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On the Dimension of Non-Abelian Tensor Squares of n-Lie Algebras

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Abstract. Let L be an n-Lie algebra over a field $\mathbb{F}$. In this paper, we introduce the notion of non-abelian tensor square $L\otimes L$ of L and define the central ideal $L\square L$ of it. Using techniques from group theory and Lie algebras, we show that that $L\square L \cong L^{ab}\square L^{ab}$. Also, we establish the short exact sequence

$$0 \longrightarrow \mathcal{M}(L) \longrightarrow \frac{L\otimes L}{L\square L} \longrightarrow L^2 \longrightarrow 0$$

and apply it to compute an upper bound for the dimension of non-abelian tensor square of L.

1 Introduction

The tensor product of algebraic structures has always been an interesting subject in the theory of tensor products among which the non-abelian tensor products have deep roots in algebraic K-theory, algebraic topology, and Homotopy theory. Historical studies illustrate that the ideas of non-abelian tensor products go back to primary works of Whitehead [18]. The notion of non-abelian tensor products first introduced officially in the works of Miller [12], and then Dennis [4] in a more comprehensive form. Later, Lue [11] introduced the notion of non-abelian tensor products of nilpotent groups independently. Non-abelian tensor products are efficient tools in studying the Schur multipliers of groups as well as their classifications.

The notion of non-abelian tensor products of Lie algebras is introduced by Ellis [5] in 1987 where he studied them through the same line of investigation as in case of groups. Ellis [6] in 1991 introduced the notion of universal exterior products too, and presented two exact sequences involving non-abelian tensor products and universal exterior products. Salemkar et. al. [15] in 2008 present conditions under which the non-abelian tensor product of two Lie algebras is

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nilpotent or solvable as well as determine their nilpotent class and solubility length, respectively. Also, using Frattini ideals, they obtain an upper bound for the dimension of non-abelian tensor product of two ideals of a given Lie algebra L. In the particular case that two ideals coincide with the Lie algebra L, the bound is known to be dk, where d is the dimension and k is the minimum size of a generating set of L. Niroomand [13] in 2011 improves this upper bound to $(d-1)k+2$ in case of non-abelian tensor squares by using the dimension of Schur multipliers and the short exact sequence of Ellis (see [5]) correlating Lie algebra Schur multipliers and non-abelian tensor squares.

Lie algebras are generalized in various ways one of the most interesting ones is introduced by Filippov [8] in 1985 known as n-Lie algebras. The n-Lie algebras coincide the ordinary Lie algebras when $n = 2$. Extensive research has carried on n-Lie algebras involving their classifications as well as computation of their Schur multipliers and dimensions (see [7, 13, 14, 15]).

The n-Lie algebras are themselves generalized to that of n-Leibniz algebras which coincide with Leibniz algebras in case $n = 2$. The Leibniz algebras are defined as the non-symmetric version of Lie algebras by Loday [10] in 1993. In 1999, Gnedbaye [9] introduced and studied the notion of non-abelian tensor products of Leibniz algebras. Later, Casas et. al. [3] in 2008 introduced the action of n-Leibniz algebras (without using their presentations) and applied them to study the crossed modules of Leibniz algebras. Recently, the authors in [1] defined the action of n-Lie algebras and free n-Lie algebras in a more accurate form. Also, they have defined the notion of non-abelian tensor products of two n-Lie algebras and studied their existence, construction, and uniqueness beside their other properties.

In Section 2, we shall recall some preliminary notions we need to prove our results. In Section 3, we establish some Lie homomorphisms and exact sequences of non-abelian tensor products of n-Lie algebras. Also, we generalize the short exact sequence of Ellis [5] correlating the Schur multipliers and non-abelian tensor squares of Lie algebras to that of n-Lie algebras. Moreover, using this short exact sequence and the same arguments as in Niroomand [13], we obtain an upper bound for the dimension of the non-abelian tensor squares of n-Lie algebras.

2 Preliminary results

In this section, we review some of the required notions and state the preliminary results and theorems we need in order to prove our main results.

Definition 1. Let L be a vector space over a field $\mathbb{F}$ and $[-,\ldots,-] : L^{\times n} \longrightarrow L$ be an n-linear map. Then L is called an n-Lie algebra if

$$[l_1,\ldots,l_i,\ldots,l_j,\ldots,l_n] = -[l_1,\ldots,l_j,\ldots,l_i,\ldots,l_n],$$

for all $l_1, \ldots, l_n \in L$, and that $[-, \ldots, -]$ satisfies the Jacoby identity, that is

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in L$. Notice that the first condition is equivalent to say that $[l_1, \ldots, l_n] = 0$ if $l_i = l_j$ for some $i \neq j$ provided that char$\mathbb{F} \neq 2$.

Let $L_1, L_2, \ldots, L_n$ be subalgebras of an n-Lie algebra L. The subalgebra of L generated by all vectors $[l_1, \ldots, l_n]$ where $l_i \in L_i$ for $i = 1, \ldots, n$ is denoted by $[L_1, L_2, \ldots, L_n]$. The subalgebra $[L, L, \ldots, L]$ is the derived algebra of L and it is denoted by L^2. If $L^2 = 0$, then L is called abelian. An ideal I of an n-Lie algebra L is a subalgebra of L satisfying $[I, L, \ldots, L] \subseteq I$. If $[I, L, \ldots, L] = 0$, the I is called an abelian ideal of L.

The set $Z(L)$ of all elements z of L satisfying $[z, L, \ldots, L] = 0$ is an ideal of L called the center of L. Let $Z_0(L) = Z(L)$, and define the ideal $Z_i(L)$ of L by $Z_i(L)/Z_{i-1}(L) = Z(L/Z_{i-1}(L))$ for all $i \geqslant 1$. It is evident that $Z_{i-1}(L) \lhd Z_i(L)$ for all $i \geqslant 1$. Hence, we have a series of ideals

$$0 \lhd Z_0(L) \lhd Z_1(L) \lhd \cdots \lhd Z_i(L) \lhd \cdots$$

of L, which is called the upper central series of L. Analogously, there is a series

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^i \supseteq \cdots$$

of ideals of L known as the lower central series of L defined in the same way. Indeed, $L^1 := L$ and $L^i := [L^{i-1}, L, \ldots, L]$ for all $i \geqslant 2$. Clearly, $L^{i+1} \lhd L^i$ for $i \geqslant 1$. The Lie algebra L is nilpotent of class c if $L^{c+1} = 0$ but $L^c \neq 0$ (or equivalently $Z_c(L) = L$ but $Z_{c-1}(L) \neq L$). The nilpotency class of L is denoted by $cl(L)$. Notice that nilpotency is closed under subalgebras and quotients, but it is not closed under extensions.

