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INVERSE NODAL PROBLEM FOR NONLOCAL DIFFERENTIAL OPERATORS

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Abstract. Inverse nodal problem consists in constructing operators from the given zeros of their eigenfunctions. The problem of differential operators with nonlocal boundary condition appears, e.g., in scattering theory, diffusion processes and the other applicable fields. In this paper, we consider a class of differential operators with nonlocal boundary condition, and show that the potential function can be determined by nodal data.

1. Introduction

In this work, we investigate the boundary value problem

$$l(y) := -y''(x) + v(x)y(1) = \lambda y(x),\ x \in (0,1), \tag{1.1}$$

subject to the boundary condition

$$y(0) = y'(1) + (y, v)_{L^2} = 0, \tag{1.2}$$

where the nonlocal potential function $v \in W_2^1[0,1] := \{f | f \in AC[0,1], f' \in L^2[0,1]\}$ and λ is the spectral parameter.

Problems of nonlocal boundary condition (1.2), e.g., in the theory of diffusion processes, where the generators of the Feller processes, or processes with Wentzell boundary conditions, usually involve nonlocal interactions both in the equation and the boundary conditions (see [12, 27] and the references therein). For some other applications and recently study of differential operators with nonlocal boundary conditions were investigated (see, for example, [2, 3, 6, 7, 9, 10, 16, 22, 32]).

In 2007, Albeverio, Hryniv and Nizhnik [1] considered the boundary value problem (1.1) and (1.2), where the potential function $v \in L^2[0,1]$. They proved that the differential operator corresponding to the boundary value problem (1.1) and (1.2) is self-adjoint and gave the

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asymptotic distribution of the eigenvalues: the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ satisfy the asymptotic distribution

$$\sqrt{\lambda_n} = \pi\left(n - \frac{1}{2}\right) + \frac{\mu_n}{n} \tag{1.3}$$

for some sequence $\{\mu_n\}_{n\ge1} \in l^2$. They also studied the inverse spectral problem and gave the algorithm to reconstruct the nonlocal potential function by all eigenvalues. Later, Nizhnik [24] considered the eigenvalue problem (1.1) with boundary conditions $y(0) = y(1)$ and $y'(1) - y'(0) + (y, v)_{L^2} = 0$. In [25] Nizhnik also considered the eigenvalue problem $-y''(x) + v(x)y(\pi) = \lambda y(x)$, $x \in (0, 2\pi)$, subject to the nonlocal boundary conditions similar to (1.2).

In this paper, we study the inverse nodal problem of the problem (1.1) and (1.2). Nodes are the zeros of eigenfunctions. Inverse nodal problem is to reconstruct potential function by nodal set, in some experiments, nodal set is easier to be observed and measured than the other spectral data. Inverse nodal problem for Sturm-Liouville operators with Dirichlet boundary conditions, see the original paper by McLaughlin [23]. and some generalizations were made in [4, 5, 8, 11, 13, 14, 15, 21, 26, 28, 29, 30, 31] and so on. For the other applications of nodal sets, for example, the paper [17] considered the zeros of Bessel functions and their application to the uniqueness of inverse acoustic scattering problem; the papers [18, 19, 20] showed that nodal sets of the Laplacian eigenfunctions play a critical role in establishing the uniqueness results for the inverse scattering problems.

The paper is organized as follows. In the next section, we study the eigenvalue asymptotics of the boundary value problem (1.1) and (1.2), and Section 3, we deal with the inverse nodal problem.

2. Eigenvalue asymptotics

Supposing $\lambda = z^2$, we define

$$\varphi(x;z) := \frac{\sin zx}{z} - \frac{\sin z}{z}\int_0^1 G_0(x,s;z)v(s)ds, \tag{2.1}$$

where

$$G_0(x,s;z) := \frac{1}{z\sin z}\begin{cases} \sin zx \sin z(1-s) & (s > x), \\ \sin z(1-x)\sin zs & (s < x), \end{cases}$$

then $\varphi(x;z), x \in [0,1]$, is a solution of equation $l(y) = z^2 y$ with the boundary condition $y(0) = 0$, and for fixed z, $\varphi(x;z)$ is the unique solution up to a scalar multiple.

