tau(1-x)}) \end{cases} \tag{1.5}$$

uniformly with respect to $x \in [0,1]$ for sufficiently large $|\lambda|$. It is easy to obtain the following equality:

$$\int_0^1 (\psi_\xi l(\varphi) - \varphi l(\psi_\xi)) = <\psi_\xi, \varphi>(1,\lambda) - <\psi_\xi,\varphi>(0,\lambda),$$

where $<\psi_\xi,\varphi>(x,\lambda) := \psi_\xi(x,\lambda)\varphi'(x,\lambda) - \psi'_\xi(x,\lambda)\varphi(x,\lambda)$ is the Wronskian of ψ_ξ and φ. Denote $\Delta_\xi(\lambda) :=<\psi_\xi,\varphi>(x,\lambda)$ which is independent of x. The set of eigenvalues $\sigma(L_\xi)$ of L_ξ consists of the zeros of the characteristic function $\Delta_\xi(\lambda)$ which can be enumerated as $\{\lambda_{\xi n}\}_{n\in\mathbf{A}}$(counting with their multiplicities), where $\mathbf{A} = \{\pm 0, \pm 1, \pm 2, \cdots\}$. For sufficiently large $|n|$, $\lambda_{\xi n}$ is real and simple, it also satisfies the following asymptotic formula(please refer to [6, 12] for details)

$$\lambda_{\xi n} = n\pi + \frac{\omega_\xi(1)}{n\pi} + o\left(\frac{1}{n}\right) \tag{1.6}$$

as $|n| \to \infty$, where

$$\omega_\xi(x) := h + H_\xi + \frac{1}{2}\int_0^x (q_0(t) + q_1^2(t))\mathrm{d}t. \tag{1.7}$$

Suppose that $x_{\xi n}^j$ are the nodal points of the eigenfunction $\varphi(x,\lambda_{\xi n})$ of the pencil L_ξ, i.e. $\varphi(x_{\xi n}^j, \lambda_{\xi n}) = 0$. Gasymov and Guseinov [12] showed that the differential pencil L_0 has a discrete spectrum consisting of simple and real eigenvalues with finitely many exceptions, and the n-th eigenfunction $u(x,\lambda_{\xi n})$ has exactly $|n|$ nodes in the interval $(0,1)$ for sufficiently large $|n|$. Then $x_{\xi n}^j$ satisfy the following relations:

$$0 < x_{\xi n}^1 < x_{\xi n}^2 < \cdots < x_{\xi n}^j < \cdots < x_{\xi n}^n < 1, \quad \text{for} \quad n > 0,$$
$$0 < x_{\xi n}^0 < x_{\xi n}^{-1} < \cdots < x_{\xi n}^{j+1} < \cdots < x_{\xi n}^{n+1} < 1, \quad \text{for} \quad n < 0.$$

and

$$x_{\xi n}^j = \frac{j-\frac{1}{2}}{n} + \frac{Q_1(x_{\xi n}^j)}{n\pi} + \frac{1}{(n\pi)^2}(\omega_\xi(x_{\xi n}^j) - \omega_\xi(1)x_{\xi n}^j - H_\xi) + o\left(\frac{1}{n^2}\right) \tag{1.8}$$

uniformly with respect to $j \in \mathbb{Z}$(refer to Theorem 4 of [6] for details). Denote $X_\xi := \{x_{\xi n}^j\}$. Clearly the nodal set X_ξ is dense on $[0,1]$ for $\xi = 0,1$. Buterin [4] firstly showed that $(q_0(x), q_1(x), h, H_0)$ of L_0 can be uniquely determined by the nodal set X_0. Analogous result for the case of Dirichlet boundary conditions was proved in [5]. The related results are also found in [6]. However, in [33] a more weak statement was established. By using the results in [16], Guo and Wei [17] proved an alternative result involving specification of nodal points on arbitrarily small subintervals having the midpoint. Note that the Guo-Wei's result in [17] excludes the case $\frac{1}{2} \notin (a,b)$. For the case $\frac{1}{2} \notin (a,b)$, one need some additional information to reconstruct the coefficients $(q_0(x), q_1(x), h, H_0)$ of L_0, for example, in [17], the authors showed all conditions together with information of eigenfunctions in some interior points can lead to a uniqueness theorem. To the best of our knowledge, this partial inverse nodal problem for differential pencils has been not completely solved.

The aim of this paper is to study a partial inverse nodal problem for differential pencils with real-valued coefficients on a finite interval. The authors show that the coefficients $(q_0(x), q_1(x), h, H_0)$ are uniquely determined by the partial nodal information on the right (or, left) arbitrary subinterval $[a,b]$ of $[0,1]$. This approach is different from that in [17].