In [1], the authors defined the non-abelian tensor products of two n-Lie algebras and the modular n-tensor products as in the following.

Definition 2 (Modular n-tensor product/n-tensor spaces)**.** Let $n \geqslant 1$ be a natural number. Let V_1 and V_2 be two vector spaces over a field $\mathbb{F}$ of finite dimensions d_1 and d_2, respectively. Also, let $V_1^{\times i} \times V_2^{\times(n-i)}$ denote the Cartesian product

$$\underbrace{V_1 \times \cdots \times V_1}_{i \text{ times}} \times \underbrace{V_2 \times \cdots \times V_2}_{n-i \text{ times}}$$

for all $1 \leqslant i \leqslant n-1$.

A function f from $V_1^{\times i} \times V_2^{\times(n-i)}$ to a vector space W is multilinear (or n-linear) if the restriction of f on every component of $V_1^{\times i} \times V_2^{\times(n-i)}$ is linear. Let $\{e_{ij} : 1 \leqslant j \leqslant d_i\}$ be a basis of V_i for $i = 1, 2$. Then there exists a unique multilinear function $f : V_1^{\times i} \times V_2^{\times(n-i)} \longrightarrow W$ admitting the legal values on the elements of

$$\{(e_{1j_1}, \ldots, e_{1j_i}, e_{2j_{i+1}}, \ldots, e_{2j_n}) : \\ 1 \leqslant j_k \leqslant d_1,\ 1 \leqslant k \leqslant i,\ 1 \leqslant j_s \leqslant d_2,\ i+1 \leqslant s \leqslant n\}. \quad (2.1)$$

Notice that the above set contains $d_1^i \times d_2^{n-i}$ elements, which exceeds the dimension of $V_1^{\times i} \times V_2^{\times(n-i)}$ most of the time.

A pair $(\mathbb{T}_i, \Phi_i)$ (where $\mathbb{T}_i$ is a vector space and Φ_i is a multilinear function from $V_1^{\times i} \times V_2^{\times(n-i)}$ to $\mathbb{T}_i$) satisfies the universal factorization property if to each vector space W and an n-linear function $f : V_1^{\times i} \times V_2^{\times(n-i)} \longrightarrow W$ there corresponds a linear function $h_i : \mathbb{T}_i \longrightarrow W$ such that $f = h_i \Phi_i$. The existence of the universal pair can be proved easily. Also, up to isomorphism, there exists a unique universal pair $(\mathbb{T}_i, \Phi_i)$ satisfying the universal factorization property. Hence, $\mathbb{T}_i$ is a modular i-tensor product which we may denote it by

$$\underbrace{V_1 \otimes \cdots \otimes V_1}_{i \text{ times}} \otimes \underbrace{V_2 \otimes \cdots \otimes V_2}_{n-i \text{ times}}$$

or simply by $V_1^{\otimes i} \otimes V_2^{\otimes(n-i)}$ for all $1 \leqslant i \leqslant n-1$. Now, let

$$V \otimes^n_{\text{mod}} W = \text{span}\{\mathbb{T}_i : 1 \leqslant i \leqslant n-1\}.$$

The vector space $V \otimes^n_{\text{mod}} W$ is called the modular tensor product (or the abelian tensor product) of V and W.

It is evident that $\mathbb{T}_i = \mathbb{T}_j = V \otimes^n_{\text{mod}} V$ for all $1 \leqslant i, j \leqslant n-1$ whenever $V = W$, and that $V \otimes^n_{\text{mod}} W$ coincides with the ordinary tensor product of two vector spaces when $n = 2$.

In order to define the non-abelian tensor product of n-Lie algebras, we should first recall the actions of n-Lie algebras as given in [1].

Definition 3 (Action). Let L and P be two n-Lie algebras over a field $\mathbb{F}$ with brackets $[-, \ldots, -]_L$ and $[-, \ldots, -]_P$, respectively. Then L acts on P if there exists a family $\{f_i\}_{0<i<n}$ of n-linear maps as

$$f_i : \underbrace{L \times \cdots \times L}_{i \text{ times}} \times \underbrace{P \times \cdots \times P}_{(n-i) \text{ times}} \longrightarrow P$$

satisfying the following conditions:

$$\begin{aligned} &f_i([l_1, \ldots, l_n]_L, l_{n+1}, \ldots, l_{n+s}, p_1, \ldots, p_{n-s}) \\ &\quad = (-1)^{n-1} f_{n-1}\big(l_2, \ldots, l_n, f_i(l_1, l_{n+1}, \ldots, l_{n+s}, p_1, \ldots, p_{n-s})\big) \end{aligned}$$

$$+\sum_{j=2}^{n}(-1)^{n-j+1}f_{n-1}\big(l_1,\dots,l_{j-1},l_{j+1},\dots,l_n,$$
$$f_i(l_j,l_{n+1},\dots,l_{n+s},p_1,\dots,p_{n-s})\big),$$

$$\begin{aligned}
&f_i(l_1,\dots,l_i,p_1,\dots,p_{n-i-1},[p_{n-i},p'_2,\dots,p'_n]_P)\\
&\quad=[f_i(l_1,\dots,l_i,p_1,\dots,p_{n-i}),p'_2,\dots,p'_n]_P\\
&\qquad-\sum_{j=1}^{i}(-1)^{n-j}f_{i-1}\big(l_1,\dots,l_{j-1},l_{j+1},\dots,l_i,p_1,\dots,p_{n-i},f_1(l_j,p'_2,\dots,p'_n)\big)\\
&\qquad-\sum_{k=1}^{n-i}(-1)^{n-k}f_i\big(l_1,\dots,l_i,p_1,\dots,p_{k-1},[p_k,p'_2,\dots,p'_n]_P,p_{k+1},\dots,p_{n-i-1}\big),
\end{aligned}$$

