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INVERSE PROBLEM FOR STURM–LIOUVILLE OPERATORS ON A CURVE

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Abstract. The inverse spectral problem for the Sturm–Liouville equation with a piecewise-entire potential function and the discontinuity conditions for solutions on a rectifiable curve $\gamma \subset \mathbf{C}$ by the transfer matrix along this curve is studied. By the method of a unit transfer matrix the uniqueness of the solution to this problem is proved with the help of studying of the asymptotic behavior of the solutions to the Sturm-Liouville equation for large values of the spectral parameter module.

1. Introduction and main results

Inverse problems for the classical Sturm–Liouville equation

$$u''(z) + (Q(z) - \lambda^2)u(z) = 0 \tag{1}$$

in the case when z is real are well studied in various statements [1, 2, 4, 3, 5, 6]. Inverse spectral problems for the Sturm–Liouville equations with three independent complex-valued coefficients appear in the spectroscopy of planar inhomogeneous media [7, 8]. These problems have not been studied enough [9].

The Sturm–Liouville equation with three independent coefficients on a line segment can be transformed into the classical Sturm–Liouville equation on a curve in the complex plane by corresponding substitution [10]. Such a transformation could be one of the effective methods for studying the inverse problems for the Sturm-Liouville equations of different forms on a segment [11]. Unfortunately, even the direct problems for the Sturm–Liouville equations on curves are studied in very limited cases [14, 15, 11, 12, 13]. Up to recently among the inverse problems on curves only the problem of monodromy-free classical Sturm–Liouville equations

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with a integrable potential function on a piecewise smooth closed curve, which is the boundary of a convex bounded domain, has been studied quite fully [16]. However, the requirement of convexity of a closed curve significantly limits the scope of possible application of the results obtained in [16].

The first results in removing restrictions on the shape of the curve when considering the classical Sturm–Liouville operators on the curves were obtained in the paper [17] by constriction of the potential functions class under consideration to piecewise-entire functions, i.e., functions that on different parts of the curve coincide with different entire functions of the complex variable z almost everywhere . This constriction allowed to set the inverse problem of recovering of the potential function in the classical Sturm–Liouville equation on an unspecified continuous rectifiable curve of arbitrary shape (including self-intersecting) from given column or row of the transfer matrix and to formulate the conditions for the uniqueness of its solution. For this purpose, along with the traditional study of the asymptotic of solutions to the Sturm–Liouville equation at a large value of the spectral parameter module, the unit matrix method was used for the first time.

In this paper, the results obtained in [17, 18] are partially generalized to the case where on an arbitrary continuous rectifiable curve γ there exists a finite number of points at which the solution to equation (1) with piecewise-entire potential function Q and/or the solution derivative along the curve undergo discontinuities independent of the spectral parameter $\rho := \lambda^2$. The case when the curve, the potential function, the position of the solution jump points on the curve and the transition matrices in them are unknown is considered.

Denote: $\hat{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{\sigma}_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let a piecewise-entire function Q is defined on a continuous rectifiable curve $\gamma \subset \mathbf{C}$, which specified parametrically by the function $z = V(t)$ ($t \in [t_0, t_f]$), and points are given at which the solutions to classical Sturm–Liouville equation (1) and (or) their derivatives have discontinuities independent of the parameter $\rho := \lambda^2$.

In other words, let the potential function Q be bounded on the curve γ, and there exist an integer $N \geq 0$ and a set of numbers $T = \{t_j\}_0^{N+1} : t_0 < t_1 < \ldots < t_{N+1} \equiv t_f$ such that

$$Q(z) \overset{\text{a.e.}}{=} Q_m(z), \text{ for } z = V(t),\ t \in [t_m, t_{m+1}] \quad (m = \overline{0, N}), \tag{2}$$

where all Q_m are entire functions. Furthermore, let the functions $u(z)$ and $u'(z)$ satisfy the discontinuity conditions at the points $z_j := V(t_j)$ ($j = \overline{0, N+1}$) of the curve γ:

$$\begin{cases} \begin{pmatrix} u(V(t_0+0)) \\ u'(V(t_0+0)) \end{pmatrix} = \hat{\eta}^{(0)} \begin{pmatrix} u(V(t_0)) \\ u'(V(t_0)) \end{pmatrix}, \\ \begin{pmatrix} u(V(t_n+0)) \\ u'(V(t_n+0)) \end{pmatrix} = \hat{\eta}^{(n)} \begin{pmatrix} u(V(t_n-0)) \\ u'(V(t_n-0)) \end{pmatrix} \quad (n = \overline{1, N} \text{ for } N \geq 1), \\ \begin{pmatrix} u(V(t_{N+1})) \\ u'(V(t_{N+1})) \end{pmatrix} = \hat{\eta}^{(N+1)} \begin{pmatrix} u(V(t_{N+1}-0)) \\ u'(V(t_{N+1}-0)) \end{pmatrix}, \end{cases} \tag{3}$$

where transition matrices $\hat{\eta}^{(j)}$ ($j = \overline{0, N+1}$) are independent of the spectral parameter ρ. Moreover, if $N \geq 1$, then the following conditions are valid for any numbers $n \in \{1, \ldots, N\}$:

$$\hat{\eta}^{(n)} \neq \hat{I} \text{ or (and) } Q_n \neq Q_{n-1} \quad (n = \overline{1, N} \text{ for } N \geq 1). \tag{4}$$