Define

$$\begin{aligned} d(z) :=& \varphi'_x(1;z) + (\varphi, v)_{L^2} \\ =& \cos z + \int_0^1 \frac{\sin zs}{z}(v(s) + \overline{v(s)})ds - \frac{\sin z}{z}\int_0^1\int_0^1 G_0(x,s;z)v(s)\overline{v(x)}dsdx. \end{aligned} \tag{2.2}$$

In fact, $d(z)$ is often called the characteristic function of the boundary value problem (1.1) and (1.2), if $\tilde{z}$ is a zero of $d(z)$, then $\tilde{\lambda} = \tilde{z}^2$ is an eigenvalue of the boundary value problem (1.1) and (1.2), and vice versa (see [1]).

Lemma 2.1. *Suppose $\{\lambda_n\}_{n\ge 1}$ are eigenvalues of the boundary value problem (1.1) and (1.2), then we have the following asymptotic expressions*

$$\sqrt{\lambda_n} = \pi\Big(n-\frac{1}{2}\Big)+\frac{2(-1)^{n+1}\operatorname{Re}v(0)}{\pi^2 n^2}+\frac{\eta_n}{n^2},$$

where $\{\eta_n\}_{n\ge 1}$ is used for various sequences from l^2 in this paper, and $\operatorname{Re}v(0)$ denotes the real part of the number $v(0)$.

Proof. Note that

$$d(z) = \cos z+\int_0^1 \frac{2\sin zs}{z}\operatorname{Re}v(s)\,ds-\frac{\sin z}{z}\int_0^1\int_0^1 G_0(x,s;z)\,v(s)\overline{v(x)}\,ds\,dx. \tag{2.3}$$

Integrating (2.3) by parts, one gets

$$\int_0^1 \frac{2\sin zs}{z}\operatorname{Re}v(s)\,ds = 2\frac{\operatorname{Re}v(0)-\operatorname{Re}v(1)\cos z}{z^2}+\frac{2}{z^2}\int_0^1 \operatorname{Re}v'(s)\cos zs\,ds, \tag{2.4}$$

and

$$\begin{aligned}
&\frac{\sin z}{z}\int_0^1\int_0^1 G_0(x,s;z)\,v(s)\overline{v(x)}\,ds\,dx\\
&\quad=\frac{1}{z^2}\int_0^1\int_0^1\begin{cases}\sin zx\sin z(1-s)\,v(s)\overline{v(x)}\,ds\,dx & (s>x)\\ \sin z(1-x)\sin zs\,v(s)\overline{v(x)}\,ds\,dx & (s<x)\end{cases}\\
&\quad=\frac{1}{z^2}\int_0^1\Big\{\int_x^1 \sin zx\,\overline{v(x)}\,v(s)\,d\Big(\frac{\cos z(1-s)}{z}\Big)\\
&\qquad+\int_0^x \sin z(1-x)\,\overline{v(x)}\,v(s)\,d\Big(-\frac{\cos zs}{z}\Big)\Big\}dx\\
&\quad=\frac{1}{z^3}\int_0^1\Big\{\sin zx\,\overline{v(x)}\,v(1)-\sin zx\cos z(1-x)\,\overline{v(x)}\,v(x)\\
&\qquad-\int_x^1 \sin zx\cos z(1-s)\,v'(s)\overline{v(x)}\,ds-\cos zx\sin z(1-x)\,\overline{v(x)}\,v(x)\\
&\qquad+\sin z(1-x)\,\overline{v(x)}\,v(0)+\int_0^x \cos zs\sin z(1-x)\,v'(s)\overline{v(x)}\,ds\Big\}dx\\
&\quad=\frac{1}{z^3}O(e^{|\operatorname{Im} z|}).
\end{aligned}\tag{2.5}$$