and

$$\begin{aligned}
&f_i(l_1,\dots,l_i,p_1,\dots,p_{n-i-1},f_j(l_{i+1},l'_2,\dots,l'_j,p'_1,\dots,p'_{n-j}))\\
&\quad=(-1)^{2n-i-2}f_{j-1}\big(l'_2,\dots,l'_j,p'_1,\dots,p'_{n-j},f_{i+1}(l_1,\dots,l_{i+1},p_1,\dots,p_{n-i-1})\big)\\
&\qquad-\sum_{k=1}^{i}(-1)^{2n-i-2}f_i\big(l_2,\dots,l_{k-1},f_j(l_k,l'_2,\dots,l'_j,p'_1,\dots,p'_{n-j}),\\
&\qquad\qquad l_{k+1},\dots,l_i,l_{i+1},p_1,\dots,p_{n-i-1}\big)\\
&\qquad-\sum_{r=1}^{n-i-1}(-1)^{n-i-r+j-1}f_i\big(l_1,\dots,l_i,l_{i+1},p_1,\dots,p_{r-1},\\
&\qquad\qquad f_{j-1}(l'_2,\dots,l'_j,p_r,p'_1,\dots,p'_{n-j})\big).
\end{aligned}$$

Definition 4 (Non-abelian tensor product)**.** Let L and P be two n-Lie algebras over a field $\mathbb{F}$ acting on each other via the families $\{f_i\}_{0<i<n}$ and $\{g_i\}_{0<i<n}$ of n-linear maps, respectively, and act on themselves by their own brackets. Then the non-abelian tensor product $L\otimes P$ of L and P is defined as an n-Lie algebra generated by all symbols

$$l_1\otimes\cdots\otimes l_i\otimes p_{i+1}\otimes\cdots\otimes p_n,$$

where $l_j\in L$ for $1\leqslant j\leqslant i$ and $p_k\in P$ for $i+1\leqslant k\leqslant n$, subject to the following relations among generators

$$\begin{aligned}
&l_1\otimes\cdots\otimes\alpha l_i+l'_i\otimes\cdots\otimes l_j\otimes p_{j+1}\otimes\cdots\otimes p_n\\
&=\alpha(l_1\otimes\cdots\otimes l_i\otimes\cdots\otimes l_j\otimes p_{j+1}\otimes\cdots\otimes p_n)+(l_1\otimes\cdots\otimes l'_i\otimes\cdots\otimes l_j\otimes p_{j+1}\otimes\cdots\otimes p_n),
\end{aligned}$$

$$l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes \alpha p_{j+i} + p'_{j+i} \otimes \cdots \otimes p_n$$
$$= \alpha(l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_{j+i} \otimes \cdots \otimes p_n) + (l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p'_{j+i} \otimes \cdots \otimes p_n),$$

$$[l_1, \ldots, l_n] \otimes l'_2 \otimes \ldots \otimes l'_i \otimes p_1 \otimes \ldots \otimes p_{n-i}$$
$$= \sum_{j=1}^{n} (-1)^{n-j} l_1 \otimes \ldots \otimes l_{j-1} \otimes l_{j+1} \otimes \ldots \otimes l_n \otimes f_i(l_j, l'_2, \ldots, l'_i, p_1, \ldots, p_{n-i}),$$

$$l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i-1} \otimes [p'_1, \ldots, p'_n]$$
$$= \sum_{j=1}^{n} (-1)^{n-j-2} g_i(p'_j, l_1, \ldots, l_i, p_1, \ldots, p_{n-i-1}) \otimes p'_1 \otimes \ldots \otimes p'_{j-1} \otimes p'_{j+1} \otimes \ldots \otimes p'_n$$

and

$$\begin{aligned}
&\Big[(l_1^1 \otimes \cdots \otimes l_{i_1}^1 \otimes p_1^1 \otimes \cdots \otimes p_{n-i_1}^1), (l_1^2 \otimes \cdots \otimes l_{i_2}^2 \otimes p_1^2 \otimes \cdots \otimes p_{n-i_2}^2), \ldots, \\
&(l_1^{n-1} \otimes \cdots \otimes l_{i_{n-1}}^{n-1} \otimes p_1^{n-1} \otimes \cdots \otimes p_{n-i_{n-1}}^{n-1}), (l_1^n \otimes \cdots \otimes l_{i_n}^n \otimes p_1^n \otimes \cdots \otimes p_{n-i_n}^n)\Big] \\
= \; & \frac{1}{2^{n-1}} \bigg((-1)^{\sum_{s=1}^{n-1} i_s(n-i_s)} \Big(g_{n-i_1}(p_1^1, \ldots, p_{n-i_1}^1, l_1^1, \ldots, l_{i_1}^1) \otimes \\
&\qquad f_{i_2}(l_1^2, \ldots, l_{i_2}^2, p_1^2, \ldots, p_{n-i_2}^2) \otimes \ldots \otimes \\
&\qquad f_{i_{n-1}}(l_1^{n-1}, \ldots, l_{i_{n-1}}^{n-1}, p_1^{n-1}, \ldots, p_{n-i_{n-1}}^{n-1}) \otimes \\
&\qquad f_{i_n}(l_1^n, \ldots, l_{i_n}^n, p_1^n, \ldots, p_{n-i_n}^n) \Big) \\
+ \; & \sum_{r=2}^{n-1} (-1)^{\sum_{\substack{s=1 \\ s \neq r}}^{n-1} i_s(n-i_s)} \Big(g_{n-i_1}(p_1^1, \ldots, p_{n-i_1}^1, l_1^1, \ldots, l_{i_1}^1) \otimes \\
&\qquad g_{n-i_r}(p_1^r, \ldots, p_{n-i_1}^r, l_1^r, \ldots, l_{i_1}^r) \otimes \\
&\qquad f_{i_2}(l_1^2, \ldots, l_{i_2}^2, p_1^2, \ldots, p_{n-i_2}^2) \otimes \ldots \otimes \\
&\qquad f_{i_{r-1}}(l_1^{r-1}, \ldots, l_{i_{r-1}}^{r-1}, p_1^{r-1}, \ldots, p_{n-i_{r-1}}^{r-1}) \otimes \\
&\qquad f_{i_{r+1}}(l_1^{r+1}, \ldots, l_{i_{r+1}}^{r+1}, p_1^{r+1}, \ldots, p_{n-i_{r+1}}^{r+1}) \otimes \ldots \otimes \\
&\qquad f_{i_{n-1}}(l_1^{n-1}, \ldots, l_{i_{n-1}}^{n-1}, p_1^{n-1}, \ldots, p_{n-i_{n-1}}^{n-1}) \otimes \\
&\qquad f_{i_n}(l_1^n, \ldots, l_{i_n}^n, p_1^n, \ldots, p_{n-i_n}^n) \Big)
\end{aligned}$$