Definition 1. If the conditions (2) and (4) are valid, equation (1) considered on the continuous rectifiable curve γ, supplemented by the conditions (3) of solution discontinuities, will be called equation of class D on the curve γ, and the points $z_j = V(t_j)$ ($j = \overline{0, N+1}$) — the characteristic points of the curve γ and equation (1) of class D on the curve γ. In this case, the ordered set

$$W := \{N, \{z_j, \hat{\eta}^{(j)}\}_0^{N+1}, \{Q_m\}_0^N\} \tag{5}$$

will be called the set of characteristic data of the curve γ and equation (1) of class D on γ.

Note that the prime symbol in equation (1) and further denotes the derivative with respect to z along a certain rectifiable curve γ given parametrically by the function $z = V(t)$, i.e., it is assumed that $f'(z) \equiv f'(V(t)) := \lim_{\delta\to 0}[f(V(t+\delta)) - f(V(t))]/[V(t+\delta) - V(t)]$. In addition, in the relations (3) $f(V(t\pm 0)) := \lim_{\delta\to 0, \delta>0} f(V(t\pm\delta))$. It is easy to make sure that if the function $f(z)$ is analytic in some domain of the complex plane, then at any point of that domain it has derivatives along any rectiable curve passing through this point, and these derivatives coincide with each other and are equal to the usual derivative $df(z)/dz$ of the function $f(z)$ at this point.

We emphasize that if the curve γ has points (parts) that are passed more than once, that is, correspond to two or more values (intervals of values) of the parameter t, then such points (parts) differ in the order of passage, and geometrically coinciding curves with different order of passage of the parts are considered different.

Definition 2. We will call $u(z)$ the solution to equation (1) of class D on a curve γ if the function $u(z)$ satisfies equation (1) almost everywhere on γ, is continuously differentiable at all its points except possibly characteristic, and satisfies all jump conditions (3).

Definition 3. Let $u_1(z), u_2(z)$ be solutions to equation (1) of class D on a curve γ with characteristic points z_j $(j = \overline{0, N+1})$ and

$$u_1(z_b) = 1,\ u_1'(z_b) = 0,\ u_2(z_b) = 0,\ u_2'(z_b) = 1\ \ (z_b \in \gamma,\ z_b \notin \{z_j\}_1^N \text{ for } N \geq 1). \tag{6}$$

The transfer matrix of equation (1) of class D between points z_b and z of the curve γ is said to be the matrix

$$\hat{P}(\gamma, z, z_b) = \begin{pmatrix} u_1(z) & u_2(z) \\ u_1'(z) & u_2'(z) \end{pmatrix}\ \ (z \in \gamma,\ z \notin \{z_j\}_1^N \text{ for } N \geq 1).$$

The transfer matrix between the start and end points of the curve will be called the transfer matrix along this curve.

Definition 4. The set of characteristic data (5) of equation (1) of class D will be called regular if $\det \hat{\eta}^{(0)} \neq 0$, $\det \hat{\eta}^{(N+1)} \neq 0$, and the following conditions are met

$$\det \hat{\eta}^{(n)} = 1,\ \ \hat{\eta}^{(n)} \notin \{\pm i \hat{\sigma}_3\}\ \ (n = \overline{1, N} \text{ for } N \geq 1), \tag{7}$$

$$\Delta z_m := z_{m+1} - z_m \neq 0\ \ (m = \overline{0, N}). \tag{8}$$

Definition 5. A curve γ is called regular if equation (1) of class D with a regular set of characteristic data is given on γ.

Theorem 1. *Each element of the transfer matrix $\hat{P}$ of equation* (1) *of class D on a regular curve is entire function of parameter ρ of order 1/2 and of normal type.*

Theorem 1 is proved in Section 3

Definition 6. The regular set of characteristic data (5) of equation (1) of class D will be called standard if $\det \hat{\eta}^{(0)} = 1$, $\hat{\eta}^{(0)} \notin \{\pm i \hat{\sigma}_3\}$ and the following conditions are met

$$\Re\{\eta_{11}^{(n)}\} > 0,\ \Im\{\eta_{11}^{(n)}\} \geq 0 \ \text{ or } \ \eta_{11}^{(n)} = 0,\ \Re\{\eta_{12}^{(n)}\} > 0,\ \Im\{\eta_{12}^{(n)}\} \geq 0\ (n = \overline{0, N}). \tag{9}$$

Lemma 1. *If the matrices $\hat{\eta}$ and $\hat{\eta}_1$ satisfy the conditions* (7) *and* (9)*, then the matrix $\hat{\eta}_1 \hat{\eta}^{-1}$ satisfies the conditions* (7).

Proof. By virtue of the first condition in (7) we have: $\det\left(\hat{\eta}_1 \hat{\eta}^{-1}\right) = \det \hat{\eta}_1 / \det \hat{\eta} = 1$.

