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INVERSE SPECTRAL PROBLEM FOR THE MATRIX STURM-LIOUVILLE OPERATOR WITH THE GENERAL SEPARATED SELF-ADJOINT BOUNDARY CONDITIONS

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Abstract. In this work, we study the matrix Sturm-Liouville operator with the separated self-adjoint boundary conditions of general type, in terms of two unitary matrices. Some properties of the eigenvalues and the normalization matrices are given. Uniqueness theorems for determining the potential and the unitary matrices in the boundary conditions from the Weyl matrix, two characteristic matrices or one spectrum and the corresponding normalization matrices are proved.

1. Introduction

Consider the matrix Sturm-Liouville equation

$$-Y''(x)+Q(x)Y(x)=\lambda Y(x),\quad 0<x<a, \tag{1.1}$$

with the general self-adjoint boundary conditions of separated type

$$T_1(Y):=A_1^\dagger Y'(0)-B_1^\dagger Y(0)=0_n,\quad T_2(Y):=A_2^\dagger Y'(a)+B_2^\dagger Y(a)=0_n, \tag{1.2}$$

where $Y(x)$ is either an $n\times n$ matrix-valued function or a column vector-valued function with n components, the matrix potential $Q(x)$ satisfies $Q(x)^\dagger=Q(x)$ and every entry in Q belongs to $L^2(0,a)$, λ is the spectral parameter, the dagger "†" denotes the matrix adjoint (complex conjugate and matrix transpose), 0_n denotes the zero matrix or the zero vector, and

$$A_j=\frac{1}{2}\left(U_j+I_n\right),\quad B_j=\frac{i}{2}\left(U_j-I_n\right),\quad j=1,2, \tag{1.3}$$

the matrix U_j is unitary, I_n denotes the $n\times n$ identity matrix. In order to make the system (1.1) and (1.2) to be self-adjoint, one can also require the boundary matrices to satisfy [2, 1, 11, 17]

$$A_j^\dagger B_j=B_j^\dagger A_j,\quad \operatorname{rank}[A_j\ B_j]=n,\quad j=1,2; \tag{1.4}$$

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or

$$B_j^\dagger A_j = A_j^\dagger B_j, \quad A_j^\dagger A_j + B_j^\dagger B_j > 0, \quad j = 1,2. \tag{1.5}$$

It was shown in [1] that the conditions (1.3), (1.4) and (1.5) are equivalent to each other. Here, for convenience, we use (1.3) in our paper. Denote by $L(Q,U_1,U_2)$ the problem (1.1) and (1.2).

The λ-values for which (1.1) has a non-trivial column vector solution satisfying (1.2) are called *eigenvalues*, and the corresponding vector solutions are called *eigenvector functions*.

The problem $L(Q,U_1,U_2)$ with a diagonal potential $Q(x)$ is connected with the spectral problems on star graph (see, e.g., [4, 6, 10, 20, 21, 27, 29, 31]). The continuity condition and Kirchhoff's condition at $x=0$ are equivalent to that U_1 has this form: the diagonal entries are all $-1/3$ and the other entries are all $2/3$. The separated boundary condition at $x=a$ yields that $U_2 = -\text{diag}\{e^{2i\alpha_1},\dots,e^{2i\alpha_n}\}$ with $\alpha_j \in (0,\pi]$. Some other types of boundary conditions (such as the δ-type condition [18, 25]) are also included here. Many scholars studied the matrix Sturm-Liouville operator (see [2, 1, 3, 5, 7, 8, 9, 22, 12, 15, 16, 23, 24, 26, 28, 33] and the references therein), whereas only a few of them include the problems on star graphs which are only for noncompact case [2, 1, 15, 16, 26].

The problem considered here is more difficult for investigating. Unlike the case that A_j ($j=1,2$) are invertible, $I_n\sqrt{\lambda}\sin\sqrt{\lambda}a$ is no longer the global main part of asymptotics for the characteristic matrix. This causes difficulties in studying the behavior of the spectrum of $L(Q,U_1,U_2)$. Furthermore, the boundary conditions (1.3) cause that the Weyl matrix is no longer $O(\frac{1}{\sqrt{\lambda}})$ for large λ, which yields that one cannot use the residue theorem as that used in [32] to prove that the Weyl matrix is uniquely recovered from the eigenvalues and the normalization matrices. Weyl matrix plays an important role in the spectral theory, which is from the Weyl function in the scalar case and has been generalized into many cases (see, e.g., [14, 30, 31, 33, 34, 35]). In this paper, we give some properties of the Weyl matrix, and show that the Weyl matrix uniquely determines the potential and the boundary conditions. As a corollary, we obtain the uniqueness theorems for determining the problem $L(Q,U_1,U_2)$ from two characteristic matrices or one spectrum and the corresponding normalization matrices.

The paper is organized as follows. In Section 2, we give some preliminaries, where the notations and some asymptotic estimates of the initial solutions are provided. Section 3 shows some properties of the eigenvalues and normalization matrices. In this section, we also introduce the so-called Weyl solutions and Weyl matrices, and establish the relations between the normalization matrices and the Weyl matrices. In the last section, the inverse problem is considered, where we show that the Weyl matrix uniquely determines the potential and the matrices in the boundary conditions.

2. Preliminaries

Let $\mathbb{C}^n$ be the space of complex column vector with n components. Denote $\mathbb{C}^{\pm}=\{k\in\mathbb{C}:\pm\mathrm{Im}\,k>0\}$, $\overline{\mathbb{C}}^{\pm}=\mathbb{C}^{\pm}\cup\mathbb{R}$ and $\mathbb{C}^{+}_{\delta}:=\{k:0<\delta\le\arg k\le\pi-\delta\}$. Let $L^2((0,a);\mathbb{C}^n)$ be the space of the column vector-valued functions with each element belonging to $L^2(0,a)$. If it does not cause misunderstanding, we may just use the notation $L^2(0,a)$. An overdot used below means the λ-derivative.

Together with (1.1), we consider the following equation

$$-Z''(x)+Z(x)Q(x)=\lambda Z(x),\quad x\in(0,a),\tag{2.1}$$

where $Z(x)$ is either an $n\times n$ matrix-valued function or a row vector-valued function with n components. Let $[Z;Y]:=ZY'-Z'Y$ denote the Wronskian. It is easy to prove that the Wronskian $[Z(x,\lambda);Y(x,\lambda)]$ does not depend on x. In addition, if $Y(x,\lambda)$ is a solution to (1.1), then $Y(x,\bar\lambda)^{\dagger}$ is the solution to (2.1), where $\bar\lambda$ means the complex conjugate of λ.