$$
\begin{aligned}
&+\cdots\\
&+(-1)^{i_1(n-i_1)}\Big(g_{n-i_1}(p_1^1,\ldots,p_{n-i_1}^1,l_1^1,\ldots,l_{i_1}^1)\otimes g_{n-i_2}(p_1^2,\ldots,p_{n-i_2}^2,l_1^2,\ldots,l_{i_2}^2)\\
&\qquad\qquad\otimes\ldots\otimes g_{n-i_{n-1}}(p_1^{n-1},\ldots,p_{n-i_{n-1}}^{n-1},l_1^{n-1},\ldots,l_{i_{n-1}}^{n-1})\otimes\\
&\qquad\qquad f_{i_n}(l_1^n,\ldots,l_{i_n}^n,p_1^n,\ldots,p_{n-i_n}^n)\Big)\Big).
\end{aligned}
\tag{2.2}
$$

In case $L = P$, the non-abelian tensor product is called the non-abelian tensor square of L and it is denoted by $L \otimes L$. We note that $L \otimes L$ is defined according to the natural action of L on itself by its bracket.

Remark 1. All over this paper, the notations $-\otimes-$ and $-\otimes_{\text{mod}}^n-$ stand for the non-abelian tensor product of n-Lie algebras and the modular tensor products, respectively. Also, L^{ab} denote the abelianization L/L^2 of any n-Lie algebra L.

Proposition 2.1 ([1, Propositon 4.4])**.** *Let L and P be two n-Lie algebras acting trivially on each other. Then $L \otimes P \cong L^{ab} \otimes_{\text{mod}}^n P^{ab}$. Moreover, if L and P are disjoint abelian algebras, then*

$$\dim(L\otimes P) = \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n},$$

where $a := \dim(L^{ab})$ and $b := \dim(P^{ab})$.

As in the theory of groups and Lie algebras, the elements of the Frattini ideal $\Phi(L)$ of an n-Lie algebra L are non-generators of L. Also, it is known that $L^2 = \Phi(L)$ for any nilpotent n-Lie algebra L, in which case L^2 is the set of non-generator elements of L. The following result describes the non-abelian tensor product of the ideal of central non-generator elements of an n-Lie algebra L with L.

Proposition 2.2. *Let L be a nilpotent n-Lie algebra of dimension d. Then*

$$M \otimes L \cong L^{ab} \otimes_{\text{mod}}^{n-1} L^{ab},$$

where M is a 1-dimensional subalgebra of L satisfying $M \subseteq L^2 \cap Z(L)$. Here $L^{ab} \otimes_{\text{mod}}^{n-1} L^{ab}$ is the modular $(n-1)$-tensor product of L^{ab}.

Proof. Let $M = \langle m \rangle$ for some $m \in L$. Since $M \subseteq L^2 \cap Z(L)$, the algebras M and L acts trivially on each other. Hence, by Proposition 2.1, $L \otimes M \cong L^{ab} \otimes M$. Furthermore, as

$$l_1 \otimes \cdots \otimes l_i \otimes m \otimes \cdots \otimes m = 0$$

when $i < n-1$ and $l_1, \ldots, l_i \in L$, it follows that the only possible non-zero elements of $L \otimes M$ are those of the form $l_1 \otimes \cdots \otimes l_{n-1} \otimes m$ for some $l_1, \ldots, l_{n-1} \in L$. Define the map ψ as

$$\begin{aligned} \psi : L^{ab} \otimes M &\longrightarrow L^{ab} \otimes_{\text{mod}}^{n-1} L^{ab} \\ \bar{l}_1 \otimes \bar{l}_2 \otimes \cdots \otimes \bar{l}_{n-1} \otimes m &\longmapsto \bar{l}_1 \otimes \bar{l}_2 \otimes \cdots \otimes \bar{l}_{n-1}. \end{aligned}$$

Analogous to ψ, one can define a map $\phi : L^{ab} \otimes_{\text{mod}}^{n-1} L^{ab} \longrightarrow L^{ab} \otimes M$ too. A simple verification shows that ϕ and ϕ are n-Lie algebra homomorphims satisfying $\psi\phi = I$ and $\phi\psi = I$, from which the result follows. □

Proposition 2.2 yields the following result immediately.

Corollary 2.1. *Let L and M be n-Lie algebras as in Proposition 2.2. Then*

$$L \otimes M \cong M \otimes L$$

and

$$\dim(L \otimes M) = \dim(M \otimes L) = \dim\left(\frac{L}{L^2} \otimes_{\text{mod}}^{n-1} \frac{L}{L^2}\right) = k^{n-1}.$$

The next theorem is a key-result in the theory of non-abelian tensor products of n-Lie algebras, which is proved by Ellis [5] for Lie algebras and the authors [1] for n-Lie algebras.

Theorem 2.2 ([1, Theorem 4.8])**.** *Let L_1 and L_2 be two n-Lie algebras. Then*

$$(L_1 \oplus L_2) \otimes (L_1 \oplus L_2) \cong (L_1 \otimes L_1) \oplus (L_1 \otimes L_2) \oplus (L_2 \otimes L_1) \oplus (L_2 \otimes L_2).$$

In what follows, $L \square L$ denotes the n-Lie algebra generated by all the elements

$$l_1 \otimes \cdots \otimes l_{i-1} \otimes l^* \otimes l_{i+1} \otimes \cdots \otimes l_{j-1} \otimes l^* \otimes l_{j+1} \otimes \cdots \otimes l_n$$

for every n-Lie algebra L and elements $l_1, \ldots, l_n, l^* \in L$. One can easily see that $L \square L$ is a central ideal of $L \otimes L$.

It is shown, by using Whitehead's functor Γ, that $L \square L \cong L^{ab} \square L^{ab}$ for every Lie algebra L such that L^2 has a complement in L (see [5] and [17, Page 48]). Here we shall use a different technique to prove this result in a more general case of n-Lie algebras.