From (2.3) – (2.5), we know that as $|z|\to\infty$,

$$d(z) = \cos z+2\frac{\operatorname{Re}v(0)-\operatorname{Re}v(1)\cos z}{z^2}+\frac{2}{z^2}\int_0^1 \operatorname{Re}v'(s)\cos zs\,ds+\frac{1}{z^3}O(e^{|\operatorname{Im} z|}). \tag{2.6}$$

In view of (1.3) we can suppose

$$z_n := \sqrt{\lambda_n} = \pi\Big(n - \frac{1}{2}\Big) + \varepsilon_n,\ \varepsilon_n = o\Big(\frac{1}{n}\Big), \tag{2.7}$$

and z_n is the zero of $d(z)$, therefore

$$\begin{aligned} 0 =& d(z_n) \\ =& d\Big(\pi\Big(n-\frac{1}{2}\Big)+\varepsilon_n\Big) \\ =& \cos\left(\pi\left(n-\frac{1}{2}\right)+\varepsilon_n\right) + 2\frac{\mathrm{Re}\,v(0) - \mathrm{Re}\,v(1)\cos(\pi(n-\frac{1}{2})+\varepsilon_n)}{(\pi(n-\frac{1}{2})+\varepsilon_n)^2} \\ &+ 2\frac{\int_0^1 \mathrm{Re}\,v'(s)\cos(\pi(n-\frac{1}{2})+\varepsilon_n)s\,ds}{(\pi(n-\frac{1}{2})+\varepsilon_n)^2} + \frac{1}{(\pi(n-\frac{1}{2})+\varepsilon_n)^3}O(1), \end{aligned}$$

which can be simplified into

$$(-1)^n \sin\varepsilon_n + \frac{2}{(\pi(n-\frac{1}{2})+\varepsilon_n)^2}[\mathrm{Re}\,v(0) - \mathrm{Re}\,v(1)(-1)^n \sin\varepsilon_n + \eta_n] = 0,$$

where $\{\eta_n\}_{n\geq 1} \in l^2$. Then

$$\sin\varepsilon_n = (-1)^{n+1}\frac{2}{(\pi(n-\frac{1}{2})+\varepsilon_n)^2}[\mathrm{Re}\,v(0) - \mathrm{Re}\,v(1)(-1)^n \sin\varepsilon_n + \eta_n], \tag{2.8}$$

which implies

$$\sin\varepsilon_n = O\Big(\frac{1}{n^2}\Big),$$

that is,

$$\varepsilon_n = O\Big(\frac{1}{n^2}\Big).$$

Taylor's expansion tells us

$$\sin\varepsilon_n = \varepsilon_n - \frac{1}{6}\varepsilon_n^3 + \cdots,$$

that is,

$$\sin\varepsilon_n = \varepsilon_n + O\Big(\frac{1}{n^6}\Big). \tag{2.9}$$

From (2.8) and (2.9), we get

$$\varepsilon_n = 2(-1)^{n+1}\frac{\mathrm{Re}\,v(0) + (-1)^{n+1}\mathrm{Re}\,v(1)\varepsilon_n}{(\pi(n-\frac{1}{2})+\varepsilon_n)^2} + \frac{\eta_n}{n^2},$$

thus

$$\varepsilon_n = 2(-1)^{n+1}\frac{\mathrm{Re}\,v(0)}{\pi^2(n-\frac{1}{2})^2} + \frac{\eta_n}{n^2},$$

that is,

$$\varepsilon_n = 2(-1)^{n+1}\frac{\mathrm{Re}\,v(0)}{\pi^2 n^2} + \frac{\eta_n}{n^2}.$$

Submitting it into (2.7), we have

$$z_n := \sqrt{\lambda_n} = \pi\Big(n-\frac{1}{2}\Big) + \frac{2(-1)^{n+1}\operatorname{Re} v(0)}{\pi^2 n^2} + \frac{\eta_n}{n^2}. \tag{2.10}$$

3. Inverse nodal problem

In this section, we suppose v is a real valued function. First we give the asymptotic estimation of eigenfunctions for the problem (1.1) and (1.2).