Suppose that $\hat{\eta}_1 \hat{\eta}^{-1} = \pm i \hat{\sigma}_3$ and hence $\hat{\eta}_1 = \pm i \hat{\sigma}_3 \hat{\eta}$, i.e., $\eta_{1,11} = \pm i \eta_{11}$, $\eta_{1,12} = \pm i \eta_{12}$. But the last two relations contradict the fact that both matrices $\hat{\eta}$ and $\hat{\eta}_1$ satisfy the conditions (9). This contradiction proves the Lemma. □

Definition 7. A curve with a given equation (1) of class D with standard set of characteristic data will be called the standard one.

The following Theorem contains the main result of the paper.

Theorem 2. *Let two equations* (1) *of class D have, respectively, standard sets of characteristic data* $W^{(1)}$, $W^{(2)}$ *and transfer matrices* $\hat{P}^{(1)}$, $\hat{P}^{(2)}$ *along two curves* $\gamma^{(1)}$, $\gamma^{(2)}$ *with a common starting point. Then* $\hat{P}^{(1)}(\rho) \equiv \hat{P}^{(2)}(\rho)$ *if and only if* $W^{(1)} = W^{(2)}$.

The proof of Theorem 2 is given in Section 4

2. Asymptotics of the transfer matrix along a regular curve

Lemma 2. *The elements of the transfer matrix* $\hat{P}$ *of equation* (1) *of class D along the curve* γ *are uniquely determined by specifying the set of characteristic data* (5) *of this curve; they are entire functions of the spectral parameter* ρ $(\rho = \lambda^2)$; $\det \hat{P} = \prod_{j=0}^{N+1} \det \hat{\eta}^{(j)}$ *and*

$$\hat{P}(\gamma, z_{N+1}, z_0) = \hat{\eta}^{(N+1)} \hat{P}^{(N)} \hat{\eta}^{(N)} \ldots \hat{P}^{(0)} \hat{\eta}^{(0)}, \tag{10}$$

where $\hat{P}^{(m)} := \lim_{\delta \to 0, \delta > 0} \hat{P}(\gamma, V(t_{m+1} - \delta), V(t_m + \delta))$ $(m \in \{0, \ldots, N\})$ *is the transfer matrix of equation* (1) *between points* z_m *and* z_{m+1} *of the curve* γ *in the absence of solutions discontinuities. If, moreover,* $\det \hat{\eta}^{(j)} \neq 0$ $(j = \overline{0, N+1})$, *then*

$$\hat{P}(\gamma, z_0, z) = \hat{P}^{-1}(\gamma, z, z_0) = \begin{pmatrix} u_2'(z) & -u_2(z) \\ -u_1'(z) & u_1(z) \end{pmatrix}.$$

Proof. Let $u_\alpha^{(m)}(z)$ $(\alpha \in \{1, 2\},\ m \in \{0, \ldots, N\})$ be entire solutions to the auxiliary Sturm–Liouville equation

$$\frac{d^2 u^{(m)}}{dz^2} + (Q_m - \lambda^2) u^{(m)} = 0 \qquad (z \in \mathbf{C}) \tag{11}$$

with initial conditions (6) at the point z_m. We define the functions $v_\alpha^{(s)}(z)$ $(s \in \{-1, 0, \ldots, N\})$ and $\widetilde{v}_\alpha^{(p)}(z)$ $(p \in \{-1, 0, \ldots, N-1\})$ by the following recurrent relations

$$\begin{cases} v_\alpha^{(-1)}(z) := u_\alpha^{(0)}(z), \\ \begin{pmatrix} \widetilde{v}_\alpha^{(m-1)}(z) \\ \dfrac{d\widetilde{v}_\alpha^{(m-1)}(z)}{dz} \end{pmatrix} = \hat{\eta}^{(m)} \begin{pmatrix} v_\alpha^{(m-1)}(z) \\ \dfrac{dv_\alpha^{(m-1)}(z)}{dz} \end{pmatrix} \ (m \in \{0, \ldots, N\}), \\ v_\alpha^{(m)}(z) := \widetilde{v}_\alpha^{(m-1)}(z_m) u_1^{(m)}(z) + \dfrac{d\widetilde{v}_\alpha^{(m-1)}(z)}{dz}\bigg|_{z=z_m} u_2^{(m)}(z). \end{cases} \tag{12}$$

Then, if γ_m $(m \in \{0, \ldots, N\})$ is a part of the curve γ connecting the points z_m and z_{m+1}, then, by definition 1, 2, the functions $u_\alpha(z)$, such that

$$\begin{cases} u_1(z_0) = 1,\ u_1'(z_0) = 0,\ u_2(z_0) = 0,\ u_2'(z_0) = 1, \\ u_\alpha(z) := v_\alpha^{(m)}(z),\ z \in \gamma_m \setminus \{z_m, z_{m+1}\}\ (m \in \{0, \ldots, N\}), \\ \begin{pmatrix} u_\alpha(z_{N+1}) \\ u_\alpha'(z_{N+1}) \end{pmatrix} = \hat{\eta}^{(N+1)} \begin{pmatrix} v_\alpha^{(N)}(z_{N+1}) \\ \dfrac{dv_\alpha^{(N)}(z)}{dz}\bigg|_{z=z_{N+1}} \end{pmatrix}, \end{cases} \tag{13}$$

are solutions to equation (1) of class D on γ, satisfying conditions (6) at the point z_0. Then formula (10) follows from definition 3 of the transfer matrix and relations (12), (13), and the remaining statements of Lemma 2 follow from (10) and the corresponding properties of solutions to linear differential equations with holomorphic coefficients [14] (§2, 24). □