Inverse spectral problems for differential pencils have been studied well(see [6, 9, 12, 13, 15, 16, 20, 21, 25, 37, 38] and references therein). We also note that inverse spectral and nodal problems for other differential pencils were studied in [1, 2, 7, 8, 31, 32, 35]. Define the Weyl m-function of L_0

$$m(x,\lambda) = -\frac{\varphi'(x,\lambda)}{\varphi(x,\lambda)}.$$

By the known method in [8, 12], one can obtain

Theorem 1.1. *The Weyl m-function $m(a_0,\lambda)$ with $0 < a_0 \le 1$ of the boundary value problem* (1.1)-(1.3) *can uniquely determine $(q_0(x), q_1(x))$ on the interval $[0, a_0]$ and h.*

Let $B_\xi = \{n_{\xi,k}\}_{k=-\infty}^{\infty}$ be a strictly increasing sequence in $\mathbb{Z}$ for $\xi = 0$ and 1 and B_ξ be almost symmetrical with respect to the origin, i.e. that means

$$B_\xi \subseteq \mathbb{Z}, \quad n \in B_\xi \Rightarrow -n \in B_\xi,$$

with finitely many possible exceptions, we may also assume $\lambda_{\xi,n_{\xi,k}} \neq 0$. The nodal subset

$$W_{B_\xi}([a,b]) := \left\{ x^{j_k}_{\xi,n_{\xi,k}} : x^{j_k}_{\xi,n_{\xi,k}} \in X_\xi, n_{\xi,k} \in B_\xi, a \le x^{j_k}_{\xi,n_{\xi,k}} \le b \right\}$$

is called a twin-dense nodal subset on $[a,b]$, i.e.

1. If $x^{j}_{\xi,n_{\xi,k}} \in W_{B_\xi}([a,b])$, then either $x^{j+1}_{\xi,n_{\xi,k}} \in W_{B_\xi}([a,b])$ or $x^{j-1}_{\xi,n_{\xi,k}} \in W_{B_\xi}([a,b])$,
2. $\overline{W_{B_\xi}([a,b])} = [a,b]$.

Denote

$$S_\xi := \{\lambda_{\xi,n_{\xi,k}} : n_{\xi,k} \in B_\xi,\ \lambda_{\xi,n_{\xi,k}} \in \sigma(L_\xi)\}$$

and define

$$n_{S_\xi}(t) := \begin{cases} \sum\limits_{0<\lambda_{\xi,n_{\xi,k}}<t,\ n_{\xi,k}\in B_\xi} 1, \\ -\sum\limits_{t<\lambda_{\xi,n_{\xi,k}}<0,\ n_{\xi,k}\in B_\xi} 1, \end{cases}$$

$$w_{B_\xi}(\lambda) := \text{p.v.} \prod_{n_{\xi,k}\in B_\xi} \left(1 - \frac{\lambda}{\lambda_{\xi,n_{\xi,k}}}\right) = \lim_{N\to+\infty} \prod_{k=-N}^{N} \left(1 - \frac{\lambda}{\lambda_{\xi,n_{\xi,k}}}\right). \tag{1.9}$$

In addition, we assume that the following two conditions

(i)

$$\lim_{t\to\infty} \frac{n_{S_\xi}(t)}{t} = \frac{2\theta_\xi}{\pi}. \tag{1.10}$$

(ii) There exist positive numbers t_0, ε, δ' and $b_\xi \in \mathbb{Z}$ such that

$$n_{S_\xi}(t) = \begin{cases} \geq 2\theta_\xi[\frac{t}{\pi}] + b_\xi + \varepsilon + \mathrm{O}\left(t^{-\delta'}\right), & t \geq t_0, \\ \leq -2\theta_\xi[-\frac{t}{\pi}] + b_\xi - 2\theta_\xi + \mathrm{O}\left(|t|^{-\delta'}\right), & t \leq -t_0 \end{cases} \tag{1.11}$$

hold for S_ξ, where $[\cdot]$ denotes the floor function.

In the next section, we present and prove some uniqueness theorems in this paper.

2. Main Results and Proofs

From now on, we denote $\widetilde{L}_\xi = L_\xi(\widetilde{q}_0, \widetilde{q}_1, h, H_\xi)$ the same form as $L_\xi = L_\xi(q_0, q_1, h, H_\xi)$ but with different coefficients. If a certain symbol ζ denotes an object related to $L_\xi(q_0, q_1, h, H_\xi)$, then the corresponding symbol $\widetilde{\zeta}$ with tilde denotes the analogous object related to $L_\xi(\widetilde{q}_0, \widetilde{q}_1, h, H_\xi)$, and $\widehat{\zeta} = \zeta - \widetilde{\zeta}$. At first, we have

Lemma 2.1. *The coefficients* $(q_0(x) - 2\omega_\xi(1), q_1(x))$ *on* $[a,b]$ *can be reconstructed by the given* $W_{B_\xi}([a,b])$ *for each* $\xi = 0,1$.

