A New Fuzzy Information Inequalities and its Applications in Establishing Relation among Fuzzy f-Divergence Measures

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Abstract. There are many fuzzy information and divergence measures exist in the literature of fuzzy Information Theory. Inequalities play important role for finding the relations. Here, we will introduce some new information inequalities on fuzzy measures and their applications in pattern recognition. Also established relations between new and well known fuzzy divergence measures with help of the fuzzy f-divergence measures and jensen's inequality.

1   Introduction

In information theory, Shannon [28] defined the entropy measure for a probability distribution. Fuzzy entropy, a fuzziness measure (cf. [21]) often used and cited in many literatures, was introduced by Zadeh [34, 35]; then De Luca and Termini [20] defined an entropy of a fuzzy set based on Shannon's function. Later on, many other researchers made more effort in this particular area. In 1975, Kaufmann [18] proposed a new fuzziness measure of fuzzy set by a distance between its membership function and its nearest Classical(Ordinary) sets. It has broad applications in many areas such as Pattern recognition, image processing, fuzzy clustering, speech recognition, feature selection, fuzzy aircraft control, bio-informatics etc.

Afterwards, a number of other researchers have studied the fuzzy divergence measures in different ways [26] and provide their applications in different areas. The fuzzy divergence measure introduced by Fan and Xie [8] is based on exponential operation and its relation with fuzzy divergence measures [2]. Prakash et al. [23] proposed two fuzzy divergence measures corresponding to Ferreri [11] probabilistic measure of directed divergence. Ghosh et al. [14] gave its application in the area of automated leukocyte recognition. The study submitted by Montes et al. [22] in 2002 was based on special classes of divergence measures and used the link between fuzzy and probabilistic uncertainty. Bhatia and Singh [4] proposed the fuzzy divergence measure corresponding to Taneja [30]. Tomar and Ohlan [32] studied a sequence of fuzzy mean difference divergence

measures by establishing inequalities among them and provided their applications in the context of consistency in linguistic variables and pattern recognition.

Entropy is the important concept of Information Theory. Shannon [28] use the entropy to measure a degree of uncertainty of randomness in a p.d. Let

$$\Gamma_n = \big\{ P = (p_1, p_2, p_3, \cdots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1, \big\}$$

be a set of complete finite discrete p.d., and

$$H(P) = \sum_{i=1}^{n} -p_i \log p_i.$$

Directed divergence is a cross entropy measure which provides a distance between the two p.d. Kullback and Leibler [19] proposed a measure of directed divergence between the two distributions $P = (p_1, p_2, \cdots, p_n)$ and $Q = (q_1, q_2, \cdots, q_n)$ as:

$$D(P \mid Q) = \sum_{i=1}^{n} p_i \log(\frac{p_i}{q_i}).$$

The Csiszar's f-divergence measure and new f-divergence measure contains divergences used in determining the affinity between two p.d using a convex function f, defined on $(0, \infty)$.

$$C_f(P, Q) = \sum_{i=1}^{n} q_i f(\frac{p_i}{q_i})$$

and

$$S_f(P, Q) = \sum_{i=1}^{n} q_i f(\frac{p_i + q_i}{2q_i}).$$

2   Fuzzy Sets and Fuzzy Measures of Information

Fuzziness is a feature of uncertainty, results from the lack of sharp difference of being or not being a member of the set. A fuzzy set A defined on a universe of discourse X is given as Zadeh [34]

$$A = \{\langle x, \mu_A(x)/x \in X\rangle\},$$

where $\mu_A(x) : X \to [0, 1]$ is the membership function of A. The membership value describes the degree of belongingness of X in A.

Fuzzy entropy corresponding to Shannon's entropy given as

$$H(N) = \sum_{i=1}^{n} -\Big(\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))\Big).$$

This idea of divergence measure was extended from probabilistic to fuzzy set theory by giving a fuzzy information measure for discrimination of a fuzzy set B relative to some other fuzzy set A. Let A and B be two fuzzy sets defined in universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$ having the membership values $\mu_A(x_i)$ and $\mu_B(x_i)$, $i = 1, 2, \cdots, n$ respectively. Then the fuzzy divergence measures : (Here $\mu_A(x_i)$ and $\mu_B(x_i)$ are denoted by μ_{A_i} and μ_{B_i}.)