Let $\varphi(x,\lambda),\psi(x,\lambda),\varphi_1(x,\lambda)$ and $\psi_1(x,\lambda)$ be the matrix solutions to (1.1) with the initial conditions, respectively,

$$\varphi(0,\lambda)=-\varphi_1'(0,\lambda)=A_1,\ \varphi'(0,\lambda)=\varphi_1(0,\lambda)=B_1,\tag{2.2}$$
$$\psi(a,\lambda)=-\psi_1'(a,\lambda)=A_2,\ \psi'(a,\lambda)=\psi_1(a,\lambda)=-B_2.\tag{2.3}$$

The above solutions are all entire matrix-valued functions of λ of order $1/2$ for each fixed $x\in[0,a]$.

Using (1.3), (2.2) and (2.3), it is easy to show

$$[\psi(x,\bar\lambda)^{\dagger};\psi(x,\lambda)]=[\psi_1(x,\bar\lambda)^{\dagger};\psi_1(x,\lambda)]=[\varphi(x,\bar\lambda)^{\dagger};\varphi(x,\lambda)]=[\varphi_1(x,\bar\lambda)^{\dagger};\varphi_1(x,\lambda)]=0_n,\tag{2.4}$$
$$[\varphi_1(x,\bar\lambda)^{\dagger};\varphi(x,\lambda)]=[\varphi(x,\bar\lambda)^{\dagger};-\varphi_1(x,\lambda)]=[\psi_1(x,\bar\lambda)^{\dagger};\psi(x,\lambda)]=[\psi(x,\bar\lambda)^{\dagger};-\psi_1(x,\lambda)]=I_k.\tag{2.5}$$

Let $\lambda=k^2$ and $\bar\lambda=\bar k^2$ with $-\bar k,k\in\overline{\mathbb{C}}^{+}$. It is known that $\varphi(x,\lambda)$ satisfies the integral equation

$$\varphi(x,\lambda)=A_1\cos kx+B_1\frac{\sin kx}{k}+\int_0^x\frac{\sin k(x-t)}{k}Q(t)\varphi(t,\lambda)\,dt,$$

from which it follows that as $|k|\to\infty$ in $\mathbb{C}$,

$$\varphi(x,\lambda)=A_1\cos kx+B_1\frac{\sin kx}{k}+\int_0^x\frac{\sin k(x-t)\cos kt}{k}Q(t)\,dtA_1+O\left(\frac{e^{|\mathrm{Im}k|x}}{k^2}\right),\tag{2.6}$$
$$\varphi'(x,\lambda)=-A_1k\sin kx+B_1\cos kx+\int_0^x\cos k(x-t)\cos ktQ(t)\,dtA_1+O\left(\frac{e^{|\mathrm{Im}k|x}}{k}\right).\tag{2.7}$$

Using the formula

$$\sin\alpha\cos\beta=\frac12[\sin(\alpha+\beta)+\sin(\alpha-\beta)],\quad\cos\alpha\cos\beta=\frac12[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

in (2.6) and (2.7), respectively, we obtain that as $|k| \to \infty$ in $\mathbb{C}$,

$$\varphi(x,\lambda) = A_1 \cos kx + B_1 \frac{\sin kx}{k} + \frac{\sin kx}{2k} \int_0^x Q(t)\,dt A_1 + o\left(\frac{e^{|\mathrm{Im}k|x}}{k}\right), \tag{2.8}$$

$$\varphi'(x,\lambda) = -A_1 k \sin kx + B_1 \cos kx + \frac{\cos kx}{2} \int_0^x Q(t)\,dt A_1 + o\left(e^{|\mathrm{Im}k|x}\right). \tag{2.9}$$

Lemma 2.1. *Let $F(x)$ be an $n\times n$ continuous matrix-valued function. Then for all $x \in [0,a]$ there hold*

$$F(x)A_j[ikA_j \pm B_j]^{-1} = O\left(\frac{1}{k}\right), \quad |k| \to \infty,\ k \in \mathbb{C}, \quad j = 1,2. \tag{2.10}$$

$$F(x)B_j[ikB_j \pm A_j]^{-1} = O\left(\frac{1}{k}\right), \quad |k| \to \infty,\ k \in \mathbb{C}, \quad j = 1,2. \tag{2.11}$$

Proof. We only prove (2.10) for $j = 1$. The other cases can be proved similarly.

Since (1.3) and U_1 is unitary, there exists an unitary matrix D such that

$$D^\dagger A_1 D = \mathrm{diag}\{\xi_1,\dots,\xi_r,0_{n-r}\} := \begin{bmatrix} \Xi_r & 0 \\ 0 & 0_{n-r} \end{bmatrix},$$

$$D^\dagger B_1 D = \mathrm{diag}\{\gamma_1,\dots,\gamma_r,-iI_{n-r}\} := \begin{bmatrix} \Gamma_r & 0 \\ 0 & -iI_{n-r} \end{bmatrix},$$

where $\xi_j (\neq 0)$ and γ_j $(j = \overline{1,s})$ are constants.

Denote

$$D^\dagger F(x) D := \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix},$$

where f_{11} is a $r\times r$ matrix. It follows that as $|k| \to \infty$ in $\mathbb{C}$,

$$D^\dagger F(x) A_1 [ikA_1 \pm B_1]^{-1} D = \begin{bmatrix} f_{11}(x)\Xi_r & 0 \\ f_{21}(x)\Xi_r & 0_{n-r} \end{bmatrix} \begin{bmatrix} (ik\Xi_r \pm \Gamma_r)^{-1} & 0 \\ 0 & \pm iI_{n-r} \end{bmatrix} = O\left(\frac{1}{k}\right),$$

which implies (2.10) for $j = 1$. □

Remark 2.1. From the above proof, we see that, if we use $[ikA_j \pm B_j]^{-1}F(x)A_j$ and $[ikB_j \pm A_j]^{-1}F(x)B_j$ instead of the left-hand sides in (2.10) and (2.11), respectively, then the results may not hold.

Using the formulas

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}, \quad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2},$$

in (2.8) and (2.9), and noting that $\left(ikA_1-B_1-\frac{1}{2}\int_0^x Q(t)dtA_1\right)$ is invertible for large k, we have that as $|k|\to\infty$ in $\mathbb{C}_\delta^+$,

$$\varphi^{(v)}(x,\lambda)=\frac{(-1)^v e^{-ikx}}{2(ik)^{1-v}}[I_n+o(1)]\left(ikA_1-B_1-\frac{1}{2}\int_0^x Q(t)dtA_1\right),\quad x\in(0,a],$$

which implies from Lemma 2.1 that

$$\varphi^{(v)}(x,\lambda)=\frac{(-1)^v e^{-ikx}}{2(ik)^{1-v}}[I_n+o(1)]\,[ikA_1-B_1],\quad v=0,1,\quad x\in(0,a]. \tag{2.12}$$

Similarly, we also have that as $|k|\to\infty$ in $\mathbb{C}_\delta^+$,

$$\psi^{(v)}(x,\lambda)=\frac{e^{-ik(a-x)}}{2(ik)^{1-v}}[I_n+o(1)]\,[ikA_2-B_2],\quad v=0,1,\quad x\in[0,a), \tag{2.13}$$

$$\varphi_1^{(v)}(x,\lambda)=\frac{(-1)^v e^{-ikx}}{2(ik)^{1-v}}[I_n+o(1)]\,[ikB_1+A_1],\quad v=0,1,\quad x\in(0,a], \tag{2.14}$$

$$\psi_1^{(v)}(k,x)=-\frac{e^{-ik(a-x)}}{2(ik)^{1-v}}[I_n+o(1)]\,[ikB_2+A_2],\quad v=0,1,\quad x\in[0,a), \tag{2.15}$$

3. Eigenvalues and normalization matrices

In this section, we give some properties of the eigenvalues and the normalization matrices. The specifical forms of the normalization matrices are presented in terms of the initial value solutions φ and ψ.