For each ξ, $\xi = 0,1$, the so-called $W_{B_\xi}([a,b]) = \widetilde{W}_{\widetilde{B}_\xi}([a,b])$ means that for any $x_{\xi,n_{\xi,k}}^{j_k} \in W_{B_\xi}([a,b])$, then at least one of two formulae holds, i.e.

$$x_{\xi,n_{\xi,k}}^{j_k} = \widetilde{x}_{\xi,\widetilde{n}_{\xi,k}}^{\widetilde{j}_k} \quad \text{and} \quad x_{\xi,n_{\xi,k}}^{j_k+1} = \widetilde{x}_{\xi,\widetilde{n}_{\xi,k}}^{\widetilde{j}_k+1},$$
$$x_{\xi,n_{\xi,k}}^{j_k} = \widetilde{x}_{\xi,\widetilde{n}_{\xi,k}}^{\widetilde{j}_k} \quad \text{and} \quad x_{\xi,n_{\xi,k}}^{j_k-1} = \widetilde{x}_{\xi,\widetilde{n}_{\xi,k}}^{\widetilde{j}_k-1},$$

where $x_{\xi,n_{\xi,k}}^{j_k+j} \in W_{B_\xi}([a,b])$ and $\widetilde{x}_{\xi,\widetilde{n}_{\xi,k}}^{\widetilde{j}_k+j} \in \widetilde{W}_{\widetilde{B}_\xi}([a,b])$ for $j = 0, \pm 1$ in this paper. Denote

$$M_0 := \max_{a\leq x\leq b}\{q_1(x)\} \quad \text{and} \quad m_0 := \min_{a\leq x\leq b}\{q_1(x)\}.$$

By using the results in [31], the Weyl m-function and the theory concerning densities of zeros of entire functions (refer to [22, 23]), we shall show that

Theorem 2.2. *Let* $\frac{1}{2} < a \le c_0 < c_1 \le b \le 1$. *Suppose that* $W_{B_1}([a,c_1]) = \widetilde{W}_{\widetilde{B}_1}([a,c_1])$, *and* $W_{B_0}([c_0,b]) = \widetilde{W}_{\widetilde{B}_0}([c_0,b])$, $\omega_1(1) = \widetilde{\omega}_1(1)$, *and* (1.10) *and* (1.11) *hold for each* S_ξ, $\xi = 0,1$, *where* $\theta_0 = 1-b$ *and* $\theta_0 + \theta_1 = a$, *and for each* $x^{j_k}_{\xi,n_{\xi,k}} = x^{\widetilde{j}_k}_{\xi,\widetilde{n}_{\xi,k}}$, *the corresponding eigenvalues* $\lambda_{\xi,n_{\xi,k}}$ *and* $\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}$ *satisfy the inequalities:*

$$\begin{cases} \lambda_{\xi,n_{\xi,k}} + \widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}} > 2M_0 & \text{for all} \quad n_{\xi,k} > 0, \\ \lambda_{\xi,n_{\xi,k}} + \widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}} < 2m_0 & \text{for all} \quad n_{\xi,k} < 0. \end{cases} \tag{2.1}$$

Then

$$q_1(x) = \widetilde{q}_1(x), \quad q_0(x) \overset{a.e.}{=} \widetilde{q}_0(x) \quad \text{on} \quad [0,1], \quad h = \widetilde{h} \quad \text{and} \quad H_\xi = \widetilde{H}_\xi, \quad \xi = 0,1. \tag{2.2}$$

Note that the length $b-a$ of the right subinterval $[a,b]$ of $[0,1]$ is arbitrarily small. Furthermore we establish a uniqueness theorem for the right arbitrary subinterval $[a,b]$ as follows:

Theorem 2.3. *Let* $\frac{1}{2} < a \le c < b \le 1$. *Suppose that* $W_{B_1}([a,c]) = \widetilde{W}_{\widetilde{B}_1}([a,c])$, *and* $W_{B_0}([c,b]) = \widetilde{W}_{\widetilde{B}_0}([c,b])$, $\omega_1(1) = \widetilde{\omega}_1(1)$, *and* (1.10) *and* (1.11) *hold for each* S_ξ, $\xi = 0,1$, *where* $\theta_0 = 1-b$ *and* $\theta_0 + \theta_1 = a$, *and for each* $x^{j_k}_{\xi,n_{\xi,k}} = x^{\widetilde{j}_k}_{\xi,\widetilde{n}_{\xi,k}}$ *the corresponding eigenvalues* $\lambda_{\xi,n_{\xi,k}}$ *and* $\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}$ *satisfy* (2.1) *and* $q_0(x) - \widetilde{q}_0(x)$ *is continuous at* $x = c$, *then* (2.2) *holds.*

Note that $0 \le \theta_0 \le \frac{1}{2}$, then $\theta_0 < a$ for the case $\frac{1}{2} < a < b \le 1$. Therefore Theorems 2.2 and 2.3 cannot be valid for the cases $S_1 = \emptyset$ and $\frac{1}{2} < a < b \le 1$. In addition, one can obtain an analogy of Theorem 2.2 and 2.3 for the case $0 \le a < b < 1/2$ by symmetry. We omit the details here.