Proposition 2.3. *Let L be an n-Lie algebra. If L^2 has a complement in L, then*

$$L \square L \cong L^{ab} \square L^{ab}.$$

Proof. By assumption, L^2 has a complement A in L, that is $L = L^2 \oplus A$. Let $a \in A$ and $l_2, \dots, l_n \in L$ be arbitrary elements. Since A is an ideal of L, it follows that $[a, l_2, \dots, l_n] \in L^2 \cap A$ so that $[a, l_2, \dots, l_n] = 0$. Hence, A is a central ideal of L. On the other hand, Proposition 2.8 yields

$$L \otimes L \cong (L^2 \otimes L^2) \oplus (A \otimes L^2) \oplus (L^2 \otimes A) \oplus (A \otimes A).$$

Furthermore, $A \otimes L^2 = L^2 \otimes A = 0$ as A is central. Thus

$$L \otimes L \cong (L^2 \otimes L^2) \oplus (A \otimes A),$$

which implies that

$$L \square L \cong (L^2 \square L^2) \oplus (A \square A).$$

Since $L^2 \square L^2 = 0$ by (2.2), it follows that

$$L \square L = A \square A \cong L^{ab} \square L^{ab}$$

for $L^{ab} = L/L^2 \cong A$. The proof is complete. □

Let L be an n-Lie algebra over a field $\mathbb{F}$ with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0,$$

where F is a free n-Lie algebra. Then the Schur multiplier $\mathcal{M}(L)$ of L is defined as

$$\mathcal{M}(L) := \frac{R \cap F^2}{[R, F, \dots, F]}.$$

Notice that the Schur multipliers of every n-Lie algebra are always abelian and isomorphic (see [2, 7, 14, 15]).

The Heisenberg Lie algebras are the ubiquities in the theory of Lie algebras, in particular nilpotent Lie algebras and their classifications. An n-Lie algebra L is called Heisenberg if $L^2 = Z(L)$ and $\dim(L^2) = 1$.

Theorem 2.3 ([7])**.**

(i) Every Heisenberg Lie algebra is isomorphic to the following Lie algebra of odd dimension:

$$H(2, m) = \langle x, x_1, \dots, x_{2m} : [x_{2i-1}, x_{2i}] = x, i = 1, \dots, m \rangle.$$

(ii) Let $H(n, m)$ be a Heisenberg n-Lie algebra of dimension $mn + 1$. Then

$$\dim(\mathcal{M}(H(n, m))) = \begin{cases} n, & m = 1, \\ \binom{mn}{n} - 1, & m > 1. \end{cases}$$

The following lemma establishes the structure of every finite dimensional nilpotent n-Lie algebra L satisfying $\dim(L^2) = 1$.

Lemma 2.1 ([7]). *Let L be a nilpotent n-Lie algebra of dimension d with $\dim(L^2) = 1$. Then there exists $m \geqslant 1$ such that*

$$L \cong H(n, m) \oplus F(d - mn - 1),$$

where, $F(d - mn - 1)$ is an abelian n-Lie algebra of dimension $d - mn - 1$.

3 Exact sequences of non-abelian tensor products of n-Lie algebras

In the rest of this paper, we shall introduce some exact sequences involving non-abelian tensor products of n-Lie algebras, which play important role in the computation of dimensions of the given algebras.

Proposition 3.1. *Let*

$$0 \longrightarrow M \xrightarrow{\delta_1} L \xrightarrow{\sigma_1} P \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\delta_2} L \xrightarrow{\sigma_2} P \longrightarrow 0$$

be two short exact sequences of n-Lie algebras. Also, let the n-Lie algebras in every of the pairs (M, N), (L, K), and (P, Q) act compatibly on each other, and, furthermore, the homomorphisms $\delta_1, \delta_2, \sigma_1, \sigma_2$ preserve the actions. Then the following sequence of n-Lie algebras is exact

$$(M \otimes K) \oplus (L \otimes N) \xrightarrow{\eta} L \otimes K \xrightarrow{\sigma_1 \otimes \sigma_2} P \otimes Q \longrightarrow 0, \tag{3.1}$$

where η is given by

$$\begin{aligned}\left(\sum_{i=1}^{n-1} m_1 \otimes \cdots \otimes m_i \otimes k_{i+1} \otimes \cdots \otimes k_n, \sum_{i=1}^{n-1} l_1 \otimes \cdots \otimes l_i \otimes n_{i+1} \otimes \cdots \otimes n_n\right)& \\ \longmapsto \sum_{i=1}^{n-1} \delta_1(m_1) \otimes \cdots \otimes \delta_1(m_i) \otimes k_{i+1} \otimes \cdots \otimes k_n& \\ + \sum_{i=1}^{n-1} l_1 \otimes \cdots \otimes l_i \otimes \delta_2(n_{i+1}) \otimes \cdots \otimes \delta_2(n_n).&\end{aligned}$$

Proof. First observe that η and $\sigma_1 \otimes \sigma_2$ are well-defined maps. Clearly, $\sigma_1 \otimes \sigma_2$ is an epimorphism as both σ_1 and σ_2 are epimorphisms. Hence, to prove the exactness of (3.1), it remains to show that $\mathrm{Im}(\eta) = \ker(\sigma_1 \otimes \sigma_2)$. Form the definition of η, it follows that

$$\mathrm{Im}(\eta) = \mathrm{Im}(\delta_1 \otimes I_K) + \mathrm{Im}(I_L \otimes \delta_2),$$

where

$$\delta_1 \otimes I_K : M \otimes K \longrightarrow L \otimes \text{ and } I_L \otimes \delta_2 : L \otimes N \longrightarrow L \otimes K$$

are defined naturally. Since

$$[A, X_2, \ldots, X_n] = \underbrace{[A_1, X_2, \ldots, X_n]}_{A_1'} + \underbrace{[A_2, X_2, \ldots, X_n]}_{A_2'} \in \mathrm{Im}(\eta)$$

for all $x_j \in L \otimes K$ ($2 \leqslant j \leqslant n$) and $A = A_1 + A_2 \in \mathrm{Im}(\eta)$ with $A_1 \in \mathrm{Im}(\delta_1 \otimes I_K)$ and $A_2 \in \mathrm{Im}(I_L \otimes \delta_2)$, one can easily see that $\mathrm{Im}(\eta)$ is an ideal of $L \otimes K$.