(i) **Fuzzy K-L Divergence [19]**-

$$K(A, B) = \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (1 - \mu_{A_i}) \log \left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right] \tag{2.1}$$

(ii) **Fuzzy Relative Information of type n [31]** :-

$$D_n(A, B) = \Big(n(n-1) \Big)^{-1} \left[\sum_{i=1}^{n} \Big((\mu_{A_i})^n (\mu_{B_i})^{1-n} + (1 - \mu_{A_i})^n (1 - \mu_{B_i})^{1-n} \Big) - 1 \right], \tag{2.2}$$

where $n \neq 0, 1$.

(iii) **Fuzzy Chi-Square Divergence [24]**:-

$$\chi^2(A, B) = \sum_{i=1}^{n} \left[\frac{(\mu_{A_i})^2}{\mu_{B_i}} + \frac{(1 - \mu_{A_i})^2}{1 - \mu_{B_i}} \right] - 1, \tag{2.3}$$

(iv) **Fuzzy Relative Jensen-Shannon Divergence [33]**:-

$$S_t(A, B) = \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{2\mu_{A_i}}{\mu_{A_i} + \mu_{B_i}} \right) + (1 - \mu_{A_i}) \log \left(\frac{2(1 - \mu_{A_i})}{2 - \mu_{A_i} - \mu_{B_i}} \right) \right] \tag{2.4}$$

(v) **Fuzzy Relative Arithmetic-Geometric Divergence [5]**:-

$$H(A, B) = \sum_{i=1}^{n} \left[\left(\frac{\mu_{A_i} + \mu_{B_i}}{2} \right) \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{A_i}} \right) + \right.$$
$$\left. \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2} \right) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{A_i})} \right) \right] \tag{2.5}$$

(vi) **Fuzzy Unified Relative Jensen-Shannon and Arithmetic-Geometric Divergence of type** n

$$MG_n(A, B) = \Big(n(n-1) \Big)^{-1} \Big(\sum_{i=1}^{n} \Big[\mu_{A_i} \Big(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \Big)^n + $$

$$(1 - \mu_{A_i}) \Big(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \Big)^n \Big] - 1 \Big), \qquad (2.6)$$

$n \neq 0, 1.$

(vii) **Fuzzy Hellinger Discrimination [1]:-**

$$M(A, B) = 1 - T(A, B)$$
$$= \frac{1}{2} \sum_{i=1}^{n} \Big[\Big(\sqrt{\mu_{A_i}} - \sqrt{\mu_{B_i}} \Big)^2 + \Big(\sqrt{(1 - \mu_{A_i})} - \sqrt{(1 - \mu_{B_i})} \Big)^2 \Big], \qquad (2.7)$$

where $T(A, B) = \sum\limits_{i=1}^{n} \Big[\sqrt{\mu_{A_i} \mu_{B_i}} + \sqrt{(1 - \mu_{B_i})(1 - \mu_{B_i})} \Big].$

(viii) **Fuzzy Triangular Discrimination [10]:-**

$$\triangle(A, B) = \sum_{i=1}^{n} \Big[\frac{(\mu_{A_i} - \mu_{B_i})^2}{(\mu_{A_i} + \mu_{B_i})} + \frac{(\mu_{B_i} - \mu_{A_i})^2}{(2 - \mu_{A_i} - \mu_{B_i})} \Big], \qquad (2.8)$$

(ix) **Fuzzy Symmetric Chi-Square Divergence [9]:-**

$$\varphi(A, B) = \chi^2(A, B) + \chi^2(B, A)$$
$$= \sum_{i=1}^{n} \Big[\frac{(\mu_{A_i} - \mu_{B_i})^2(\mu_{A_i} + \mu_{B_i})}{(\mu_{A_i} \mu_{B_i})} + \frac{(\mu_{B_i} - \mu_{A_i})^2(2 - \mu_{A_i} - \mu_{B_i})}{(1 - \mu_{A_i})((1 - \mu_{A_i})} \Big] \qquad (2.9)$$