Lemma 3. *For any $m \in \{0,\dots,N\}$ and $K \in \boldsymbol{N}$, there exist positive numbers $\lambda_{m,K}, C^{(0)}_{m,K}$ and two continuously differentiable solutions $F^{(m)}_{\pm K}(z,\lambda)$ to corresponding equation (11), which for all $\lambda \neq 0$ and $z \in \boldsymbol{C}$ can be represented as:*

$$F^{(m)}_{\pm K} = C^{(m)}_{\pm K}(z,\lambda)\exp\{\pm\lambda(z-z_m)\}, \frac{dF^{(m)}_{\pm K}}{dz} = \pm\lambda E^{(m)}_{\pm K}(z,\lambda)\exp\{\pm\lambda(z-z_m)\}, \tag{14}$$

$$C^{(m)}_{\pm K}(z,\lambda) := 1 + \sum_{k=1}^{K}\left(\pm\frac{1}{\lambda}\right)^k C_{m,k}(z) + \frac{B^{(m)}_{\pm K}(z,\lambda)}{\lambda^{K+1}}, \tag{15}$$

$$E^{(m)}_{\pm K}(z,\lambda) := 1 + \sum_{k=1}^{K}\left(\pm\frac{1}{\lambda}\right)^k \left(C_{m,k}(z) + \frac{dC_{m,k-1}(z)}{dz}\right) + \frac{H^{(m)}_{\pm K}(z,\lambda)}{\lambda^{K+1}}, \tag{16}$$

where $B^{(m)}_{\pm K}(z,\lambda)$, $H^{(m)}_{\pm K}(z,\lambda)$ and all $C_{m,k}(z)$ are entire functions of z, $C_{m,0}(z) := 1$,

$$C_{m,k}(z_m) := 0, \ \frac{dC_{m,k}}{dz} := -\frac{1}{2}\left(\frac{d^2C_{m,k-1}}{dz^2} + Q_m(z)C_{m,k-1}(z)\right) \ (k = \overline{1,K}). \tag{17}$$

Moreover, if $|\lambda| \geq \lambda_{m,K}$, then $F^{(m)}_{\pm K}(z,\lambda)$ are linearly independent solutions to equation (11) in $\boldsymbol{C}$ and the following inequalities are valid for any $z \in L_m$ (L_m — the line segment connecting the points z_m and z_{m+1}):

$$|B^{(m)}_{\pm K}(z,\lambda)| \leq C^{(0)}_{m,K}, \quad |H^{(m)}_{\pm K}(z,\lambda)| \leq C^{(0)}_{m,K}. \tag{18}$$

Proof. Formulas (14) – (17) are checked by substituting into (11), and estimates (18) on the line segment L_m follow from the well-known results on asymptotic expansions of solutions to equations of the form (11) on a segment of real axis for large values of the parameter λ [19].□

The symbols $O(1)$ and $\hat{O}(1)$ denote, respectively, the functions and matrices of the functions of the parameter λ, the form of which is not important to us, bounded for $|\lambda| > \lambda_{cr}$, where λ_{cr} is a finite value different for different functions and matrices.

Lemma 4. *Let $\hat{P}(\gamma, z_{N+1}, z_0)$ be the transfer matrix of equation (1) of class D along the curve γ with the set of characteristic data (5) satisfying conditions (7). Then there exists an integer $K_0 \geq 2$ such that for any integer $K \geq K_0$ there exists a finite number $\lambda_K > 0$ such that for $|\lambda| \geq \lambda_K$ the matrix $\hat{P}$ can be written as*

$$\hat{P}(\gamma, z_{N+1}, z_0) = \hat{\eta}^{(N+1)}\hat{C}^{(f)}\hat{T}^{(N)}\hat{T}^{(N-1)}\dots\hat{T}^{(2)}\hat{T}^{(1)}\hat{T}^{(0)}\hat{A}^{(0)}\hat{\eta}^{(0)}, \tag{19}$$

where $\hat{T}^{(0)} := \hat{I}$ *(unit matrix),*

$$\hat{A}^{(0)} := -\frac{1}{D_K^{(0)}}\begin{pmatrix} \lambda E_{-K}^{(0)}(z_0,\lambda) & C_{-K}^{(0)}(z_0,\lambda) \\ \lambda E_{+K}^{(0)}(z_0,\lambda) & -C_{+K}^{(0)}(z_0,\lambda)\end{pmatrix}, \tag{20}$$

$$\hat{C}^{(f)} := \begin{pmatrix} C_{+K}^{(N)}(z_{N+1},\lambda)\exp(\lambda\Delta z_N) & C_{-K}^{(N)}(z_{N+1},\lambda)\exp(-\lambda\Delta z_N) \\ \lambda E_{+K}^{(N)}(z_{N+1},\lambda)\exp(\lambda\Delta z_N) & -\lambda E_{-K}^{(N)}(z_{N+1},\lambda)\exp(-\lambda\Delta z_N)\end{pmatrix}, \tag{21}$$

$$\hat{T}^{(n)} := \begin{pmatrix} t_{1+}^{(n)}\exp(\lambda\Delta z_{n-1}) & t_{-}^{(n)}\exp(-\lambda\Delta z_{n-1}) \\ t_{+}^{(n)}\exp(\lambda\Delta z_{n-1}) & t_{1-}^{(n)}\exp(-\lambda\Delta z_{n-1})\end{pmatrix}\ (n=\overline{1,N}\, for\, N\geq 1). \tag{22}$$