Denote

$$\Delta(\lambda):=[\psi(x,\bar\lambda)^\dagger;\varphi(x,\lambda)]=\psi(0,\bar\lambda)^\dagger B_1-\psi'(0,\bar\lambda)^\dagger A_1=A_2^\dagger\varphi'(a,\lambda)+B_2^\dagger\varphi(a,\lambda), \tag{3.1}$$

which is called the *characteristic matrix*. Then we have

$$\Delta(\bar\lambda)^\dagger=-[\varphi(x,\bar\lambda)^\dagger;\psi(x,\lambda)]=\varphi'(a,\bar\lambda)^\dagger A_2+\varphi(a,\bar\lambda)^\dagger B_2=B_1^\dagger\psi(0,\lambda)-A_1^\dagger\psi'(0,\lambda). \tag{3.2}$$

Proposition 3.1. *λ_0 is an eigenvalue of the problem $L(Q,U_1,U_2)$ if and only if $\ker\Delta(\lambda_0)$ is nontrivial, i.e., $\det\Delta(\lambda_0)=0$. Moreover, any eigenvector function must be $\varphi(x,\lambda_0)\xi$ (or $\psi(x,\lambda_0)\eta$) for some nonzero column vector $\xi\in\ker\Delta(\lambda_0)$ (or $\eta\in\ker\Delta(\lambda_0)^\dagger$). For the eigenvalue λ_0, there is a bijection $\xi\to\eta$ between $\ker\Delta(\lambda_0)$ and $\ker[\Delta(\lambda_0)^\dagger]$ in such a way that $\varphi(x,\lambda_0)\xi=\psi(x,\lambda_0)\eta$.*

Proof. If $\det\Delta(\lambda_0)=0$ for some $\lambda_0\in\mathbb{C}$, then there exists at least one nonzero column vector $\xi\in\ker\Delta(\lambda_0)$, namely,

$$0_n=[\psi(x,\bar\lambda_0)^\dagger;\varphi(x,\lambda_0)\xi]_{x=a}=A_2^\dagger\varphi'(0,\lambda_0)\xi+B_2^\dagger\varphi(0,\lambda_0)\xi.$$

Note that $\varphi(x,\lambda_0)\xi$ satisfies the left boundary condition in (1.2). Thus, $\varphi(x,\lambda)\xi$ is the nonzero column solution to (1.1) and satisfies (1.2). This implies that λ_0 is a certain eigenvalue of the problem $L(Q,U_1,U_2)$.

Assume that λ_0 is an certain eigenvalue, and the corresponding eigenvector function is $\omega(x,\lambda_0)$ which satisfies (1.2). From (1.3), (2.2) and (2.3) we see that $\{\varphi,\varphi_1\}$ and $\{\psi,\psi_1\}$ are both the fundamental solutions to (1.1). Thus, there exist column vectors $\xi,\xi_1,\eta,\eta_1\in\mathbb{C}^n$ such that

$$\omega(x,\lambda_0)=\varphi(x,\lambda_0)\xi+\varphi_1(x,\lambda_0)\xi_1=\psi(x,\lambda_0)\eta+\psi_1(x,\lambda_0)\eta_1. \tag{3.3}$$

Since $T_1(\omega)=0_n=T_2(\omega)$, we have

$$0_n=A_1^\dagger\omega'(0,\lambda_0)-B_1^\dagger\omega(0,\lambda_0)=-\xi_1,\quad 0_n=A_2^\dagger\omega'(\pi,\lambda_0)+B_2^\dagger\omega(\pi,\lambda_0)=-\eta_1. \tag{3.4}$$

From (3.1)-(3.4), we get

$$0_n=A_1^\dagger\psi'(0,\lambda_0)\eta-B_1^\dagger\psi(0,\lambda_0)\eta=[\varphi(x,\bar\lambda_0)^\dagger;\psi(x,\lambda_0)]\eta=-\Delta(\bar\lambda_0)^\dagger\eta,$$

and

$$0_n=A_2^\dagger\varphi'(0,\lambda_0)\xi+B_2^\dagger\varphi(0,\lambda_0)\xi=[\psi(x,\bar\lambda_0)^\dagger;\varphi(x,\lambda_0)]\xi=\Delta(\lambda_0)\xi,$$

which imply, respectively, $0_n\neq\eta\in\ker[\Delta(\lambda_0)^\dagger]$ (the following proposition shows $\lambda_0=\bar\lambda_0$) and $0_n\neq\xi\in\ker\Delta(\lambda_0)$. Thus, $\det\Delta(\lambda_0)=0$. □

Proposition 3.2. *The eigenvalues of the problem $L(Q,U_1,U_2)$ are all real. Eigenvector functions related to different eigenvalues are orthogonal in $L^2((0,a);\mathbb{C}^n)$.*

Proof. Let λ_1 be an certain eigenvalues of the problem $L(Q,U_1,U_2)$, and $\omega_1(x)$ be the corresponding eigenvector functions. From the proof of Proposition 2.1, we see $\omega_1(x)=\varphi(x,\lambda_1)\xi_1=\psi(x,\lambda_1)\eta_1$ for some $\xi_1,\eta_1\in\mathbb{C}^n$. By integration by parts and, and using (2.2) and (2.3), we have

$$\int_0^a[\omega_1''(x)^\dagger+\omega_1(x)^\dagger Q(x)]\omega_1(x)dx=\int_0^a\omega_1(x)^\dagger[\omega_1''(x)+Q(x)\omega_1(x)]dx,$$

which implies

$$(\lambda_1-\bar\lambda_1)\int_0^a\omega_1(x)^\dagger\omega_1(x)dx=0. \tag{3.5}$$

It follows that if $\lambda_1\neq\bar\lambda_1$ then $\omega_1(x)=0$ a.e. on $(0,a)$. This is impossible.