(x) **Fuzzy Symmetric J-Divergence Measure [7]:-**

$$N(A, B) = K(A, B) + K(B, A)$$
$$= \sum_{i=1}^{n} \Big[(\mu_{A_i} - \mu_{B_i}) \log \Big(\frac{\mu_{A_i}}{\mu_{B_i}} \Big) + (\mu_{B_i} - \mu_{A_i}) \log \Big(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \Big) \Big]. \qquad (2.10)$$

(xi) **Fuzzy Relative J-Divergence Measure [29]:-**

$$J_m(A, B) = \sum_{i=1}^{n} \Big[(\mu_{A_i} - \mu_{B_i}) \log \Big(\frac{(\mu_{A_i} + \mu_{B_i})}{2\mu_{B_i}} \Big) +$$
$$(\mu_{B_i} - \mu_{A_i}) \log \Big(\frac{(2 - \mu_{A_i} - \mu_{B_i})}{2(1 - \mu_{B_i})} \Big) \Big]. \qquad (2.11)$$

(xii) **Fuzzy Symmetric Arithmetic-Geometric Divergence [30] –**

$$F(A, B) = \frac{1}{2}(H(A, B) + H(B, A)) = \sum_{i=1}^{n} \Big[\Big(\frac{\mu_{A_i} + \mu_{B_i}}{2} \Big) \log \Big(\frac{\mu_{A_i} + \mu_{B_i}}{2\sqrt{\mu_{A_i} \mu_{B_i}}} \Big) +$$
$$\Big(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2} \Big) \log \Big(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \Big) \Big] \qquad (2.12)$$

(xiii) **Fuzzy Symmetric Chi-Square, Arithmetic and Geometric Mean Divergence Measure [17]:-**

$$E_1^*(A, B) = \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{\mu_{A_i}\mu_{B_i}}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right] \qquad (2.13)$$

(xiv) **Fuzzy Symmetric Jensen-Shannon Divergence Measure [5, 27]:-**

$$I(A, B) = S_t(A, B) + S_t(B, A)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{2\mu_{A_i}}{\mu_{A_i} + \mu_{B_i}} \right) + \mu_{B_i} \log \left(\frac{2\mu_{B_i}}{\mu_{A_i} + \mu_{B_i}} \right) \right] +$$

$$\frac{1}{2} \sum_{i=1}^{n} \left[(1 - \mu_{A_i}) \log \left(\frac{2(1 - \mu_{A_i})}{2 - \mu_{A_i} - \mu_{B_i}} \right) + (1 - \mu_{B_i}) \log \left(\frac{2(1 - \mu_{B_i})}{2 - \mu_{A_i} - \mu_{B_i}} \right) \right]. \quad (2.14)$$

3 New Fuzzy Information Inequality

Theorem 3.1. *Let $f : R_+ \to R_+$ be a convex function and normalized i.e. f(1) = 0. Then $\forall$ A and B be two fuzzy sets defined in universe of discourse $X = \{x_1, x_2, .., x_n\}$ having the membership values μ_{A_i} and μ_{B_i}, $i = 1, 2, .., n$ respectively,*

$$\sum_{i=1}^{n} \left[\mu_{B_i} f \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \right) + (1 - \mu_{B_i}) f \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right) \right]$$

$$\leq \frac{1}{2} \sum_{i=1}^{n} \left[\mu_{B_i} f \left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (1 - \mu_{B_i}) f \left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right],$$

$$S_f(A, B) \leq \frac{1}{2} C_f(A, B), \qquad (3.1)$$

and equality holds in (3.1) iff $\frac{\mu_{A_1}}{\mu_{B_1}} = \frac{\mu_{A_2}}{\mu_{B_2}} = \cdots = \frac{\mu_{A_n}}{\mu_{B_n}}$.