Here Δz_m $(m=\overline{0,N})$ *are defined in* (8), $D_K^{(0)} = -\lambda(C_{+K}^{(0)}(z_0,\lambda)E_{-K}^{(0)}(z_0,\lambda)+C_{-K}^{(0)}(z_0,\lambda)\,E_{+K}^{(0)}(z_0,\lambda))$ $=-2\lambda(1+O(1)/\lambda)\neq 0$, *and if* $N\geq 1$, *then for* $t_{1\pm}^{(n)}$, $t_{\pm}^{(n)}$ $(n=\overline{1,N})$ *the following relations are valid:*

$$t_{1\pm}^{(n)} = \pm\eta_{12}^{(n)}\lambda\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right)+\left(\eta_{11}^{(n)}+\eta_{22}^{(n)}\right)\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right)\pm\frac{\eta_{21}^{(n)}}{\lambda}\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right), \tag{23}$$

$$t_{\pm}^{(n)} = \begin{cases} \pm\eta_{12}^{(n)}\lambda\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right)\mp\frac{\eta_{21}^{(n)}}{\lambda}\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right)+ & \\ \qquad +(\eta_{11}^{(n)}-\eta_{22}^{(n)})\left(\frac{1}{2}+\frac{O(1)}{\lambda}\right), & \hat{\eta}^{(n)}\neq\hat{I}; \\ \qquad -\left(\mp\frac{1}{2\lambda}\right)^{m_n+2}\delta_n\left(1+\frac{O(1)}{\lambda}\right), & \hat{\eta}^{(n)}=\hat{I}, \end{cases} \tag{24}$$

where the integers $m_n\in[0,K_0-2]$, *the complex numbers* $\delta_n\neq 0$ *and don't depend on* λ.

Proof. By Lemma 3, for any integer $K\in\mathbf{N}$ and $m\in\{0,\dots,N\}$ there exist numbers $\lambda_{m,K}>0$ such that for $|\lambda|\geq\lambda_{m,K}$ the solutions $u_1^{(m)}(z), u_2^{(m)}(z)$ to corresponding equation (11) can be represented as a linear combination of functions $F_{\pm K}^{(m)}(z,\lambda)$. Therefore, for $|\lambda|\geq\lambda_K := \max\{\lambda_{m,K}, m=\overline{0,N}\}$, formula (10) can be written as (19), where the matrices $\hat{T}^{(n)}(n=\overline{1,N}$ for $N\geq 1)$, $\hat{A}^{(0)}$, $\hat{C}^{(f)}$ satisfy formulas (20) – (22), and $D_K^{(n)} := -\lambda(C_{+K}^{(n)}(z_n,\lambda)E_{-K}^{(n)}(z_n,\lambda)+C_{-K}^{(n)}(z_n,\lambda)E_{+K}^{(n)}(z_n,\lambda))$,

$$\begin{pmatrix} t_{1+}^{(n)} & t_{-}^{(n)} \\ t_{+}^{(n)} & t_{1-}^{(n)}\end{pmatrix} = -\frac{1}{D_K^{(n)}}\begin{pmatrix} C_{+K}^{(n-1)}(z_n,\lambda)\tau_{1+}^{(n)} & C_{-K}^{(n-1)}(z_n,\lambda)\tau_{-}^{(n)} \\ C_{+K}^{(n-1)}(z_n,\lambda)\tau_{+}^{(n)} & C_{-K}^{(n-1)}(z_n,\lambda)\tau_{1-}^{(n)}\end{pmatrix}+\frac{\hat{O}(1)}{\lambda^{K+1}}, \tag{25}$$

$$\tau_{1\pm}^{(n)} := \left(\pm\lambda^2\eta_{12}^{(n)}\varphi_{\pm}^{(n-1)}(z_n)\varphi_{\mp}^{(n)}(z_n)+\lambda\eta_{11}^{(n)}\varphi_{\mp}^{(n)}(z_n)+\lambda\eta_{22}^{(n)}\varphi_{\pm}^{(n-1)}(z_n)\pm\eta_{21}^{(n)}\right), \tag{26}$$

$$\tau_{\pm}^{(n)} := \left(\pm\lambda^2\eta_{12}^{(n)}\varphi_{\pm}^{(n-1)}(z_n)\varphi_{\pm}^{(n)}(z_n)+\lambda\eta_{11}^{(n)}\varphi_{\pm}^{(n)}(z_n)-\lambda\eta_{22}^{(n)}\varphi_{\pm}^{(n-1)}(z_n)\mp\eta_{21}^{(n)}\right), \tag{27}$$

$$\varphi_{\pm}^{(n)}(z) := \frac{E_{\pm K}^{(n)}(z,\lambda)}{C_{\pm K}^{(n)}(z,\lambda)} = 1+\frac{O(1)}{\lambda^2}\quad (z\in\{z_n,z_{n+1}\}). \tag{28}$$