Let λ_2 be a different eigenvalue from λ_1, and $\omega_2(x)$ be the corresponding eigenvector functions. Using a similar method to (3.5) and noting $\lambda_2=\bar\lambda_2$, we obtain

$$(\lambda_1-\lambda_2)\int_0^a\omega_1(x)^\dagger\omega_2(x)dx=0, \tag{3.6}$$

which implies $\int_0^a\omega_1(x)^\dagger\omega_2(x)dx=0$. □

Let $\{\lambda_j\}_{j\geq0}$ be all the eigenvalues of the problem $L(Q,U_1,U_2)$.

Proposition 3.3. *The inverse of the the characteristic matrix $\Delta(\lambda)^{-1}$ has a simple poles at λ_j $(j \geq 0)$, namely,*

$$\Delta(\lambda)^{-1} = \frac{N_{-1,j}}{\lambda - \lambda_j} + N_0 + O(\lambda - \lambda_j), \quad \lambda \to \lambda_j, \tag{3.7}$$

where $N_{-1,j}$ is a nonzero constant matrix.

Proof. Differentiating (1.1) for $Y = \varphi$ with respect to λ and evaluating it at $\lambda = \lambda_j$ yield

$$-\dot{\varphi}''(x,\lambda_j) + Q(x)\dot{\varphi}(x,\lambda_j) = \lambda\dot{\varphi}(x,\lambda_j) + \varphi(x,\lambda_j). \tag{3.8}$$

Premultiplying (3.8) by $\varphi(x,\lambda_j)^\dagger$, postmultiplying (2.1) for $Z = \varphi(x,\lambda_j)^\dagger$ by $\dot{\varphi}(x,\lambda_j)$, taking the difference, and integrating on $[0,a]$, we get

$$\begin{aligned}\int_0^a \varphi(x,\lambda_j)^\dagger\varphi(x,\lambda_j) =& [\varphi'(x,\lambda_j)^\dagger\dot{\varphi}(x,\lambda_j) - \varphi(x,\lambda_j)^\dagger\dot{\varphi}'(x,\lambda_j)]|_0^a \\ =& \varphi'(a,\lambda_j)^\dagger\dot{\varphi}(a,\lambda_j) - \varphi(a,\lambda_j)^\dagger\dot{\varphi}'(a,\lambda_j).\end{aligned} \tag{3.9}$$

Here, we have used the fact that $\dot{\varphi}(0,\lambda) \equiv 0_n \equiv \dot{\varphi}'(0,\lambda)$ for all $\lambda \in \mathbb{C}$.

Let $\xi \in \ker\Delta(\lambda_j)$ and $\eta \in \ker[\Delta(\lambda_j)^\dagger]$ such that $\varphi(x,\lambda_j)\xi = \psi(x,\lambda_j)\eta$. It follows from (3.9) and (3.1) that

$$\int_0^a \xi^\dagger\varphi(x,\lambda_j)^\dagger\varphi(x,\lambda_j)\xi\,dx = -\eta^\dagger\dot{\Delta}(\lambda_j)\xi. \tag{3.10}$$

Since $\det\Delta(\lambda_j) = 0$, we have that as $\lambda \to \lambda_j$,

$$\Delta(\lambda) = \Delta(\lambda_j) + (\lambda - \lambda_j)\dot{\Delta}(\lambda_j) + O((\lambda - \lambda_j)^2), \tag{3.11}$$

$$\Delta(\lambda)^{-1} = \frac{N_{-p,j}}{(\lambda - \lambda_j)^p} + \frac{N_{-p+1,j}}{(\lambda - \lambda_j)^{p-1}} + \cdots + \frac{N_{-1,j}}{\lambda - \lambda_j} + N_0 + O(\lambda - \lambda_j), \tag{3.12}$$

where $\{N_{-s,j}\}_{s=1}^p$ are constant matrices and at least one of them is nonzero.

Since $\Delta(\lambda)\Delta(\lambda)^{-1} = I_n$, we get that if $p \geq 2$ then

$$\Delta(\lambda_j)N_{-p,j} = 0_n, \quad \Delta(\lambda_j)N_{-p+1,j} + \dot{\Delta}(\lambda_j)N_{-p,j} = 0_n. \tag{3.13}$$

It follows that each column in $N_{-p,j}$, denoted also by ξ, belongs to $\ker\Delta(\lambda_j)$. Let $\eta \in \ker\Delta(\lambda_j)^\dagger$ such that $\varphi(x,\lambda_j)\xi = \psi(x,\lambda_j)\eta$. It follows from $\eta^\dagger\Delta(\lambda_j) = 0_n$ and (3.13) that

$$\eta^\dagger\dot{\Delta}(\lambda_j)\xi = 0_n, \tag{3.14}$$

which implies from (3.10) that $\xi = 0_n$. Thus, $N_{-p,j} = 0_n$ if $p \geq 2$. The proof is complete. □

Denote

$$C_j := P_j\left[P_j\int_0^a \varphi(x,\lambda_j)^\dagger\varphi(x,\lambda_j)\,dx P_j + I_n - P_j\right]^{-1}, \tag{3.15}$$

and

$$C_{1,j} := P_{1,j}\left[P_{1,j}\int_0^a \psi(x,\lambda_j)^\dagger\psi(x,\lambda_j)dxP_{1,j}+I_n-P_{1,j}\right]^{-1}, \tag{3.16}$$

which are called *normalization matrices*, where P_j and $P_{1,j}$ are the orthogonal projections onto $\ker\Delta(\lambda_j)$ and $\ker[\Delta(\lambda_j)^\dagger]$, respectively, $j\geq 0$.

Consider the *Weyl solutions* $\Phi(x,\lambda)$ and $\Phi_1(x,\lambda)$ of the problem $L(Q,U_1,U_2)$, which satisfy (1.1) with the conditions $T_1(\Phi)=I_n$, $T_2(\Phi)=0_n$ and $T_1(\Phi_1)=0_n$, $T_2(\Phi_1)=I_n$, respectively. It is easy to show that the Weyl solutions Φ and Φ_1 uniquely exist. Denote

$$M(\lambda):=V_1(\Phi):=A_1^\dagger\Phi(0,\lambda)+B_1^\dagger\Phi'(0,\lambda),\ M_1(\lambda):=V_2(\Phi_1):=A_2^\dagger\Phi_1(a,\lambda)-B_2^\dagger\Phi_1'(a,\lambda), \tag{3.17}$$

which are called *Weyl matrices*. Using (1.3), (2.2), (2.3), (3.1) and (3.2), it is easy to verify

$$\Phi(x,\lambda) = -\psi(x,\lambda)[\Delta(\bar\lambda)^\dagger]^{-1} = -\varphi_1(x,\lambda)+\varphi(x,\lambda)M(\lambda), \tag{3.18}$$

$$M(\lambda) = -V_1(\psi)[\Delta(\bar\lambda)^\dagger]^{-1} = T_2(\varphi)^{-1}T_2(\varphi_1) = \Delta(\lambda)^{-1}T_2(\varphi_1), \tag{3.19}$$

$$\Phi_1(x,\lambda) = \varphi(x,\lambda)\Delta(\lambda)^{-1} = -\psi_1(x,\lambda)+\psi(x,\lambda)M_1(\lambda), \tag{3.20}$$

$$M_1(\lambda) = V_2(\varphi)\Delta(\lambda)^{-1} = T_1(\psi)^{-1}T_1(\psi_1) = -[\Delta(\bar\lambda)^\dagger]^{-1}T_1(\psi_1). \tag{3.21}$$