Lemma 3.1. *When n is large enough, the eigenfunction corresponding to the eigenvalue λ_n of the boundary value problem (1.1) and (1.2) satisfies the following asymptotic estimation*

$$\begin{aligned}\varphi(x, z_n) =& \frac{\sin\pi\left(n-\frac{1}{2}\right)x}{\pi(n-\frac{1}{2})} + \frac{1}{\pi^3 n^3}\Big\{2(-1)^{n+1}\operatorname{Re} v(0)\, x\cos\pi\Big(n-\frac{1}{2}\Big)x \\ &- v(1)\sin\pi\Big(n-\frac{1}{2}\Big)x - v(0)\sin\pi\Big(n-\frac{1}{2}\Big)(1-x) - (-1)^n v(x)\Big\} + o\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Proof. For simplicity, let $c_n = 2(-1)^{n+1}\operatorname{Re} v(0)$, then

$$z_n = \pi\Big(n-\frac{1}{2}\Big) + \frac{c_n}{\pi^2 n^2} + \frac{\eta_n}{n^2}.$$

Plugging z_n into (2.1), we obtain

$$\varphi(x; z_n) = \frac{\sin z_n x}{z_n} - \frac{\sin z_n}{z_n}\int_0^1 G_0(x, s; z_n)\, v(s)\, ds. \tag{3.1}$$

First we note that

$$\sin z_n x = \sin\pi\Big(n-\frac{1}{2}\Big)x + \frac{c_n x}{\pi^2 n^2}\cos\pi\Big(n-\frac{1}{2}\Big)x + \frac{\eta_n}{n^2},$$

and

$$\frac{1}{z_n} = \frac{1}{\pi(n-\frac{1}{2})} - \frac{c_n}{\pi^4 n^4} + \frac{\eta_n}{n^4}.$$

Combining the above two asymptotic estimations, we have

$$\frac{\sin z_n x}{z_n} = \frac{\sin\pi\left(n-\frac{1}{2}\right)x}{\pi(n-\frac{1}{2})} + \frac{c_n x}{\pi^3 n^3}\cos\pi\Big(n-\frac{1}{2}\Big)x + \frac{\eta_n}{n^3}. \tag{3.2}$$

Second we have

$$\begin{aligned}
&\frac{\sin z_n}{z_n}\int_0^1 G_0(x,s;z_n)v(s)ds\\
&=\frac{1}{z_n^3}\Big\{\sin z_n x\Big[v(1)-\cos z_n(1-x)v(x)-\int_x^1\cos z_n(1-s)v'(s)ds\Big]\\
&\quad+\sin z_n(1-x)\Big[v(0)-\cos z_n x v(x)+\int_0^x\cos z_n s v'(s)ds\Big]\Big\}\\
&=\frac{1}{\pi^3n^3}\Big\{\sin\pi\left(n-\frac{1}{2}\right)x\Big[v(1)-v(x)\cos\pi\left(n-\frac{1}{2}\right)(1-x)\Big]\\
&\quad+\sin\pi\left(n-\frac{1}{2}\right)(1-x)\Big[v(0)-v(x)\cos\pi\left(n-\frac{1}{2}\right)x\Big]\Big\}+o\Big(\frac{1}{n^3}\Big)\\
&=\frac{1}{\pi^3n^3}\Big\{v(1)\sin\pi\left(n-\frac{1}{2}\right)x+v(0)\sin\pi\left(n-\frac{1}{2}\right)(1-x)\\
&\quad+(-1)^n v(x)\Big\}+o\Big(\frac{1}{n^3}\Big).
\end{aligned}\tag{3.3}$$