Proof. Let $f : k \subset R \to R$ be differentiable convex on the interval k and $y \in k^0$, where k^0 is interior of k. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in X_n$, then Jensen inequality is

$$f \left(\sum_{i=n}^{n} \lambda_i y_i \right) \leq \sum_{i=n}^{n} \lambda_i f(y_i).$$

if $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\lambda_3 = \lambda_4 = \cdots = \lambda_n = 0$, then

$$f(\frac{y_1 + y_2}{2}) \le \frac{1}{2}[f(y_1) + f(y_2)].$$

Putting $y_1 = y$ and $y_2 = 1$ then

$$f(\frac{y + 1}{2}) \le \frac{1}{2}[f(y) + f(1)] \le \frac{1}{2}f(y),$$

since $f(1) = 0$. Put $y = \frac{\mu_{A_i}}{\mu_{B_i}}$ and apply fuzzy, then

$$f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \le \frac{1}{2}\left[f\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}}\right)\right]$$

Multiplying by μ_{B_i} and taking summation both side, we have

$$\sum_{i=1}^{n}\left[\mu_{B_i}f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + (1 - \mu_{B_i})f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right)\right]$$
$$\le \frac{1}{2}\sum_{i=1}^{n}\left[\mu_{B_i}f\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{B_i})f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}}\right)\right],$$

that is, $S_f(A, B) \le \frac{1}{2}C_f(A, B)$. Equation (3.1) is a relation between fuzzy new f-divergence and fuzzy Csiszar f-divergence functional.   □

4   Results Analysis:-

In this section, we shall show relationship in the form of inequality using the results (3.1) of theorem (3.1) among the various known measures such as Csiszar's, new f-divergence, Hellinger, Chi-square, and J-divergence etc.

Result 1. Let $A, B \in X_n$ then we have the following relations

$$S_t(A, B) \le \frac{1}{2}K(A, B), \tag{4.1}$$

$$S_t(B, A) \le \frac{1}{2}K(B, A), \tag{4.2}$$

where $S_t(A, B)$ and $K(A, B)$ are Fuzzy Relative Jensen Shannon Divergence Measure and Fuzzy K-L Divergence Measure given in (2.1) and (2.4).

Proof. Consider the mapping $f : (0, \infty) \longrightarrow R$,

$$f(y) = -\log y, f^{''}(y) = \frac{1}{y^2} > 0 \ \forall y > 0, f(1) = 0,$$

so f is convex and normalized. Then

$$S_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \right]$$

$$= \sum_{i=1}^{n} \left[\mu_{B_i} \log\left(\frac{2\mu_{B_i}}{\mu_{A_i} + \mu_{B_i}}\right) + (1 - \mu_{B_i}) \log\left(\frac{2(1 - \mu_{B_i})}{2 - \mu_{A_i} - \mu_{B_i}}\right) \right] = S_t(B, A). \quad (4.3)$$

Interchange by $A \mapsto B$,

$$S_f(B, A) = \sum_{i=1}^{n} \left[\mu_{A_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{A_i}}\right) + (1 - \mu_{A_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{A_i})}\right) \right]$$

$$= \sum_{i=1}^{n} \left[\mu_{A_i} \log\left(\frac{2\mu_{A_i}}{\mu_{B_i} + \mu_{A_i}}\right) + (1 - \mu_{A_i}) \log\left(\frac{2(1 - \mu_{A_i})}{2 - \mu_{B_i} - \mu_{A_i}}\right) \right] = S_t(A, B) \quad (4.4)$$

Similarly,

$$C_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}}\right) \right]$$

$$= \sum_{i=1}^{n} \left[\mu_{B_i} \log\left(\frac{\mu_{B_i}}{\mu_{A_i}}\right) + (1 - \mu_{B_i}) \log\left(\frac{1 - \mu_{B_i}}{1 - \mu_{A_i}}\right) \right] = K(B, A) \quad (4.5)$$

Interchange by $A \mapsto B$,

$$C_f(B, A) = \sum_{i=1}^{n} \left[\mu_{A_i} f\left(\frac{\mu_{B_i}}{\mu_{A_i}}\right) + (1 - \mu_{A_i}) f\left(\frac{1 - \mu_{B_i}}{1 - \mu_{A_i}}\right) \right]$$

$$= \sum_{i=1}^{n} \left[\mu_{A_i} \log\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{A_i}) \log\left(\frac{1 - \mu_{A_i}}{(1 - \mu_{B_i})}\right) \right] = K(A, B) \quad (4.6)$$

$\square$

From equations (3.1), (4.3), (4.4), (4.5) and (4.6) give the results (4.1) and (4.2).