Remark 3.1. From the above discussion, we see that if $U_1=U_2$ and $Q(x)=Q(a-x)$, then $A_1=A_2$ and $B_1=B_2$, $\varphi(x,,\lambda)=\psi(a-x,\lambda)$. It follows from (3.1) and (3.2) that $\Delta(\lambda)=\Delta(\bar\lambda)^\dagger$, and so $P_j=P_{1,j}$ and $C_j=C_{1,j}$. It also follows from (3.18)-(3.21) that $\Phi(x,\lambda)=-\Phi_1(a-x,\lambda)$ and $M(\lambda)=-M_1(\lambda)$.

The following theorem gives the relations between the Weyl solutions and normalization matrices.

Theorem 3.1. *For all $j\geq 0$, the matrices C_j and $C_{1,j}$ defined in (3.15) and (3.16) are positive semi-definite. Moreover, the following equations hold:*

$$\operatorname*{Res}_{\lambda=\lambda_j}\Phi(x,\lambda) = \varphi(x,\lambda_j)C_j,\quad \operatorname*{Res}_{\lambda=\lambda_j}M(\lambda) = \operatorname*{Res}_{\lambda=\lambda_j}V_1(\Phi)=C_j, \tag{3.22}$$

$$\operatorname*{Res}_{\lambda=\lambda_j}\Phi_1(x,\lambda) = -\psi(x,\lambda_j)C_{1,j},\quad \operatorname*{Res}_{\lambda=\lambda_j}M_1(\lambda) = \operatorname*{Res}_{\lambda=\lambda_j}V_2(\Phi_1)=-C_{1,j}. \tag{3.23}$$

Proof. We only prove (3.22), and one can prove (3.23) similarly. From (3.7), we have

$$[\Delta(\bar\lambda)^\dagger]^{-1} = \frac{N_{-1,j}^\dagger}{\lambda-\lambda_j}+N_0^\dagger+O(\lambda-\lambda_j),\quad \lambda\to\lambda_j, \tag{3.24}$$

which implies from (3.18) that

$$\operatorname*{Res}_{\lambda=\lambda_j}\Phi(x,\lambda) = -\lim_{\lambda\to\lambda_j}\psi(x,\lambda)(\lambda-\lambda_j)[\Delta(\bar\lambda)^\dagger]^{-1} = -\psi(x,\lambda)N_{-1,j}^\dagger, \tag{3.25}$$

Using (1.3), (3.1) and (3.2), by a direct calculation, one can verify

$$\begin{cases} A_2U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j=[i\Delta(\lambda_j)+\varphi(a,\lambda_j)]P_j=\varphi(a,\lambda_j)P_j, \\ -B_2U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j=[-\Delta(\lambda_j)+\varphi'(a,\lambda_j)]P_j=\varphi'(a,\lambda_j)P_j, \end{cases} \tag{3.26}$$

which implies from (2.3) that

$$\psi(x,\lambda_j)U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(0,\lambda_j)]P_j=\varphi(x,\lambda_j)P_j. \tag{3.27}$$

Note that $[\varphi(x,\bar{\lambda})^\dagger;\varphi(x,\lambda)]=0_n$. It follows from (3.26) and (3.2) that

$$\Delta(\lambda_j)^\dagger U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j=0_n. \tag{3.28}$$

From (3.7) and (3.11), and noting $\Delta(\lambda)\Delta(\lambda)^{-1}=I_n$, we have

$$\Delta(\lambda_j)N_{-1,j}=0_n,\quad N_{0,j}^\dagger\Delta(\lambda_j)^\dagger+N_{-1,j}^\dagger\dot{\Delta}(\lambda_j)^\dagger=I_n. \tag{3.29}$$

It follows from the first equation in (3.29) that $P_jN_{-1,j}=N_{-1,j}$, or equivalently,

$$N_{-1,j}^\dagger P_j=N_{-1,j}^\dagger. \tag{3.30}$$

Using (3.28) and the second equation in (3.29), together with (3.1), (3.9) and (3.26), we get

$$\begin{aligned} U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j=&N_{-1,j}^\dagger\dot{\Delta}(\lambda_j)U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j \\ =&N_{-1,j}^\dagger[\dot{\varphi}'(a,\lambda_j)^\dagger A_2+\dot{\varphi}(a,\lambda_j)^\dagger B_2]U_2^\dagger[\varphi(a,\lambda_j)+i\varphi'(a,\lambda_j)]P_j \\ =&N_{-1,j}^\dagger[\dot{\varphi}'(a,\lambda_j)^\dagger\varphi(a,\lambda_j)-\dot{\varphi}(a,\lambda_j)^\dagger\varphi'(a,\lambda_j)]P_j \\ =&-N_{-1,j}^\dagger\int_0^a\varphi(x,\lambda_j)^\dagger\varphi(x,\lambda_j)dxP_j. \end{aligned} \tag{3.31}$$

Going back to (3.27), and using (3.31) and (3.30), we obtain

$$\varphi(x,\lambda_j)P_j=-\psi(x,\lambda_j)N_{-1,j}^\dagger\int_0^a\varphi(t,\lambda_j)^\dagger\varphi(t,\lambda_j)dtP_j=-\psi(x,\lambda_j)N_{-1,j}^\dagger F_j, \tag{3.32}$$

where $F_j=P_j\int_0^a\varphi(t,\lambda_j)^\dagger\varphi(t,\lambda_j)dtP_j+I_n-P_j$, and we have used

$$N_{-1,j}^\dagger(I_n-P_j)=N_{-1,j}^\dagger-N_{-1,j}^\dagger P_j=0_n.$$

It is obvious that F_j is positive definite, and satisfies $F_jP_j=P_jF_j=P_jF_jP_j$. Hence $P_jF_j^{-1}=F_j^{-1}P_j=P_jF_j^{-1}P_j=C_j$, which is obviously positive semi-definite. It follows from (3.32) that

$$\varphi(x,\lambda_j)C_j=-\psi(x,\lambda_j)N_{-1,j}^\dagger,$$

which, together with (3.25) and (3.19), implies (3.22).