Combining (3.2) and (3.3), together with $c_n=2(-1)^{n+1}\mathrm{Re}\,v(0)$, we have

$$\begin{aligned}
\varphi(x,z_n)=\frac{\sin\pi\left(n-\frac{1}{2}\right)x}{\pi(n-\frac{1}{2})}&+\frac{1}{\pi^3n^3}\Big\{2(-1)^{n+1}\mathrm{Re}\,v(0)x\cos\pi\left(n-\frac{1}{2}\right)x\\
&-v(1)\sin\pi\left(n-\frac{1}{2}\right)x-v(0)\sin\pi\left(n-\frac{1}{2}\right)(1-x)\\
&-(-1)^n v(x)\Big\}+o\Big(\frac{1}{n^3}\Big).
\end{aligned}\tag{3.4}$$

□

Next, we provide the nodal asymptotic estimation by the above lemma. For convenience, we set

$$\begin{aligned}
u(x,n):=&2(-1)^{n+1}v(0)x\cos\pi\left(n-\frac{1}{2}\right)x-v(1)\sin\pi\left(n-\frac{1}{2}\right)x\\
&-v(0)\sin\pi\left(n-\frac{1}{2}\right)(1-x)-(-1)^n v(x).
\end{aligned}$$

Then Eq. (3.4) can be written as

$$\varphi(x,z_n)=\frac{\sin\pi\left(n-\frac{1}{2}\right)x}{\pi(n-\frac{1}{2})}+\frac{1}{\pi^3n^3}u(x,n)+o\Big(\frac{1}{n^3}\Big).\tag{3.5}$$

From the above equation, we have

$$\varphi(x,z_n)=\frac{\sin\pi\left(n-\frac{1}{2}\right)x}{\pi(n-\frac{1}{2})}+O(\frac{1}{n^3}).$$

For n large enough, we can see that

$$\varphi\left(\frac{k-\frac{1}{2}}{n-\frac{1}{2}}\right)\varphi\left(\frac{k+\frac{1}{2}}{n-\frac{1}{2}}\right)<0,$$

where $k=1,2,\cdots,n-1$. So $\varphi(x,z_n)$ have at least one zero in every interval $\left[\frac{k-\frac{1}{2}}{n-\frac{1}{2}},\frac{k+\frac{1}{2}}{n-\frac{1}{2}}\right]$, for $k=1,2,\cdots,n-1$. Then, we will calculate the asymptotic estimations of these zeros.

Suppose $\varphi(x,z_n)$ have a zero $\widetilde{x}_n$ for the variable x, that is, $\varphi(\widetilde{x}_n,z_n)=0$, then

$$\sin\pi\left(n-\frac{1}{2}\right)\widetilde{x}_n=-\frac{1}{\pi^2n^2}u(\widetilde{x}_n,n)+o\Big(\frac{1}{n^2}\Big).$$

Note that $u(\widetilde{x}_n,n)=O(1)$, we obtain

$$\sin\pi\left(n-\frac{1}{2}\right)\widetilde{x}_n=O\Big(\frac{1}{n^2}\Big).$$

Let

$$\pi\Big(n-\frac{1}{2}\Big)\widetilde{x}_n=j\pi+\xi_n,$$

where $\xi_n=O(\frac{1}{n^2})$ and j is an integer. According to Taylor's expansion,

$$\begin{aligned}\sin(j\pi+\xi_n)&=(-1)^j\Big(\xi_n-\frac{1}{6}\xi_n^3+\cdots\Big)\\&=-\frac{1}{\pi^2n^2}u(\widetilde{x}_n,n)+o\Big(\frac{1}{n^2}\Big),\end{aligned}$$

we obtain

$$\xi_n=(-1)^{j+1}\frac{1}{\pi^2n^2}u(\widetilde{x}_n,n)+o\Big(\frac{1}{n^2}\Big),$$

then

$$\pi\left(n-\frac{1}{2}\right)\widetilde{x}_n=j\pi+(-1)^{j+1}\frac{1}{\pi^2n^2}u(\widetilde{x}_n,n)+o\Big(\frac{1}{n^2}\Big),$$

that is,

$$\widetilde{x}_n=\frac{j}{n-\frac{1}{2}}+(-1)^{j+1}\frac{u(\widetilde{x}_n,n)}{\pi^3n^3}+o\Big(\frac{1}{n^3}\Big).$$