If $q_1(x) \equiv 0$ on $[0,1]$, then the problem (1.1)-(1.3) becomes a classical Sturm-Liouville operator and such problems are well studied (please refer to [3, 10, 11, 14, 15, 18, 24, 25, 26, 27, 28, 29, 30, 34, 36, 38] and the references therein).

In the remaining of this section, we shall present proofs of Theorems 2.2 and 2.3. At first, we show the proof of Lemma 2.1.

Proof of Lemma 2.1. For each $\xi = 0,1$ and each fixed $x \in [a,b]$, choose $\{x^{j_k}_{\xi,n_{\xi,k}}\}$ such that

$$\lim_{|k|\to\infty} x^{j_k}_{n_{\xi,k}} = x.$$

By virtue of (1.8), then there exist the following finite limits and the corresponding equalities hold:

$$f_\xi(x) := \lim_{|k|\to\infty} n_{\xi,k}\pi \left(x^{j_k}_{\xi,n_{\xi,k}} - \frac{j_k - \frac{1}{2}}{n_{\xi,k}} \right) = Q_1(x), \tag{2.3}$$

$$g_\xi(x) := \lim_{|k|\to\infty} n^2_{\xi,k}\pi^2 \left(x^{j_k}_{\xi,n_{\xi,k}} - \frac{j_k - \frac{1}{2}}{n_{\xi,k}} - \frac{Q_1(x^{j_k}_{\xi,n_{\xi,k}})}{n_{\xi,k}\pi} \right)$$

$$=\omega_\xi(x)-\omega_\xi(1)x-H_\xi. \tag{2.4}$$

Thus we find the functions $f_\xi(x)$ and $g_\xi(x)$ via (2.3) and (2.4). Moreover we reconstruct $(q_0(x)-2\omega_\xi(1), q_1(x))$ on $[a,b]$ by

$$q_1(x)=f'_\xi(x),\quad x\in[a,b], \tag{2.5}$$

$$q_0(x)-2\omega_\xi(1)=2g'_\xi(x)-q_1^2(x),\quad x\in[a,b], \tag{2.6}$$

This complete the proof of Lemma 2.1. □

The following are the proofs of our main results.

Proof of Theorem 2.2. Since $W_{B_1}([a,c_1])=\widetilde{W}_{\widetilde{B}_1}([a,c_1])$, and $W_{B_0}([c_0,b])=\widetilde{W}_{\widetilde{B}_0}([c_0,b])$, we have

$$\begin{cases} f_1(x)=\widetilde{f}_1(x),\quad x\in[a,c_1], & (2.7)\\ f_0(x)=\widetilde{f}_0(x),\quad x\in[c_0,b], & (2.8)\\ g_1(x)=\widetilde{g}_1(x),\quad x\in[a,c_1], & (2.9)\\ g_0(x)=\widetilde{g}_0(x),\quad x\in[c_0,b]. & (2.10)\end{cases}$$

Therefore (2.7)-(2.10) together with the assumption $\omega_1(1)=\widetilde{\omega}_1(1)$ lead to that

$$\begin{cases} \widehat{q}_1(x)=0 \quad \text{on} \quad [a,b],\\ \widehat{q}_0(x)\overset{a.e.}{=}0 \quad \text{on} \quad [a,c_1],\\ \widehat{q}_0(x)\overset{a.e.}{=}2\widehat{\omega}_0(1) \quad \text{on} \quad [c_0,b],\end{cases}$$

which together with $a\le c_0<c_1\le b$ imply

$$\begin{cases} \widehat{q}_1(x)=0 \quad \text{on} \quad [a,b], & (2.11)\\ \widehat{q}_0(x)\overset{a.e.}{=}0 \quad \text{on} \quad [a,b]. & (2.12)\end{cases}$$