Using a similar method, one can prove (3.23). The proof is complete. □

4. Inverse problems

In this section, we consider the inverse problem. We will show that the Weyl matrix uniquely determines the potential and the boundary conditions. Under the additional information on the the boundary conditions (see (4.1) or (4.2)), we show that the eigenvalues and the normalization matrices uniquely determine the potential and the matrices in the boundary conditions. If A_1 (or A_2) is known to be invertible or to be zero matrix like the cases considered in [7, 8, 9, 24, 22, 23], then the addition conditions (4.1) (or (4.2)) obviously holds. However, it is not clear that whether the results hold without these restrictions.

Together with the problem $L(Q,U_1,U_2)$, we consider the problem $L(\widetilde{Q},\widetilde{U}_1,\widetilde{U}_2)$ of the same form but with different $\widetilde{U_1}$, $\widetilde{U_2}$ and $\widetilde{Q}(x)$. We agree that if a certain symbol δ denotes an object related to $L(Q,U_1,U_2)$, then $\widetilde{\delta}$ will denote an analogous object related to $L(\widetilde{Q},\widetilde{U}_1,\widetilde{U}_2)$.

Lemma 4.1. *If $M(\lambda)=\widetilde{M}(\lambda)$ then as $|k|\to\infty$ in $\mathbb{C}_\delta^+$,*

$$[ikA_1-B_1][ik\widetilde{A}_1-\widetilde{B}_1]^{-1}=O(1),\quad [ikA_1^\dagger-B_1^\dagger]^{-1}[ik\widetilde{A}_1^\dagger-\widetilde{B}_1^\dagger]=O(1). \tag{4.1}$$

If $M_1(\lambda)=\widetilde{M}_1(\lambda)$ then as $|k|\to\infty$ in $\mathbb{C}_\delta^+$,

$$[ikA_2-B_2][ik\widetilde{A}_2-\widetilde{B}_2]^{-1}=O(1),\quad [ikA_2^\dagger-B_2^\dagger]^{-1}[ik\widetilde{A}_2^\dagger-\widetilde{B}_2^\dagger]=O(1). \tag{4.2}$$

Proof. We agree that the following "$o(1)$" and "$O(1)$" are under $|k|\to\infty$ in $\mathbb{C}_\delta^+$. With the help of (2.12) and (3.1), we get

$$\Delta(\lambda)=\frac{e^{-ika}}{2ik}\left[ikA_2^\dagger-B_2^\dagger\right][I_n+o(1)][B_1-ikA_1]. \tag{4.3}$$

It follows from (2.14) and the last equation in (3.19) that

$$M(\lambda)=[ikA_1-B_1]^{-1}[I_n+o(1)][A_1+ikB_1]. \tag{4.4}$$

Similar, we also obtain

$$M_1(\lambda)=-[ikA_2-B_2]^{-1}[I_n+o(1)][A_2+ikB_2]. \tag{4.5}$$

We only prove (4.1) with the help of (4.4). Using (4.5), one can also prove (4.2).

Since (4.4), and $M(\lambda)=\widetilde{M}(\lambda)$, we have

$$[ikA_1-B_1]^{-1}[I_n+o(1)][A_1+ikB_1]=[ik\widetilde{A}_1-\widetilde{B}_1]^{-1}[I_n+o(1)][\widetilde{A}_1+ik\widetilde{B}_1],$$

which implies

$$I_n+o(1)=[ikA_1-B_1][ik\widetilde{A}_1-\widetilde{B}_1]^{-1}[I_n+o(1)][\widetilde{A}_1+ik\widetilde{B}_1][A_1+ikB_1]^{-1}. \tag{4.6}$$

Since A_1 and B_1 has the form of (1.3), there exists an unitary matrix N such that

$$N^\dagger A_1 N = \operatorname{diag}\left\{\frac{e^{2i\theta_1}+1}{2},\dots,\frac{e^{2i\theta_s}+1}{2}, I_t, 0_p\right\} := \begin{bmatrix} \Theta_s & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & 0_p \end{bmatrix},$$

$$N^\dagger B_1 N = \operatorname{diag}\left\{\frac{i(e^{2i\theta_1}-1)}{2},\dots,\frac{i(e^{2i\theta_s}-1)}{2}, 0_t, -iI_p\right\} := \begin{bmatrix} \Gamma_s & 0 & 0 \\ 0 & 0_t & 0 \\ 0 & 0 & -iI_p \end{bmatrix},$$

where $\theta_j \in (0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$ and $s+t+p=n$. The similar equations hold for $\widetilde{A}_1$ and $\widetilde{B}_1$ with $\widetilde{\theta}_j, \widetilde{s}, \widetilde{t}$ and $\widetilde{p}$ satisfying $\widetilde{s}+\widetilde{t}+\widetilde{p}=n$. It follows that

$$[ikA_1-B_1][ik\widetilde{A}_1-\widetilde{B}_1]^{-1} = N\begin{bmatrix} ik\Theta_s-\Gamma_s & 0 & 0 \\ 0 & ikI_t & 0 \\ 0 & 0 & iI_p \end{bmatrix} N^\dagger\widetilde{N}\begin{bmatrix} (ik\widetilde{\Theta}_{\widetilde{s}}-\widetilde{\Gamma}_{\widetilde{s}})^{-1} & 0 & 0 \\ 0 & -ik^{-1}I_{\widetilde{t}} & 0 \\ 0 & 0 & -iI_{\widetilde{p}} \end{bmatrix}\widetilde{N}^\dagger, \tag{4.7}$$

$$[\widetilde{A}_1+ik\widetilde{B}_1][A_1+ikB_1]^{-1} = \widetilde{N}\begin{bmatrix} \widetilde{\Theta}_{\widetilde{s}}+ik\widetilde{\Gamma}_{\widetilde{s}} & 0 & 0 \\ 0 & I_{\widetilde{t}} & 0 \\ 0 & 0 & kI_{\widetilde{p}} \end{bmatrix}\widetilde{N}^\dagger N\begin{bmatrix} (\Theta_s+ik\Gamma_s)^{-1} & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & k^{-1}I_p \end{bmatrix} N^\dagger \tag{4.8}$$

Let us first show $s=\widetilde{s},\ t=\widetilde{t},\ p=\widetilde{p}$. Indeed, taking the determinants on both sides of (4.6), and using (4.7) and (4.8), we see that

$$\frac{k^{t-\widetilde{t}}}{k^{p-\widetilde{p}}}=O(1), \quad \frac{k^{p-\widetilde{p}}}{k^{t-\widetilde{t}}}=O(1),$$

which implies $t-\widetilde{t}=p-\widetilde{p}$. Since (4.4), we have

$$N^\dagger M(\lambda)N = \begin{bmatrix} (ik\Theta_s-\Gamma_s)^{-1} & 0 & 0 \\ 0 & -ik^{-1}I_t & 0 \\ 0 & 0 & -iI_p \end{bmatrix}[I_n+o(1)]\begin{bmatrix} \Theta_s+ik\Gamma_s & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & kI_p \end{bmatrix}. \tag{4.9}$$

Letting $|k|\to\infty$ in (4.9) for $k\in\mathbb{C}_\delta^+$, we get that the rank of the limit of the matrix $M(\lambda)k^{-1}$ is p. Since $M(\lambda)=\widetilde{M}(\lambda)$, we conclude $p=\widetilde{p}$, and so $t=\widetilde{t}$ and $s=\widetilde{s}$.