Result 2. Consider $A, B \in X_n$, then we have the following relations

$$H(B, A) \leq \frac{1}{2} K(A, B) \quad (4.7)$$

where $K(A, B)$ and $H(A, B)$ are Fuzzy K-L divergence and Fuzzy Relative Arithmetic-Geometric divergence measure given in (2.1) and (2.5).

Proof. Consider the mapping $f : (0, \infty) \longrightarrow R$

$$f(y) = y \log y, f''(y) = \frac{1}{y} > 0, \forall y > 0, f(1) = 0,$$

so f is convex and normalized. Then

$$
\begin{aligned}
S_f(A, B) &= \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \right] \\
&= \sum_{i=1}^{n} \left[\left(\frac{\mu_{A_i} + \mu_{B_i}}{2}\right) \log\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2}\right) \log\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \right] \\
&= H(B, A), \tag{4.8}
\end{aligned}
$$

$$
\begin{aligned}
C_f(A, B) &= \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}}\right) \right] \\
&= \sum_{i=1}^{n} \left[\mu_{A_i} \log\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{A_i}) \log\left(\frac{1 - \mu_{A_i}}{(1 - \mu_{B_i})}\right) \right] = K(A, B.) \tag{4.9}
\end{aligned}
$$

The equation (3.1) using (4.8) and (4.9) gives the result (4.7).   □

Result 3. Consider $A, B \in X_n$ then we have the following relations

$$
J_m(A, B) \leq N(A, B), \tag{4.10}
$$

$$
I(A, B) + F(A, B) \leq \frac{1}{2} N(A, B), \tag{4.11}
$$

where $N(A, B), J_m(A, B), F(A, B),$ and $I(A, B)$ are Fuzzy J-divergence, Fuzzy relative J-divergence, Fuzzy arithmetic-geometric divergence and Fuzzy jensen-Shannon divergence measure given in (2.10), (2.11),(2.12) and (2.14) respectively.

Proof. Consider the mapping $f : (0, \infty) \longrightarrow R$

$$
f(y) = (y - 1) \log y, \, f''(y) = \frac{y + 1}{y^2} > 0, \forall y > 0, f(1) = 0,
$$

so f is convex and normalized. Then

$$
\begin{aligned}
S_f(A, B) &= \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \right] \\
&= \frac{1}{2} \sum_{i=1}^{n} \left[(\mu_{A_i} - \mu_{B_i}) \log\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}}\right) + (\mu_{B_i} - \mu_{A_i}) \log\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})}\right) \right] \tag{4.12} \\
&= \frac{1}{2} J_m(A, B).
\end{aligned}
$$

$$
C_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i}}{\mu_{B_i}}\right) + (1 - \mu_{B_i}) f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}}\right) \right]
$$

$$= \sum_{i=1}^{n} \left[(\mu_{A_i} - \mu_{B_i}) \log \left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (\mu_{B_i} - \mu_{A_i}) \log \left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right] = N(A, B) \qquad (4.13)$$

The equation (3.1) using (4.12) and (4.13) gives (4.10).