From the above asymptotic expression, $\widetilde{x}_n$ is related to j, so we can define the zeros as

$$x_n^j:=\frac{j}{n-\frac{1}{2}}+(-1)^{j+1}\frac{u(x_n^j,n)}{\pi^3n^3}+o\Big(\frac{1}{n^3}\Big).$$

Plugging $x_n^j=\frac{j}{n-\frac{1}{2}}+O(\frac{1}{n^3})$ into $u(x,n)$, we can obtain

$$\begin{aligned}u(x_n^j,n)&=u\Big(\frac{j}{n-\frac{1}{2}}+O\Big(\frac{1}{n^3}\Big),n\Big)\\&=2(-1)^{n+j+1}v(0)\frac{j}{n-\frac{1}{2}}+(-1)^{n+j}v(0)-(-1)^n v\Big(\frac{j}{n-\frac{1}{2}}\Big)+O\Big(\frac{1}{n^2}\Big),\end{aligned}$$

which yields

$$x_n^j=\frac{j}{n-\frac{1}{2}}+\frac{(-1)^{n+1}v(0)\left[1-\frac{2j}{n-\frac{1}{2}}\right]+(-1)^{n+j}v\left(\frac{j}{n-\frac{1}{2}}\right)}{\pi^3n^3}+o\Big(\frac{1}{n^3}\Big). \tag{3.6}$$

It's easy to see that for large enough n and $1 \le j \le n-1$, the nodes $x_n^j \in (0,1)$. So, for n large enough, $\varphi(x, z_n)$ at least have $n-1$ zeros $x_n^j (1 \le j \le n-1)$ in the segment $(0,1)$. Then we can define the nodal subset $X_N = \{x_n^j | n \in \mathrm{N}, n > N, j = \overline{1, n-1}\}$, where N is large enough, and obviously, X_N is dense in $[0,1]$.

Next, we give the uniqueness theorem of inverse nodal problem for the problem (1.1) and (1.2).

Theorem 3.2. *If the following two boundary value problems*

$$\begin{cases} l(y) := -y''(x) + v(x)y(1) = \lambda y(x), x \in [0,1], \\ y(0) = y'(1) + (y, v)_{L^2} = 0, \end{cases} \qquad \text{(I)}$$

$$\begin{cases} \hat{l}(y) := -y''(x) + \hat{v}(x)y(1) = \lambda y(x), x \in [0,1], \\ y(0) = y'(1) + (y, \hat{v})_{L^2} = 0 \end{cases} \qquad \text{(II)}$$

have the same nodal subset, that is, $X_N = \hat{X}_N$, where N is large enough, then $v(x) \equiv \hat{v}(x), x \in [0,1]$.

Proof. First we shall proof $v(0) = \hat{v}(0)$.

Choose the nodal set of the boundary value problem (I): $\{x_n^1 | n > N\} \subset X_N$, when $n \to \infty$,

$$x_n^1 \to 0.$$

By (3.6) we have

$$(-1)^{n+1}\left[x_n^j - \frac{j}{n-\frac{1}{2}}\right]\pi^3 n^3 = v(0)\left[1 - \frac{2j}{n-\frac{1}{2}}\right] + (-1)^{j+1} v\left(\frac{j}{n-\frac{1}{2}}\right) + o(1),$$

therefore, adopting $\{x_n^1 | n > N\}$ and taking the limit in the above equation as $n \to \infty$, we have

$$\lim_{n\to\infty} (-1)^{n+1}\left[x_n^1 - \frac{1}{n-\frac{1}{2}}\right]\pi^3 n^3 = 2v(0). \tag{3.7}$$