Now, for every element

$$\sum_{i=1}^{n-1} \delta_1(m_1) \otimes \cdots \otimes \delta_1(m_i) \otimes k_{i+1} \otimes \cdots \otimes k_n + \sum_{i=1}^{n-1} l_1 \otimes \cdots \otimes l_i \otimes \delta_2(n_{i+1}) \otimes \cdots \otimes \delta_2(n_n)$$

of $\mathrm{Im}(\eta)$, we have

$$\begin{aligned}
(\sigma_1 \otimes \sigma_2) & \left(\sum_{i=1}^{n-1} \delta_1(m_1) \otimes \cdots \otimes \delta_1(m_i) \otimes k_{i+1} \otimes \cdots \otimes k_n \right. \\
& \quad \left. + \sum_{i=1}^{n-1} l_1 \otimes \cdots \otimes l_i \otimes \delta_2(n_{i+1}) \otimes \cdots \otimes \delta_2(n_n) \right) \\
&= (\sigma_1 \otimes \sigma_2) \left(\sum_{i=1}^{n-1} \delta_1(m_1) \otimes \cdots \otimes \delta_1(m_i) \otimes k_{i+1} \otimes \cdots \otimes k_n \right) \\
& \quad + (\sigma_1 \otimes \sigma_2) \left(\sum_{i=1}^{n-1} l_1 \otimes \cdots \otimes l_i \otimes \delta_2(n_{i+1}) \otimes \cdots \otimes \delta_2(n_n) \right) \\
&= \sum_{i=1}^{n-1} \left(\underbrace{\sigma_1\delta_1(m_1)}_{0} \otimes \cdots \otimes \underbrace{\sigma_1\delta_1(m_i)}_{0} \otimes \sigma_2(k_{i+1}) \otimes \cdots \otimes \sigma_2(k_n) \right) \\
& \quad + \sum_{i=1}^{n-1} \left(\sigma_1(l_1) \otimes \cdots \otimes \sigma_1(l_i) \otimes \underbrace{\sigma_2\delta_2(n_{i+1})}_{0} \otimes \cdots \otimes \underbrace{\sigma_2\delta_2(n_n)}_{0} \right) = 0
\end{aligned}$$

so that $\mathrm{Im}(\eta) \subseteq \ker(\sigma_1 \otimes \sigma_2)$. Hence, we get a natural epimorphism

$$\theta : \frac{L \otimes K}{\mathrm{Im}(\eta)} \longrightarrow P \otimes Q.$$

On the other hand, one can define a map $\phi : P \otimes Q \longrightarrow (L \otimes K)/\mathrm{Im}(\eta)$ by means of the family $\{\phi_i\}_{0<i<n}$ of maps, where ϕ is defined as

$$\phi_i : \underbrace{P \times \cdots \times P}_{i \text{ times}} \times \underbrace{Q \times \cdots \times Q}_{n-i \text{ times}} \longrightarrow \frac{L\otimes K}{\mathrm{Im}(\eta)}$$

$$(p_1, \ldots, p_i, q_{i+1}, \ldots, q_n) \longmapsto (l_1 \otimes \cdots l_i \otimes k_{i+1} \otimes \cdots \otimes k_n) + \mathrm{Im}(\eta),$$

where $\sigma_1(l_j) = p_j$ and $\sigma_2(k_r) = q_r$ for all $1 \leqslant j \leqslant i \leqslant n-1$ and $i+1 \leqslant r \leqslant n$.

A simple verification shows that $\phi\theta = I$ and $\theta\phi = I$ from which it follows that

$$\frac{L \otimes K}{\mathrm{Im}(\eta)} \cong P \otimes Q.$$

The proof is complete. □

Proposition 3.1 yields the following results immediately.

Corollary 3.1. *Let*

$$0 \longrightarrow M \xrightarrow{\delta} L \xrightarrow{\sigma} P \longrightarrow 0$$

be a short exact sequence of n-Lie algebras. For every n-Lie algebra K acting on M, L, and P and vice versa compatibly, we have an exact sequence of n-Lie algebras

$$M \otimes K \longrightarrow L \otimes K \longrightarrow P \otimes K \longrightarrow 0.$$

Proof. Put $N = Q = K$ in proposition 3.1. □

In the sequel, we apply Proposition 3.1 and its subsequent Corollary 3.2 to obtain a few more exact sequences.

Corollary 3.2. *Let L be an n-Lie algebra and N be an ideal of L. Then the following sequences are exact:*

(i) $(N \otimes L) \oplus (L \otimes N) \xrightarrow{\eta} L \otimes L \xrightarrow{\pi\otimes\pi} L/N \otimes L/N \longrightarrow 0$;

(ii) $N \otimes N \longrightarrow L \otimes N \xrightarrow{\pi\otimes I_N} L/N \otimes N \longrightarrow 0$;

(iii) $N \otimes L/N \longrightarrow L \otimes L/N \xrightarrow{\pi\otimes I_{L/N}} L/N \otimes L/N \longrightarrow 0$;

(iv) $N \otimes L \longrightarrow L \otimes L \xrightarrow{\pi\otimes I_L} L/N \otimes L \longrightarrow 0$; *and*

(v) $N \otimes L \longrightarrow L \otimes L \xrightarrow{\pi\otimes\pi} L/N \otimes L/N \longrightarrow 0$ *if* $N \subseteq L^2$.

Proof. Part (i) follows from Proposition 3.1 by putting $P = Q = L/N$, $L = K$, and $M = N$. Also, putting $K = N$, $K = L/N$, and $K = L$ in Corollary 3.2 yields parts (ii), (iii), and (iv), respectively. Finally, part (v) follows from (3) and (4). □

Proposition 3.2. *To each n-Lie algebra L there corresponds a short exact sequence*

$$0 \longrightarrow \mathcal{M}(L) \longrightarrow \frac{L \otimes L}{L \square L} \longrightarrow L^2 \longrightarrow 0.$$

Proof. We use a free presentation

$$0 \longrightarrow R \longrightarrow L \longrightarrow F \longrightarrow 0$$

of L to prove the result. Since $L \cong F/R$, we get

$$\frac{L \otimes L}{L \square L} \cong \frac{F/R \otimes F/R}{F/R \square F/R}.$$

Clearly, $L^2 = (F^2 + R)/R \cong F^2/(R \cap F^2)$ and that $\mathcal{M}(L) = (R \cap F^2)/[R, F, \ldots, F]$ by the definition. Let ψ be the map defined as

$$\begin{array}{rcl} \psi : \dfrac{F/R \otimes F/R}{F/R \square F/R} & \longrightarrow & \dfrac{F^2}{R \cap F^2} \\ (\bar{f}_1 \otimes \cdots \otimes \bar{f}_n) + (F/R \square F/R) & \longmapsto & [f_1, \ldots, f_n] + (R \cap F^2), \end{array}$$

where $\bar{f}_i := f_i + R$ denotes the image of $f_i \in F$ on F/R. It is not difficult to see that ψ is a well-defined surjection. Also, every element of $R \cap F^2$ can be expressed as $[f_1, \ldots, f_n]$, where $f_1, \ldots, f_n \in F$.