Next we shall show $\lambda_{\xi,n_{\xi,k}}=\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}$ for all $n_{\xi,k}\in B_\xi$. Note that

$$\begin{cases} -\varphi''(x,\lambda_{\xi,n_{\xi,k}})+(q_0(x)+2\lambda_{\xi,n_{\xi,k}}q_1(x))\varphi(x,\lambda_{\xi,n_{\xi,k}})=\lambda^2_{\xi,n_{\xi,k}}\varphi(x,\lambda_{\xi,n_{\xi,k}}), & (2.13)\\ \varphi(x^{j_k}_{\xi,n_{\xi,k}},\lambda_{\xi,n_{\xi,k}})=\varphi(x^{j_k+1}_{\xi,n_{\xi,k}},\lambda_{\xi,n_{\xi,k}})=0, & (2.14)\end{cases}$$

and

$$\begin{cases} -\widetilde{\varphi}''(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})+(\widetilde{q}_0(x)+2\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}\widetilde{q}_1(x))\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})=\widetilde{\lambda}^2_{\xi,\widetilde{n}_{\xi,k}}\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}), & (2.15)\\ \widetilde{\varphi}(x^{j_k}_{\xi,n_{\xi,k}},\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})=\widetilde{\varphi}(x^{j_k+1}_{\xi,n_{\xi,k}},\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})=0. & (2.16)\end{cases}$$

Equations (2.13)-(2.16) yield to

$$\int_{x^{j_k}_{\xi,n_{\xi,k}}}^{x^{j_k+1}_{\xi,n_{\xi,k}}}[\widehat{q}_0(x)+2(\lambda_{\xi,n_{\xi,k}}q_1(x)-\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}\widetilde{q}_1(x))-(\lambda^2_{\xi,n_{\xi,k}}-\widetilde{\lambda}^2_{\xi,\widetilde{n}_{\xi,k}})]\varphi(x,\lambda_{\xi,n_{\xi,k}})\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})\mathrm{d}x=0. \tag{2.17}$$

By virtue of (2.17) together with (2.11) and (2.12), this yields

$$\left(\lambda_{\xi,n_{\xi,k}}-\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}\right)\int_{x^{j_k}_{\xi,n_{\xi,k}}}^{x^{j_k+1}_{\xi,n_{\xi,k}}}\left(2q_1(x)-\lambda_{\xi,n_{\xi,k}}-\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}\right)\varphi(x,\lambda_{\xi,n_{\xi,k}})\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})\mathrm{d}x=0. \tag{2.18}$$

Since both $\varphi(x,\lambda_{\xi,n_{\xi,k}})$ and $\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})$ have no zero in the interval $(x^{j_k}_{n_k},x^{j_k+1}_{n_k})$ together with the assumption (2.1), we obtain

$$\int_{x^{j_k}_{\xi,n_{\xi,k}}}^{x^{j_k+1}_{\xi,n_{\xi,k}}}\left(2q_1(x)-\lambda_{\xi,n_{\xi,k}}-\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}}\right)\varphi(x,\lambda_{\xi,n_{\xi,k}})\widetilde{\varphi}(x,\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}})\mathrm{d}x\neq 0. \tag{2.19}$$

Therefore (2.18) and (2.19) show that

$$\lambda_{\xi,n_{\xi,k}}=\widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}},\quad \forall\ n_{\xi,k}\in B_\xi. \tag{2.20}$$

For each $\lambda_{\xi,n_{\xi,k}}$, by (2.17) and (2.20), we get

$$\begin{aligned}&\int_a^{x^{j_k}_{1,n_{1,k}}}(\widehat{q}_0(x)+2\lambda_{1,n_{1,k}}\widehat{q}_1(x))\varphi(x,\lambda_{1,n_{1,k}})\widetilde{\varphi}(x,\lambda_{1,n_{\xi,k}})\mathrm{d}x\\ &\quad=<\widetilde{\varphi},\varphi>(x^{j_k}_{1,n_{1,k}},\lambda_{1,n_{1,k}})-<\widetilde{\varphi},\varphi>(a,\lambda_{1,n_{1,k}}),\end{aligned} \tag{2.21}$$

By virtue of (2.14), (2.16), (2.11) and (2.12), then (2.21) implies

$$<\widetilde{\varphi},\varphi>(a,\lambda_{1,n_{1,k}})=0,\quad \forall n_{1,k}\in B_1. \tag{2.22}$$

Applying the same arguments as the proof of (2.22), we obtain

$$<\widetilde{\varphi},\varphi>(c_0,\lambda_{0,n_{0,k}})=0,\quad \forall n_{0,k}\in B_0. \tag{2.23}$$

By virtue of (2.11), (2.12) and (2.23), this yields

$$<\widetilde{\varphi},\varphi>(a,\lambda_{0,n_{0,k}})=0,\quad \forall n_{0,k}\in B_0. \tag{2.24}$$

Furthermore (2.11), (2.12), (2.23) and (2.24) show that

$$<\widetilde{\varphi},\varphi>(a,\lambda_{\xi,n_{\xi,k}})=0,\quad \forall n_{\xi,k}\in B_\xi, \tag{2.25}$$

$$<\widetilde{\varphi},\varphi>(b,\lambda_{\xi,n_{0,k}})=0,\quad \forall n_{0,k}\in B_0. \tag{2.26}$$