The last equality follows from relations (15) – (18) with $K\geq 1$. From formulas (15), (16), (18) we also have:

$$D_K^{(m)} = -2\lambda\left(1+\frac{O(1)}{\lambda}\right)\neq 0\ \ (m=\overline{0,N}). \tag{29}$$

Taking into account conditions (7), we obtain that for $N \geq 1$ substitution of formulas (26) — (29) into relation (25) fully proves formula (23), as well as formula (24) for the case $\hat{\eta}^{(n)} \neq \hat{I}$ $(n \in \{1,\dots,N\})$. If $\hat{\eta}^{(n)} = \hat{I}$, then by definition 1 (by virtue of (4)) $Q_n \neq Q_{n-1}$, and relation (24) follows from the comparison of formulas (19), (20) of paper [17] with formulas (22), (24) of this paper. □

Note that the numbers m_n and δ_n included in formula (24) for the case $\hat{\eta}^{(n)} = \hat{I}$ $(n = \overline{1,N}$ for $N \geq 1)$ are equal, respectively, to the minimal order of the derivative of the potential function along the curve γ, which has a jump at the point z_n, and the magnitude of this jump ([18], formulas (18), (19))

Lemma 5. *Let* $\det\hat{\eta}^{(0)} \neq 0$, $\det\hat{\eta}^{(N+1)} \neq 0$. *Then the elements* $c^{(f)}_{\eta,\alpha\nu}$ $(\nu \in \{1,2\})$ *and* $a^{(0)}_{\eta,\kappa\beta}$ $(\kappa \in \{1,2\})$, *respectively, of the matrices* $\hat{C}^{(f)}_{\eta} := \hat{\eta}^{(N+1)}\hat{C}^{(f)}$ *and* $\hat{A}^{(0)}_{\eta} := \hat{A}^{(0)}\hat{\eta}^{(0)}$ *for large values of the parameter* λ *can be represented as*

$$c^{(f)}_{\eta,\alpha\nu}(\lambda) = \exp\left\{-(-1)^{\nu}\lambda\Delta z_N\right\}\left[\eta^{(N+1)}_{\alpha 1}\left(1+\frac{O(1)}{\lambda}\right) - (-1)^{\nu}\eta^{(N+1)}_{\alpha 2}\lambda\left(1+\frac{O(1)}{\lambda}\right)\right] \neq 0 \ (\nu \in \{1,2\}), \quad (30)$$

$$a^{(0)}_{\eta,\kappa\beta}(\lambda) = \frac{\eta^{(0)}_{1\beta}}{2}\left(1+\frac{O(1)}{\lambda}\right) - \frac{(-1)^{\kappa}}{2\lambda}\eta^{(0)}_{2\beta}\left(1+\frac{O(1)}{\lambda}\right) \neq 0 \quad (\kappa \in \{1,2\}). \quad (31)$$

Proof. Taking into account Lemma conditions and relations (15), (16), (18), we find that formulas (30) and (31) follow from formulas (21) and (20), (29), respectively. □

We denote $(N+1)$-dimensional vectors by letters with an arrow on top, and their scalar product by round brackets. For example, $\overrightarrow{\Delta z} := (\Delta z_0, \Delta z_1, \dots, \Delta z_N)$; $\vec{\alpha}_s := (\alpha^{(s)}_0, \alpha^{(s)}_1, \dots, \alpha^{(s)}_N)$, where $\alpha^{(s)}_m \in \{\pm 1\}$ $(m = \overline{0,N})$, $s = 1 + \sum\limits_{m=0}^{N}(1+\alpha^{(s)}_m)2^{m-1}$, i.e., $s \in \{1,\dots,2^{N+1}\}$ (as in the binary number system); $(\vec{\alpha}_s, \overrightarrow{\Delta z}) := \sum\limits_{m=0}^{N}\alpha^{(s)}_m\Delta z_m$.

The following corollary can be obtained from Lemmas 4, 5.

Corollary 3. *Let* γ *be a regular curve. Then, under the conditions of Lemma 4, for* $K \geq K_0$ *and* $|\lambda| \geq \lambda_K$, *all elements of the matrix* $\hat{P}(\gamma, z_{N+1}, z_0)$ *can be written as:*

$$p_{\alpha\beta} = \sum_{s=1}^{2^{N+1}} d^{(s)}_{\alpha\beta}(\lambda)\exp\{\lambda h_s\} \quad (\alpha,\beta \in \{1,2\}). \quad (32)$$

Here the coefficients $h_s := \left(\vec{\alpha}_s, \overrightarrow{\Delta z}\right)$ *don't depend on* λ *and functions* $d^{(s)}_{\alpha\beta}(\lambda)$ *can be represented as:*

$$d^{(s)}_{\alpha\beta}(\lambda) = \left(\frac{1}{\lambda}\right)^{m^{(s)}_{\alpha\beta}}\delta^{(s)}_{\alpha\beta}\left(1+\frac{O(1)}{\lambda}\right) \neq 0 \quad (\alpha,\beta \in \{1,2\}, s = \overline{1,2^{N+1}}),$$

where integers $m^{(s)}_{\alpha\beta} \in [-N-1, NK_0+1]$, *complex numbers* $\delta^{(s)}_{\alpha\beta} \neq 0$ *and don't depend on* λ.