Denote

$$N^\dagger\widetilde{N} := \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix},$$

where n_{11}, n_{22} and n_{33} are, respectively, $s\times s$, $t\times t$ and $p\times p$ matrices. Then we have

$$N^\dagger[ikA_1-B_1][ik\widetilde{A}_1-\widetilde{B}_1]^{-1}\widetilde{N}$$

$$= \begin{bmatrix} (ik\Theta_s-\Gamma_s)n_{11} & (ik\Theta_s-\Gamma_s)n_{12} & (ik\Theta_s-\Gamma_s)n_{13} \\ ikn_{21} & ikn_{22} & ikn_{23} \\ in_{31} & in_{32} & in_{33} \end{bmatrix} \begin{bmatrix} (ik\widetilde{\Theta}_{\tilde{s}}-\widetilde{\Gamma}_{\tilde{s}})^{-1} & 0 & 0 \\ 0 & -ik^{-1}I_{\tilde{t}} & 0 \\ 0 & 0 & -iI_{\tilde{p}} \end{bmatrix}$$

$$= \begin{bmatrix} (ik\Theta_s-\Gamma_s)n_{11}(ik\widetilde{\Theta}_{\tilde{s}}-\widetilde{\Gamma}_{\tilde{s}})^{-1} & -ik^{-1}(ik\Theta_s-\Gamma_s)n_{12} & -i(ik\Theta_s-\Gamma_s)n_{13} \\ ikn_{21}(ik\widetilde{\Theta}_{\tilde{s}}-\widetilde{\Gamma}_{\tilde{s}})^{-1} & n_{22} & kn_{23} \\ in_{31}(ik\widetilde{\Theta}_{\tilde{s}}-\widetilde{\Gamma}_{\tilde{s}})^{-1} & k^{-1}n_{32} & n_{33} \end{bmatrix}, \tag{4.10}$$

and

$$\widetilde{N}^\dagger[\widetilde{A}_1+ik\widetilde{B}_1][A_1+ikB_1]^{-1}N$$

$$= \begin{bmatrix} (\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{11}^\dagger & (\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{21}^\dagger & (\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{31}^\dagger \\ n_{12}^\dagger & n_{22}^\dagger & n_{32}^\dagger \\ kn_{13}^\dagger & kn_{23}^\dagger & kn_{33}^\dagger \end{bmatrix} \begin{bmatrix} (\Theta_s+ik\Gamma_s)^{-1} & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & k^{-1}I_p \end{bmatrix}$$

$$= \begin{bmatrix} (\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{11}^\dagger(\Theta_s+ik\Gamma_s)^{-1} & (\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{21}^\dagger & k^{-1}(\widetilde{\Theta}_{\tilde{s}}+ik\widetilde{\Gamma}_{\tilde{s}})n_{31}^\dagger \\ n_{12}^\dagger(\Theta_s+ik\Gamma_s)^{-1} & n_{22}^\dagger & k^{-1}n_{32}^\dagger \\ kn_{13}^\dagger(\Theta_s+ik\Gamma_s)^{-1} & kn_{23}^\dagger & n_{33}^\dagger \end{bmatrix}. \tag{4.11}$$

Together with (4.6), (4.10) and (4.11), and letting $|k|\to\infty$ in $\mathbb{C}_\delta^+$, we get $n_{13}=0, n_{21}=0, n_{23}=0$, which implies from (4.10) and (4.11) that $[ikA_1-B_1][ik\widetilde{A}_1-\widetilde{B}_1]^{-1}=O(1)$ and $[ikA_1^\dagger+B_1^\dagger][ik\widetilde{A}_1^\dagger+\widetilde{B}_1^\dagger]^{-1}=O(1)$, respectively. Using a similar method, one can prove the equations in (4.2). The proof is complete. □

Theorem 4.2. *If either $M(\lambda)=\widetilde{M}(\lambda)$ or $M_1(\lambda)=\widetilde{M}_1(\lambda)$, then $U_1=\widetilde{U}_1$, $U_2=\widetilde{U}_2$ and $Q(x)=Q(x)$ a.e. on* $[0,a]$.

Proof. We only give the proof for $M(\lambda)$, and the proof for $M_1(\lambda)$ is similar. Using (2.13) and (3.2), we get

$$\Delta(\bar{\lambda})^\dagger=\frac{e^{-ika}}{2ik}[B_1^\dagger-ikA_1^\dagger][I_n+o(1)]\,[ikA_2-B_2]\,,\quad |k|\to\infty,\quad k\in\mathbb{C}_\delta^+, \tag{4.12}$$

which implies from (2.13) and (3.18) that

$$\Phi^{(\upsilon)}(x,\lambda)=(ik)^\upsilon e^{ikx}[I_n+o(1)][ikA_1^\dagger-B_1^\dagger]^{-1},\quad x\in[0,a),\quad \upsilon=1,2. \tag{4.13}$$

Using (2.4) and (2.5), it is easy to check that

$$[\varphi(x,\bar{\lambda})^\dagger;\Phi(x,\lambda)]=-[\Phi(x,\bar{\lambda})^\dagger;\varphi(x,\lambda)]=I_n,\ [\Phi(x,\bar{\lambda})^\dagger;\Phi(x,\lambda)]=0_n. \tag{4.14}$$

This, together with (2.4), implies that

$$\begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}^{-1} = \begin{bmatrix} \Phi'(x,\bar{\lambda})^\dagger & -\Phi(x,\bar{\lambda})^\dagger \\ -\varphi'(x,\bar{\lambda})^\dagger & \varphi(x,\bar{\lambda})^\dagger \end{bmatrix}. \tag{4.15}$$

Define the $2n \times 2n$ matrix

$$\begin{bmatrix} P_{11}(x,\lambda) & P_{12}(x,\lambda) \\ P_{21}(x,\lambda) & P_{22}(x,\lambda) \end{bmatrix} := \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix} \begin{bmatrix} \widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{bmatrix}^{-1}. \tag{4.16}$$