Since the functions $\varphi(x,\lambda_{0,n_{0,k}})$ and $\psi_0(x,\lambda_{0,n_{0,k}})$ are both eigenfunctions corresponding to the $n_{0,k}$-th eigenvalue $\lambda_{0,n_{0,k}}$ of L_0, there exists a constant $\beta_0(\lambda_{0,n_{0,k}})\neq 0$ such that

$$\psi_0(x,\lambda_{0,n_{0,k}})=\beta_0(\lambda_{0,n_{0,k}})\varphi(x,\lambda_{0,n_{0,k}}),\quad \forall x\in[0,1]. \tag{2.27}$$

Consequently (2.26) and (2.27) imply

$$<\widetilde{\psi}_0,\psi_0>(b,\lambda_{0,n_{0,k}})=0,\quad \forall n_{0,k}\in B_0. \tag{2.28}$$

It is easy to prove

$$\left|<\widetilde{\varphi},\varphi>(a,\lambda)\right|=\mathrm{O}(\mathrm{e}^{2a\tau}) \tag{2.29}$$

for sufficiently large λ. Define the function

$$K_1(\lambda):=\frac{<\widetilde{\varphi},\varphi>(a,\lambda)}{w_{B_0}(\lambda)w_{B_1}(\lambda)}. \tag{2.30}$$

Note that $\lambda_{\xi,n_{\xi,k}}$ satisfy (1.6), and

$$\begin{aligned}\left(1-\frac{\lambda}{\lambda_{\xi,n_{\xi,k}}}\right)\left(1-\frac{\lambda}{\lambda_{\xi,-n_{\xi,k}}}\right)=&\left(1-\frac{\lambda}{n_{\xi,k}\pi+\mathrm{O}(1)}\right)\left(1+\frac{\lambda}{n_{\xi,k}\pi-\mathrm{O}(1)}\right)\\ =&1-\frac{\lambda^2+\mathrm{O}(1)\lambda+\mathrm{O}(1)}{(n_{\xi,k}\pi+\mathrm{O}(1))(n_{\xi,k}\pi-\mathrm{O}(1))}.\end{aligned} \tag{2.31}$$

Therefore (2.31) implies that the locally uniform convergence of the products (1.9) holds. Since $H_0\neq H_1$, then $\sigma(L_0)\cap\sigma(L_1)=\emptyset$, which guarantees that

$$S_0\cap S_1=\emptyset. \tag{2.32}$$

Thus (2.25), (1.9) and (2.32) show that the function $K_1(\lambda)$ is an entire function in λ. Next we shall prove $K_1(\lambda)\equiv 0$. By the classical estimate of Levinson in [23] together with the assumptions in Theorem 2.2 and (1.9), there exists a constant C_ξ such that

$$\frac{1}{|w_{B_\xi}(\lambda)|}=\mathrm{O}\left(\mathrm{e}^{-2\theta_\xi\tau+\varepsilon r}\right),\quad \forall\lambda\in G_{C_\xi},\quad r=|\lambda|, \tag{2.33}$$

where $\varepsilon>0$, $G_{C_\xi}:=\{\lambda:|\lambda-\lambda_{n_{\xi,k}}|\geq\frac{1}{8}C_\xi,\ \lambda_{\xi,n_{\xi,k}}\in S_\xi\}$. Consequently (2.33) and (2.29) imply

$$|K_1(\lambda)|=\mathrm{O}\left(\mathrm{e}^{-2(\theta_0+\theta_1-a)\tau+2\varepsilon r}\right),\quad \forall\lambda\in G_{C_0}\bigcap G_{C_1}$$

for sufficiently large $|\lambda|$, where ε is arbitrary. Since $\theta_0+\theta_1-a=0$, the maximum modulus principle shows that

$$|K_1(\lambda)|\leq c_2\mathrm{e}^{2\varepsilon|\lambda|},\quad \lambda\in\mathbb{C}, \tag{2.34}$$

where c_2 is constant. Therefore (2.34) implies that $K_1(z)$ is of zero exponential type. We say that the notation $\asymp$ means that both $\frac{|w_{B_\xi}(\lambda)|}{\left|w^*_{B_\xi}(\lambda)\right|}$ and $\frac{\left|w^*_{B_\xi}(\lambda)\right|}{|w_{B_\xi}(\lambda)|}$ are bounded, where

$$w^*_{B_\xi}(\lambda)=\mathrm{p.v.}\prod_{n_{\xi,k}\in B_\xi}\left(1-\frac{\lambda}{n_{\xi,k}}\right).$$

By Lemmas 2.5-2.7 in [19], for each $\delta > 0$, we have

$$\left|w_{B_\xi}(\lambda)\right| \asymp \left|w^*_{B_\xi}(\lambda)\right| \quad \text{if} \quad \left|\lambda - \lambda_{\xi,n_{\xi,k}}\right| \geq \delta, \quad \left|\lambda - n_{\xi,k}\right| \geq \delta \quad \text{for} \quad n_{\xi,k} \in B_\xi.$$