The part of the exponential functions in the right-hand sides of equalities (32) may coincide, and the question arises: are not the exponential functions that are most rapidly growing with increasing $|\lambda|$ mutually destroyed in (32)? The negative answer to this question for the elements of the transfer matrix of equation (1) of class D along a regular curve is given by Lemma 6, similar to Lemma 4 of the paper [18].

Lemma 6. *Let $h_{max} := \max\{|h_s|, s = \overline{1, 2^{N+1}}\}$, where $h_s = (\vec{\alpha}_s, \overrightarrow{\Delta z})$. In this case there exist at least two different numbers $s_0 \in \{1, \dots, 2^{N+1}\}$ such that $|h_{s_0}| = h_{max}$, and if $\overrightarrow{\Delta z} \neq \vec{0}$, then $h_{max} > 0$. Moreover, if conditions (8) hold, then the inequality $h_{s_0} \neq h_s$ is valid for any coefficient h_{s_0} such that $|h_{s_0}| = h_{max}$ and for any number $s \in \{1, \dots, 2^{N+1}\} \setminus \{s_0\}$.*

3. Proof of Theorem 1

By the condition of the Theorem 1, the curve γ is regular. Corollary 3 and Lemma 6 imply that in this case there exists at least one straight line passing through zero of the complex plane of the parameter λ (and therefore, at least one ray going out of the zero of the complex plane of the spectral parameter ρ), such that among the 2^{N+1} summands in (32) for each element of the matrix $\hat{P}$ there exists exactly one summand having for $\lambda \to \infty$ ($\rho \to \infty$) along this line (ray) the greatest exponential growth with exponent $h_{max} > 0$. Therefore, each element of the transfer matrix $\hat{P}$ along a regular curve is entire function ρ of order $1/2$ and of the normal type [20] (Chapter I, §1).

4. Proof of Theorem 2

If $W^{(1)} = W^{(2)}$, then $\hat{P}^{(1)} = \hat{P}^{(2)}$ by Lemma 2.

Let $\hat{P}^{(1)} = \hat{P}^{(2)}$ and $W^{(1)} := \left\{N^{(1)}, \{z_j^{(1)}, \hat{\eta}_1^{(j)}\}_0^{N^{(1)}+1}, \{Q_m^{(1)}\}_0^{N^{(1)}}\right\}$, $W^{(2)} := \left\{N^{(2)}, \{z_j^{(2)}, \hat{\eta}_2^{(j)}\}_0^{N^{(2)}+1}, \{Q_m^{(2)}\}_0^{N^{(2)}}\right\}$. Since the curves $\gamma^{(1)}$ and $\gamma^{(2)}$ are standard, then $\det\hat{\eta}_1^{(j)} \neq 0$ ($j = \overline{0, N^{(1)} + 1}$) and $\det\hat{\eta}_2^{(j)} \neq 0$ ($j = \overline{0, N^{(2)} + 1}$). Therefore, by Lemma 2, the transfer matrix of equation (1) of class D along the curve γ obtained by successive walk first over the curve $\gamma^{(1)}$ from the point $z^{(1)}_{N^{(1)}+1}$ to the point z_0, and then over the curve $\gamma^{(2)}$ from the point z_0 to the point $z^{(2)}_{N^{(2)}+1}$, will be equal to

$$\begin{aligned}\hat{P} = \hat{\eta}_2^{(N^{(2)}+1)} \hat{P}_2^{(N^{(2)})} \hat{\eta}_2^{(N^{(2)})} \dots \hat{P}_2^{(0)} \hat{\eta}_2^{(0)} \left(\hat{\eta}_1^{(0)}\right)^{-1} \left(\hat{P}_1^{(0)}\right)^{-1} \dots \left(\hat{\eta}_1^{(N^{(1)})}\right)^{-1} \\ \left(\hat{P}_1^{(N^{(1)})}\right)^{-1} \left(\hat{\eta}_1^{(N^{(1)}+1)}\right)^{-1} = \hat{P}^{(2)} (\hat{P}^{(1)})^{-1}.\end{aligned} \tag{33}$$

On the other hand, by assumption $\hat{P}^{(1)} = \hat{P}^{(2)}$ and hence $\hat{P} = \hat{I}$.

By the condition of the Theorem 2 $z_0^{(1)} = z_0^{(2)} = z_0$. Therefore, there exists an integer $i_0 \geq 0$ such that $z_i^{(2)} = z_i^{(1)} = z_i$ ($i = \overline{0, i_0}$), $Q_i^{(2)} = Q_i^{(1)}$, $\eta_2^{(i)} = \eta_1^{(i)}$ ($i = \overline{0, i_0 - 1}$ for $i_0 \geq 1$), and if $i_0 \le \min\{N^{(1)}, N^{(2)}\}$, then the ordered sets $\{z^{(2)}_{i_0+1}, Q^{(2)}_{i_0}, \eta^{(i_0)}_2\}$ and $\{z^{(1)}_{i_0+1}, Q^{(1)}_{i_0}, \eta^{(i_0)}_1\}$ are different. Further, for definiteness, we assume that $N^{(1)} \le N^{(2)}$.