Now we define the map

$$\begin{array}{rcl} \phi : \dfrac{R \cap F^2}{[R, F \ldots, F]} & \longrightarrow & \dfrac{F/R \otimes F/R}{F/R \square F/R} \\ [f_1, \ldots, f_n] + (R \cap F^2) & \longmapsto & (\bar{f}_1 \otimes \cdots \otimes \bar{f}_n) + (F/R \square F/R) \end{array}.$$

It is evident that ϕ is a monomorphism of n-Lie algebras.

From the definitions of ψ and ϕ one observe that $\mathrm{Im}(\phi) \cong \ker(\psi)$, from which we obtain the required exact sequence. □

4 Upper bounds for the dimension of non-abelian tensor squares

In this section, we aim to obtain an upper bound for the dimension of $L \otimes L$ by invoking the same argument as in [13]. The following result has a central role in the proof of our main theorem.

Proposition 4.1. *Let $H(n,m)$ be a Heisenberg n-Lie algebra of dimension $mn+1$. Then*

$$H(n,m) \otimes H(n,m) \cong \frac{H(n,m)}{H^2(n,m)} \otimes \frac{H(n,m)}{H^2(n,m)}$$

if $m \geqslant 2$, and $H(n,1) \otimes H(n,1)$ is an abelian n-Lie algebra of dimension $n^n + n$.

Proof. We know, from Proposition 3.4, that

$$\mathcal{M}(L) \cong \ker \left(\frac{L \otimes L}{L \square L} \xrightarrow{\theta} L^2 \right). \tag{4.1}$$

Hence L^2 is the quotient of $(L \otimes L)/(L \square L)$ by $\ker(\theta)$ for a suitable epimorphism θ. This implies that

$$\begin{aligned} \dim \left(\frac{L \otimes L}{L \square L} / \ker \theta \right) = \dim(L^2) &\Longrightarrow \dim \left(\frac{L \otimes L}{L \square L} \right) - \dim(\ker \theta) = \dim(L^2) \\ &\Longrightarrow \dim(\ker \theta) = \dim \left(\frac{L \otimes L}{L \square L} \right) - \dim(L^2). \end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we get

$$\begin{aligned} \dim(\mathcal{M}(L)) = \dim(\ker \theta) &= \dim \left(\frac{L \otimes L}{L \square L} \right) - \dim(L^2) \\ &= \dim(L \otimes L) - \dim(L \square L) - \dim(L^2), \end{aligned}$$

hence,

$$\dim(L \otimes L) = \dim(\mathcal{M}(L)) + \dim(L \square L) + \dim(L^2). \tag{4.3}$$

Substituting L by $H(n,m)$ in (4.3) yields

$$\begin{aligned} &\dim(H(n,m)) \otimes H(n,m) \\ &\qquad = \dim(\mathcal{M}(H(n,m))) + \dim(H(n,m) \square H(n,m)) + \dim(H(n,m))^2. \end{aligned}$$

Since $\dim(H(n,m))^2 = 1$,

$$\dim(\mathcal{M}(H(n,m))) = \begin{cases} n & m = 1, \\ \binom{mn}{n} - 1 & m > 1. \end{cases}$$

by Theorem 2.10, and

$$\begin{aligned}\dim(H(n,m)\square H(n,m)) &= \dim\left(\frac{H(n,m)}{H^2(n,m)}\square\frac{H(n,m)}{H^2(n,m)}\right)\\ &= (mn)^n - \binom{mn}{n}.\end{aligned}$$

by Proposition 2.9, it follows that

$$\dim(H(n,m)\otimes H(n,m)) = (mn)^n - \binom{mn}{n} + \binom{mn}{n} - 1 + 1 = (mn)^n \tag{4.4}$$

for all $m \geqslant 2$.

Also, from the definition of non-abelian tensor squares of n-Lie algebras, it follows that

$$\dim\left(\frac{H(n,m)}{(H(n,m))^2}\otimes\frac{H(n,m)}{(H(n,m))^2}\right) = (mn)^n. \tag{4.5}$$

On the other hand, we have an epimorphism

$$\eta : H(n,m)\otimes H(n,m) \longrightarrow \frac{H(n,m)}{H^2(n,m)}\otimes\frac{H(n,m)}{H^2(n,m)},$$

which together with (4.4) and (4.5) imply that

$$H(n,m)\otimes H(n,m) \cong H(n,m)/H^2(n,m)\otimes H(n,m)/H^2(n,m).$$

Finally, assume that $m = 1$. The same argument as above shows that

$$\begin{aligned}\dim(H(n,1)\otimes H(n,1)) =& \dim(H(n,1)\square H(n,1))\\ &+ \dim(\mathcal{M}(H(n,1)\otimes H(n,1))) + \dim(H(n,1))^2\\ =&(1\times n)^n - \binom{n}{n} + n + 1 = n^n + n.\end{aligned}$$

Also, from the definiton of Heisenberg n-Lie algebras, one can easily see that $H(n,1)\otimes H(n,1)$ is abelian, from which the result follows. □

Theorem 4.1. *Let L be a non-abelian nilpotent n-Lie algebra of dimension d.*

(i) If $n = 2$, then

$$\dim(L\otimes L) \leqslant (d-1)^2 + 2.$$

(ii) If $n \geqslant 3$, then

$$\dim(L\otimes L) \leqslant ([k/n]n)^n + 2\left(\binom{k}{n} - \binom{[k/n]n}{n}\right) + (k - [k/n]n)^n,$$

where $k := \dim(L/L^2)$ and $[k/n]$ denotes the integer part of k/n.

Proof. (i) See [13].