This implies

$$|w_{B_\xi}(iy)| \asymp |w^*_{B_\xi}(iy)|, \quad |y| \to \infty. \tag{2.35}$$

By calculating(refer to [16, 31] for details), we obtain

$$\begin{aligned}\ln\left|w^*_{B_\xi}(iy)\right| &= \int_{-\infty}^{\infty} \frac{n_{S_\xi}(t)}{t} \frac{y^2}{y^2+t^2}\mathrm{d}t \\ &= \int_{-\infty}^{1} \frac{n_{S_\xi}(t)}{t} \frac{y^2}{y^2+t^2}\mathrm{d}t + \int_{1}^{\infty} \frac{n_{S_\xi}(t)}{t} \frac{y^2}{y^2+t^2}\mathrm{d}t + O(1) \\ &= 2\theta_\xi|y| + \varepsilon\ln|y| + O(1).\end{aligned}$$

This shows that

$$\left|w^*_{B_\xi}(iy)\right| \asymp |y|^\varepsilon \mathrm{e}^{2\theta_\xi|y|}. \tag{2.36}$$

Therefore (2.34)-(2.36) show that

$$|K_1(iy)| = O\left(\frac{1}{|y|^{2\varepsilon}}\right). \tag{2.37}$$

By the Phragmén-Lindelöf-type result in [22] together with (2.34) and (2.37), we get

$$K_1(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}.$$

This implies

$$m(a,\lambda) = \widetilde{m}(a,\lambda). \tag{2.38}$$

By virtue of Theorem 1.1 together with (2.38), we have

$$\widehat{q}_1(x) = 0, \quad \text{and} \quad \widehat{q}_0(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0,a] \quad \text{and} \quad h = \widetilde{h}. \tag{2.39}$$

Define the function

$$K_2(\lambda) := \frac{<\widetilde{\psi}_0, \psi_0>(b,\lambda)}{w_{B_0}(\lambda)}, \tag{2.40}$$

Consequently (2.23), (2.28) and (2.40) show that the function $K_2(\lambda)$ is an entire function in λ. Applying the same arguments as the proof of (2.34) and (2.37), we obtain

$$\begin{cases} |K_2(\lambda)| \leq c\mathrm{e}^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}, & (2.41) \\ |K_2(iy)| = O\left(\dfrac{1}{|y|^\varepsilon}\right). & (2.42)\end{cases}$$

By the Phragmén-Lindelöf-type result in [22] together with (2.41) and (2.42) again, we have

$$K_2(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}.$$

This implies

$$< \widetilde{\psi}_0, \psi_0 > (b,\lambda) \equiv 0, \quad \lambda \in \mathbb{C}. \tag{2.43}$$

The function

$$m_+(b,\lambda) = \frac{\psi_0'(b,\lambda)}{\psi_0(b,\lambda)}, \quad 0 \le b < 1$$

is called the Weyl m−function of L_0. Thus (2.43) shows that

$$m_+(b,\lambda) = \frac{\psi_0'(b,\lambda)}{\psi_0(b,\lambda)} = \frac{\widetilde{\psi}_0'(b,\lambda)}{\widetilde{\psi}_0(b,\lambda)} = \widetilde{m}_+(b,\lambda), \tag{2.44}$$

Similar to Theorem 1.1, then it follows from (2.44)

$$\widehat{q}_1(x) = 0, \quad \widehat{q}_0(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [b,1] \quad \text{and} \quad H_0 = \widetilde{H}_0. \tag{2.45}$$

Moreover we have

$$H_1 = \widetilde{H}_1.$$

This together with (2.11), (2.12), (2.39) and (2.45) implies Theorem 2.2 holds. □

Next we prove Theorem 2.3.

Proof of Theorem 2.3. By virtue of $W_{B_1}([a,c]) = \widetilde{W}_{\widetilde{B}_1}([a,c])$, and $W_{B_0}([c,b]) = \widetilde{W}_{\widetilde{B}_0}([c,b])$, this together with Lemma 2.1 and the assumption $\omega_1(1) = \widetilde{\omega}_1(1)$ yields

$$\begin{cases} \widehat{q}_1(x) = 0 & \text{on} \quad [a,b], & (2.46)\\ \widehat{q}_0(x) \overset{a.e.}{=} 0 & \text{on} \quad [a,c], & (2.47)\\ \widehat{q}_0(x) \overset{a.e.}{=} 2\widehat{\omega}_0(1) & \text{on} \quad [c,b]. & (2.48) \end{cases}$$