Suppose that $i_0 \le N^{(1)}$, and the ordered sets $\{Q^{(2)}_{i_0}, \eta^{(i_0)}_2\}$ and $\{Q^{(1)}_{i_0}, \eta^{(i_0)}_1\}$ are different. Then $\hat{P}^{(i)}_2 = \hat{P}^{(i)}_1$ ($i = \overline{0, i_0 - 1}$ for $i_0 \ge 1$), and the formula (33) after simplification takes the form:

$$\hat{P} = \hat{\eta}_2^{(N^{(2)}+1)} \hat{P}_2^{(N^{(2)})} \hat{\eta}_2^{(N^{(2)})} \dots \hat{P}_2^{(i_0)} \hat{\eta}_2^{(i_0)} \left(\hat{\eta}_1^{(i_0)}\right)^{-1} \left(\hat{P}_1^{(i_0)}\right)^{-1} \dots \left(\hat{\eta}_1^{(N^{(1)})}\right)^{-1} \left(\hat{P}_1^{(N^{(1)})}\right)^{-1} \left(\hat{\eta}_1^{(N^{(1)}+1)}\right)^{-1},$$

i.e., by Lemma 2, the matrix $\hat{P}$ will be equal to the transfer matrix along the curve γ_{min}, which first coincides with the part of the curve $\gamma^{(1)}$ from point $z^{(1)}_{N^{(1)}+1}$ to point z_{i_0}, and then with the part of curve $\gamma^{(2)}$ from point z_{i_0} to point $z^{(2)}_{N^{(2)}+1}$. Moreover, the transition matrix $\hat{\hat{\eta}}^{(i_0)}$ at point z_{i_0} is equal to $\hat{\eta}_2^{(i_0)} \left(\hat{\eta}_1^{(i_0)}\right)^{-1}$, and point z_{i_0} is a characteristic point of the curve γ_{min}, since ordered sets $\{Q^{(2)}_{i_0}, \eta^{(i_0)}_2\}$ and $\{Q^{(1)}_{i_0}, \eta^{(i_0)}_1\}$ are different by assumption. Since by the condition of the Theorem 2 the sets of characteristic data $W^{(1)}$ and $W^{(2)}$ are standard and, by Lemma 1, the transition matrix $\hat{\hat{\eta}}^{(i_0)}$ at point z_{i_0} satisfies conditions (7), then the curve γ_{min} is regular, which by virtue of Theorem 1 contradicts the fact that $\hat{P} = \hat{I}$. So

$$Q^{(1)}_{i_0} = Q^{(2)}_{i_0},\ \eta^{(i_0)}_1 = \eta^{(i_0)}_2. \tag{34}$$

Let $i_0 \le N^{(1)}$, conditions (34) are valid, and $z^{(2)}_{i_0+1} \ne z^{(1)}_{i_0+1}$. Then, by virtue of (34), point z_{i_0} will not be a characteristic point of a curve γ_{min}, and various points $z^{(1)}_{i_0+1}$, $z^{(2)}_{i_0+1}$ will be its successive characteristic points. Therefore, the curve γ_{min} will be standard and hence regular, which contradicts the fact that $\hat{P} = \hat{I}$. Consequently,

$$z^{(1)}_{i_0+1} = z^{(2)}_{i_0+1}. \tag{35}$$

If $i_0 \le \min\{N^{(1)}, N^{(2)}\}$, then formulas (34), (35) contradict the definition of the number i_0 and hence $i_0 = N^{(1)} + 1$. If, in addition, $N^{(1)} < N^{(2)}$, then formula (33), after simplification, takes the form:

$$\hat{P} = \hat{\eta}_2^{(N^{(2)}+1)} \hat{P}_2^{(N^{(2)})} \hat{\eta}_2^{(N^{(2)})} \dots \hat{P}_2^{(N^{(1)}+1)} \hat{\hat{\eta}}^{(N^{(1)}+1)}.$$

where $\hat{\hat{\eta}}^{(N^{(1)}+1)} = \hat{\eta}_2^{(N^{(1)}+1)} \left(\hat{\eta}_1^{(N^{(1)}+1)}\right)^{-1}$. By definitions 4, 6 we have: $\det \hat{\hat{\eta}}^{(N^{(1)}+1)} \ne 0$. Thus, in this case, by virtue of Lemma 2, the matrix $\hat{P}$ will be equal to the transfer matrix along the regular curve that coincides with the part of the curve $\gamma^{(2)}$ connecting the points $z^{(1)}_{N^{(1)}+1} = z^{(2)}_{N^{(1)}+1}$ and $z^{(2)}_{N^{(2)}+1}$. By virtue of Theorem 1, this contradicts the fact that $\hat{P} = \hat{I}$ and hence $N^{(1)} = N^{(2)}$, $i_0 = N^{(1)} + 1$. So $W^{(1)} = W^{(2)}$. Theorem 2 is proven.

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