(ii) We use induction on the dimension of L^2. Let $s := \dim(L^2)$. If $s = 1$ then Lemma 2.7 yields

$$L \cong H(n,m) \oplus F(d-mn-1),$$

where $H(n,m)$ is a Heisenberg n-Lie algebra of dimension $mn+1$, and $F(d-mn-1)$ is the free abelian n-Lie algebra of dimension $d-mn-1$. From [1, Theorem 4.8], we get

$$\begin{aligned} L\otimes L \cong & \big(H(n,m)\otimes H(n,m)\big) \oplus \big(H(n,m)\otimes F(d-mn-1)\big) \\ & \oplus \big(F(d-mn-1)\otimes H(n,m)\big) \oplus \big(F(d-mn-1)\otimes F(d-mn-1)\big), \end{aligned}$$

which implies that

$$\begin{aligned} \dim(L\otimes L) \cong & \dim\big(H(n,m)\otimes H(n,m)\big) \\ & + \dim\big(H(n,m)\otimes F(d-mn-1)\big) \\ & + \dim\big(F(d-mn-1)\otimes H(n,m)\big) \\ & + \dim\big(F(d-mn-1)\otimes F(d-mn-1)\big). \end{aligned}$$

On the other hand, from Proposition 2.5 we obtain

$$H(n,m)\otimes F(d-mn-1) \cong H(n,m)^{ab} \otimes^n_{\text{mod}} F(d-mn-1).$$

The last two equations imply that

$$\begin{aligned} \dim(L\otimes L) \cong & \dim\big(H(n,m)\otimes H(n,m)\big) \\ & + 2\dim\big(H(n,m)^{ab}\otimes F(d-mn-1)\big) \\ & + \dim\big(F(d-mn-1)\otimes F(d-mn-1)\big). \end{aligned} \tag{4.6}$$

We have two cases to consider:

(i) $m = 1$. Following Proposition 4.1 and (4.6), we get

$$\begin{aligned} \dim(L\otimes L) =& (n^n+n) + 2\binom{d-n-1+n}{n} - 2\binom{d-n-1}{n} \\ & - 2\binom{n}{n} + (d-n-1)^n \\ =& n^n + n + 2\binom{d-1}{n} - 2\binom{d-n-1}{n} - 2 + (d-n-1)^n. \end{aligned} \tag{4.7}$$

(ii) $m \geqslant 2$. From (4.6), we obtain

$$\begin{aligned}\dim(L \otimes L) =&(mn)^n + 2\binom{d-mn-1+mn}{n} - 2\binom{d-mn-1}{n} \\ &- 2\binom{mn}{n} + (d-mn-1)^n \\ =&(mn)^n + 2\binom{d-1}{n} - 2\binom{d-mn-1}{n} \\ &- 2\binom{mn}{n} + (d-mn-1)^n.\end{aligned} \tag{4.8}$$

A simple computation shows that the number in (4.8) is greater or equal than the number in (4.7), and that it is an increasing function of m. Also, $m \leqslant (d-1)/n$ as L is of dimension d. Hence, it follows from (4.8) that

$$\begin{aligned}&(mn)^n + 2\binom{d-1}{n} - 2\binom{d-mn-1}{n} - 2\binom{mn}{n} + (d-mn-1)^n \\ &\leqslant ([(d-1)/n]n)^n + 2\binom{d-1}{n} - 2\cancelto{0}{\binom{d-1-[(d-1)/n]n}{n}} \\ &\qquad - 2\binom{[(d-1)/n]n}{n} + (d-1-[(d-1)/n]n)^n\end{aligned}$$

giving rise to an upper bound for the dimension of $L \otimes L$.

Finally, suppose that $s = \dim(L^2) > 1$. Utilizing an inductive argument, it follows that

$$\dim(L \otimes L) \leqslant ([(d-s)/n]n)^n + 2\left(\binom{d-s}{n} - \binom{[(d-s)/n]n}{n}\right) + (d-s-[(d-s)/n]n)^n.$$

Since $k = \dim(L/L^2) = d - s$, the above result can be written as

$$\dim(L \otimes L) \leqslant ([k/n]n)^n + 2\left(\binom{k}{n} - \binom{[k/n]n}{n}\right) + (k-[k/n]n)^n,$$

as required. The proof is complete. $\square$

If there exists a algebra with conditions in Theorem 4.2, then the following example illustrates the sharpness of the obtained above bound.

Example 1. Let the 3-Lie algebra

$$A_3 = \langle x_1, x_2, x_3, x_4, x_5;\ [x_1, x_2, x_3] = x_4,\ [x_2, x_3, x_4] = x_5\rangle.$$

It is clear that $cl(A_3) = 2$, $A_3^2 = \langle x_4, x_5\rangle$ and hence $s = \dim A_3^2 = 2$ and $k = \dim A_3^{ab} = 3$. Using Theorem 4.2, we have

$$\dim A_3 \otimes A_3 \leqslant 3^3 = 27.$$

On the other hand, by calculating the basic elements of $A_3 \otimes A_3$ we obtain that

$$\begin{aligned} A_3 \otimes A_3 = \langle & x_1 \otimes x_1 \otimes x_1,\ x_1 \otimes x_1 \otimes x_2,\ \ldots,\ x_1 \otimes x_1 \otimes x_5, \\ & x_1 \otimes x_2 \otimes x_1,\ x_1 \otimes x_2 \otimes x_2,\ \ldots,\ x_1 \otimes x_2 \otimes x_5, \\ & x_1 \otimes x_3 \otimes x_1,\ x_1 \otimes x_3 \otimes x_2,\ \ldots,\ x_1 \otimes x_3 \otimes x_5, \\ & x_1 \otimes x_4 \otimes x_1,\ x_1 \otimes x_4 \otimes x_2,\ x_1 \otimes x_4 \otimes x_3, \\ & x_1 \otimes x_5 \otimes x_1,\ x_1 \otimes x_5 \otimes x_2,\ x_1 \otimes x_5 \otimes x_3, \\ & x_2 \otimes x_2 \otimes x_2,\ x_2 \otimes x_2 \otimes x_3, \\ & x_2 \otimes x_3 \otimes x_3,\ x_2 \otimes x_3 \otimes x_4,\ x_2 \otimes x_3 \otimes x_5, \\ & x_3 \otimes x_3 \otimes x_3\rangle. \end{aligned}$$

Therefore, dim $A_3 \otimes A_3 = 27$.

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