Therefore (2.47) and (2.48) together with the function $q_0(x) - \widetilde{q}_0(x)$ is continuous at $x = c$, this yields

$$\widehat{\omega}_0(1) = 0. \tag{2.49}$$

Consequently (2.47), (2.48) and (2.49) imply that

$$\widehat{q}_0(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [a,b]. \tag{2.50}$$

Similar to the proof of (2.20) in Theorem 2.2, we have

$$\lambda_{\xi,n_{\xi,k}} - \widetilde{\lambda}_{\xi,\widetilde{n}_{\xi,k}} = 0, \quad \forall\ n_{\xi,k} \in B_\xi, \quad \xi = 0,1. \tag{2.51}$$

Modifying the proof in Theorem 2.2 simply together with (2.46), (2.50) and (2.51), we obtain

$$\widehat{q}_1(x)=0,\quad \widehat{q}_0(x)\overset{a.e.}{=}0\quad \text{on}\quad [0,1],\quad h=\widetilde{h},\quad \text{and}\quad H_\xi=\widetilde{H}_\xi,\quad \xi=0,1. \qquad \square$$

Finally, we shall present an example for reconstruction of $(q_0(x),q_1(x),h,H_0)$ from the twin-dense nodal subset $W_{B_0}([0,1])$.

Example 2.4. Let $W_{B_0}([0,1])=\{x_{n_k}^j\}, B_0\subseteq\mathbb{Z}$, be the twin-dense nodal subset of the pencil $L(q_0,q_1,h,H_0)$ and $\int_0^1 q_0(t)\mathrm{d}t=\frac{1}{2}$, where

$$x_{n_k}^j=\frac{j-\frac{1}{2}}{n_k}-\frac{j-\frac{1}{2}}{{n_k}^2\pi}+\frac{(j-\frac{1}{2})^2}{{n_k}^3\pi}+\frac{\omega_0(x_{n_k}^j)}{(n_k\pi)^2}+o\left(\frac{1}{n_k^2}\right),\quad \forall n_k\in B_0, \tag{2.52}$$

where

$$\omega_0(x_{n_k}^j)=\frac{2(j-\frac{1}{2})^3}{3{n_k}^3}-\frac{3(j-\frac{1}{2})^2}{4{n_k}^2}-\frac{23(j-\frac{1}{2})}{12n_k}+1,$$

reconstruct $(q_0(x),q_1(x),h,H_0)$.

For each fixed $x\in[0,1]$, we choose $x_{n_k}^{j_{n_k}}$ such that $\lim\limits_{k\to\infty}\frac{j-\frac{1}{2}}{n_k}=x$. By (2.52), we have

$$\begin{aligned}
f(x):=&\lim_{|k|\to\infty} n_k\pi\left(x_{n_k}^{j_k}-\frac{j_k-\frac{1}{2}}{n_k}\right)\\
=&\lim_{|k|\to\infty}\left(\frac{(j-\frac{1}{2})^2}{{n_k}^2}-\frac{j_k-\frac{1}{2}}{n_k}+\mathrm{O}\left(\frac{1}{n_k}\right)\right)\\
=&x^2-x=\int_0^x q_1(x)\mathrm{d}t,
\end{aligned}$$

which implies

$$q_1(x)=2x-1,\quad x\in[0,1]. \tag{2.53}$$

By (2.53), we obtain

$$\begin{aligned}
g(x):=&\lim_{|k|\to\infty} n_k^2\pi^2\left(x_{n_k}^{j_k}-\frac{j_k-\frac{1}{2}}{n_k}+\frac{j-\frac{1}{2}}{{n_k}^2\pi}-\frac{(j-\frac{1}{2})^2}{{n_k}^3\pi}\right)\\
=&\lim_{|k|\to\infty}\left(\frac{2(j-\frac{1}{2})^3}{3{n_k}^3}-\frac{3(j-\frac{1}{2})^2}{4{n_k}^2}-\frac{23(j-\frac{1}{2})}{12n_k}+1+o(1)\right)\\
=&\frac{2}{3}x^3-\frac{3}{4}x^2-\frac{23}{12}x+1\\
=&\omega_0(x)-\omega_0(1)x-H_0.
\end{aligned} \tag{2.54}$$

Therefore (2.54) shows that

$$h=g(0)=1,\quad \text{and}\quad H_0=-g(1)=1. \tag{2.55}$$

By (2.53)-(2.55) together with $\int_0^1 q_0(t)\mathrm{d}t = \frac{1}{2}$, we get

$$q_0(x) \overset{a.e.}{=} 4 + 2g'(x) - (2x-1)^2 + \int_0^1 (2x-1)^2 \mathrm{d}x + \frac{1}{2} = x, \quad x \in [0,1]. \tag{2.56}$$

Thus the coefficients $(q_0(x), q_1(x), h, H_0)$ are reconstructed by (2.53), (2.55) and (2.56).

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