Using (4.15) in (4.16), we get

$$\begin{cases} P_{11}(x,\lambda) = \varphi(x,\lambda)\widetilde{\Phi}'(x,\bar{\lambda})^{\dagger} - \Phi(x,\lambda)\widetilde{\varphi}'(x,\bar{\lambda})^{\dagger}, \\ P_{12}(x,\lambda) = \Phi(x,\lambda)\widetilde{\varphi}(x,\bar{\lambda})^{\dagger} - \varphi(x,\lambda)\widetilde{\Phi}(x,\bar{\lambda})^{\dagger}. \end{cases} \tag{4.17}$$

It follows from (4.13) and (2.12) that as $|k| \to \infty$ in $\mathbb{C}_{\delta}^{+}$,

$$\begin{aligned} P_{11}(x,\lambda) = &\frac{1}{2}[I_n + o(1)][ikA_1 - B_1][ik\widetilde{A}_1 - \widetilde{B}_1]^{-1}[I_n + o(1)] \\ &+ \frac{1}{2}[I_n + o(1)][ikA_1^{\dagger} - B_1^{\dagger}]^{-1}[ik\widetilde{A}_1^{\dagger} - \widetilde{B}_1^{\dagger}][I_n + o(1)], \end{aligned} \tag{4.18}$$

$$\begin{aligned} P_{12}(x,\lambda) = &\frac{1}{2ik}[I_n + o(1)][ikA_1^{\dagger} - B_1^{\dagger}]^{-1}[ik\widetilde{A}_1^{\dagger} - \widetilde{B}_1^{\dagger}][I_n + o(1)] \\ &- \frac{1}{2ik}[I_n + o(1)][ikA_1 - B_1][ik\widetilde{A}_1 - \widetilde{B}_1]^{-1}[I_n + o(1)]. \end{aligned} \tag{4.19}$$

On the other hand, using (4.14) and (3.18), we have

$$M(\bar{\lambda})^{\dagger} = M(\lambda). \tag{4.20}$$

Substituting (3.18) into (4.17) and using $M(\lambda) = \widetilde{M}(\lambda)$ and (4.20), we obtain

$$\begin{cases} P_{11}(x,\lambda) = -\varphi(x,\lambda)\widetilde{\varphi}_1'(x,\bar{\lambda})^{\dagger} + \varphi_1(x,\lambda)\widetilde{\varphi}'(x,\bar{\lambda})^{\dagger}, \\ P_{12}(x,\lambda) = -\varphi_1(x,\lambda)\widetilde{\varphi}(x,\bar{\lambda})^{\dagger} + \varphi(x,\lambda)\widetilde{\varphi}_1(x,\bar{\lambda})^{\dagger}, \end{cases}$$

which implies that $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are both entire functions of λ of order $1/2$. Note that when $k \in \mathbb{C}_{\delta}^{+}$, we have $\lambda \in \{\lambda \in \mathbb{C} : \arg\lambda \in [2\delta, 2\pi - 2\delta]\}$. From Lemma 4.1, we see that

$$P_{11}(x,\lambda) = O(1), \quad P_{12}(x,\lambda) = O\left(\frac{1}{k}\right), \quad |k| \to \infty, \quad k \in \mathbb{C}_{\delta}^{+}.$$

Thus, by the Phragmén-Lindelöf theorem, we conclude $P_{11}(x,\lambda) \equiv C$ and $P_{12}(x,\lambda) \equiv 0_n$, which implies from (4.16) that $\varphi(x,\lambda) = C\widetilde{\varphi}(x,\lambda)$, here C is some constant matrix. Using (1.3) and (2.2), we have

$$U_1 + I_n = C(\widetilde{U}_1 + I_n), \quad U_1 - I_n = C(\widetilde{U}_1 - I_n),$$

which implies that $C = I_n$ and $U_1 = \widetilde{U}_1$. Thus, $\varphi(x,\lambda) = \widetilde{\varphi}(x,\lambda)$ and $Q(x) = \widetilde{Q}(x)$ a.e. on $(0,a)$. It follows from (4.16) that $\Phi(x,\lambda) = \widetilde{\Phi}(x,\lambda)$. Using the fact that $\widetilde{T}_2(\widetilde{\Phi}) = 0_n = T_2(\Phi)$, together with (1.3), we obtain that

$$(\widetilde{U}_2^{\dagger} - U_2^{\dagger})[\Phi'(a,\lambda) - i\Phi(a,\lambda)] = 0_n. \tag{4.21}$$

It is easy to show that the matrix $\Phi'(a,\lambda) - i\Phi(a,\lambda)$ is invertible (cf. [26]). Thus, $U_2 = \widetilde{U}_2$. The proof is finished. □

Corollary 4.1. *The characteristic matrix of the problem $L(Q,U_1,U_2)$, together with the characteristic matrix of either the problem (1.1) with the boundary conditions $T_2(Y) = 0_n$ and $V_1(Y) = 0_n$ or the problem (1.1) with the boundary conditions $T_1(Y) = 0_n$ and $V_2(Y) = 0_n$, uniquely determines the unitary matrices U_1 and U_2 and the potential $Q(x)$ a.e. on $[0,a]$, where $T_j(\cdot)$ and $V_j(\cdot)$ are respectively defined in (1.2) and (3.17), $j = 1,2$.*

Theorem 4.3. *If for all $j \geq 0$, $\lambda_j = \widetilde{\lambda}_j$ and either $C_j = \widetilde{C}_j$ and (4.1) holds or $C_{1,j} = \widetilde{C}_{1,j}$ and (4.2) holds, then $U_1 = \widetilde{U}_1$, $U_2 = \widetilde{U}_2$ and $Q(x) = Q(x)$ a.e. on $[0,a]$.*

Proof. We only prove this theorem for C_j and omit the proof for $C_{1,j}$. From (4.17), we see that $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are analytic in $\lambda \in \mathbb{C}$ except for the simple poles $\{\lambda_j\}_{j\geq 0}$. Since $C_j = \widetilde{C}_j$ for $j \geq 0$, by Theorem 3.1, we see that (4.17) implies

$$\begin{cases} \operatorname*{Res}_{\lambda=\lambda_j} P_{11}(x,\lambda) = \varphi(x,\lambda_j)C_j\widetilde{\varphi}'(x,\lambda_j)^\dagger - \varphi(x,\lambda_j)C_j\widetilde{\varphi}'(x,\lambda_j)^\dagger = 0_n, \\ \operatorname*{Res}_{\lambda=\lambda_j} P_{12}(x,\lambda) = \varphi(x,\lambda_j)C_j\widetilde{\varphi}(x,\lambda_j)^\dagger - \varphi(x,\lambda_j)C_j\widetilde{\varphi}(x,\lambda_j)^\dagger = 0_n. \end{cases}$$

It follows that $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire matrix-valued functions of λ. Following the proof of Theorem 4.1, we finish the proof of this theorem. □

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