Similarly, for the problem (II), choose $\{\hat{x}_n^1 | n > N\} \subset \hat{X}_N$, we also have

$$\lim_{n\to\infty} (-1)^{n+1}\left[\hat{x}_n^1 - \frac{1}{n-\frac{1}{2}}\right]\pi^3 n^3 = 2\hat{v}(0). \tag{3.8}$$

Since $X_N = \hat{X}_N$, we have

$$x_n^j = \hat{x}_n^j,\ for\ n > N\ and\ j = \overline{1, n-1}. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), then

$$v(0) = \hat{v}(0). \tag{3.10}$$

Second, since $\overline{X} = [0,1]$, for all $x \in [0,1]$, there exists $\{x_n^{j(n)}\} \subset X_N$ such that

$$\lim_{n\to\infty} x_n^{j(n)} = x.$$

Then, by (3.6), and taking into account v being continuous we obtain

$$\begin{aligned}
&\lim_{n\to\infty} (-1)^{n+j(n)} \left\{ \left[x_n^{j(n)} - \frac{j(n)}{n-\frac{1}{2}} \right] \pi^3 n^3 + (-1)^n v(0) \left[1 - \frac{2j(n)}{n-\frac{1}{2}} \right] \right\} \\
&\quad = \lim_{n\to\infty} \left[v\left(\frac{j(n)}{n-\frac{1}{2}} \right) + o(1) \right] \\
&\quad = v(x).
\end{aligned} \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we have

$$\begin{aligned}
v(x) &= \lim_{n\to\infty} (-1)^{n+j(n)} \left\{ \left[x_n^{j(n)} - \frac{j(n)}{n-\frac{1}{2}} \right] \pi^3 n^3 + (-1)^n v(0) \left[1 - \frac{2j(n)}{n-\frac{1}{2}} \right] \right\} \\
&= \lim_{n\to\infty} (-1)^{n+j(n)} \left\{ \left[\hat{x}_n^{j(n)} - \frac{j(n)}{n-\frac{1}{2}} \right] \pi^3 n^3 + (-1)^n \hat{v}(0) \left[1 - \frac{2j(n)}{n-\frac{1}{2}} \right] \right\} \\
&= \hat{v}(x).
\end{aligned}$$

Therefore, we get $v(x) \equiv \hat{v}(x)$. □

From the above theorem, we can give a reconstruct algorithm of inverse nodal problem.

Corollary 3.3. *Given the nodal subset X_N of the boundary value problem* (1.1) *and* (1.2)*, where N is large enough, we can reconstruct the potential function v.*

Reconstruct algorithm consists of the two steps as follows.

(1) Choose the nodal subset $\{x_n^1 | n > N\} \subset X_N$, reconstruct $v(0)$:

$$v(0) = \frac{1}{2} \lim_{n\to\infty} (-1)^{n+1} \left[x_n^1 - \frac{1}{n-\frac{1}{2}} \right] \pi^3 n^3.$$

(2) For all $x \in [0,1]$, choose a nodal subset $\{x_n^{j(n)} | n > N\}$ such that

$$\lim_{n\to\infty} x_n^{j(n)} = x.$$

Then reconstruct $v(x)$:

$$v(x) = \lim_{n\to\infty} (-1)^{n+j(n)} \left\{ \left[x_n^{j(n)} - \frac{j(n)}{n-\frac{1}{2}} \right] \pi^3 n^3 + (-1)^n v(0) \left[1 - \frac{2j(n)}{n-\frac{1}{2}} \right] \right\}.$$

Remark 3.1. In fact, if $Y \subset X_N$ satisfies

(1) $\overline{Y} = [0,1]$;

(2) $\exists\{x_n^{j(n)}|j(n)\ is\ odd\}\subset Y$ such that

$$\lim_{n\to\infty} x_n^{j(n)} = 0.$$

Then, similar to the proof of Theorem 3.2, Y is sufficient to recover the potential function v.

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