Now again we get

$$S_f(A, B)$$
$$= \frac{1}{2} \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{A_i}} \right) - \mu_{B_i} \log \left(\frac{\mu_{A_i}\mu_{B_i}}{2\mu_{B_i}} \right) \right]$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \left[(1 - \mu_{A_i}) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{A_i})} \right) - (1 - \mu_{B_i}) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right) \right]$$
$$= \frac{1}{2} \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{2\mu_{A_i}}{\mu_{A_i} + \mu_{B_i}} \right) + \mu_{B_i} \log \left(\frac{2\mu_{B_i}}{\mu_{A_i} + \mu_{B_i}} \right) \right]$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \left[(1 - \mu_{A_i}) \log \left(\frac{2(1 - \mu_{A_i})}{2 - \mu_{A_i} - \mu_{B_i}} \right) + (1 - \mu_{B_i}) \log \left(\frac{2(1 - \mu_{B_i})}{2 - \mu_{A_i} - \mu_{B_i}} \right) \right]$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{A_i}} \right) \right.$$
$$\left. + (1 - \mu_{B_i}) \log \left(\frac{(2 - \mu_{A_i} - \mu_{B_i})}{2(1 - \mu_{B_i})} \frac{(2 - \mu_{A_i} - \mu_{B_i})}{2(1 - \mu_{A_i})} \right) \right]$$
$$= I(A, B) + \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\sqrt{\mu_{A_i}\mu_{B_i}}} \right) + (1 - \mu_{A_i}) \log \left(\frac{(2 - \mu_{A_i} - \mu_{B_i})}{2\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right].$$
$$(4.14)$$

Using equations (3.1), (4.13) and (4.14), we get

$$I(A, B) + \sum_{i=1}^{n} \left[\mu_{A_i} \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\sqrt{\mu_{A_i}\mu_{B_i}}} \right) \right] +$$
$$\sum_{i=1}^{n} \left[(1 - \mu_{A_i}) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right] \leq \frac{1}{2} N(A, B) \qquad (4.15)$$

Interchanging A and B we get

$$I(B, A) + \sum_{i=1}^{n} \left[\mu_{B_i} \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\sqrt{\mu_{A_i}\mu_{B_i}}} \right) \right] +$$
$$\sum_{i=1}^{n} \left[(1 - \mu_{B_i}) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right] \leq \frac{1}{2} N(B, A). \qquad (4.16)$$

Adding inequalities (4.15) and (4.16)

$$I(A,B) + I(B,A) + \sum_{i=1}^{n} \left[(\mu_{A_i} + \mu_{B_i}) \log \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\sqrt{\mu_{A_i}\mu_{B_i}}} \right) \right] +$$

$$\sum_{i=1}^{n} \left[(2 - \mu_{A_i} - \mu_{B_i}) \log \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2\sqrt{(1-\mu_{A_i})(1-\mu_{B_i})}} \right) \right] \le \frac{1}{2} N(A,B) + \frac{1}{2} N(B,A)$$

$$2I(A,B) + 2F(A,B) \le$$

$$\sum_{i=1}^{n} \left[(\mu_{A_i} - \mu_{B_i}) \log \left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) \right] + \sum_{i=1}^{n} \left[(\mu_{B_i} - \mu_{A_i}) \log \left(\frac{(1-\mu_{A_i})}{(1-\mu_{B_i})} \right) \right]$$

Therefore $I(A,B) = I(B,A)$ and $2I(A,B) + 2F(A,B) \le N(A,B)$. $\square$

Note that, from equation (4.11), we get

(a$_1$) $I(A,B) \le \frac{1}{2} N(A,B)$

(a$_2$) $F(A,B) \le \frac{1}{2} N(A,B)$, with equality if $A = B$, where $I(A,B)$ and $F(A,B)$ both are positive.

Result 4. Let $A, B \in X_n$ then we have the following relations

$$\triangle(A,B) < E_1^*(A,B), \tag{4.17}$$

where $\triangle(A,B)$ and $E_1^*(A,B)$ are Fuzzy Triangular discrimination and Fuzzy Symmetric Chi square, Arithmetic and Geometric mean divergence measure give in (2.8) and (2.13) respectively.

Proof. Consider the mapping $f : (0,\infty) \longrightarrow R$

$$f(y) = \frac{(y-1)^2}{\sqrt{y}}, f''(y) = \frac{3}{4} y^{\frac{-1}{2}} + \frac{3}{4} y^{\frac{-5}{2}} + \frac{1}{2} y^{\frac{-3}{2}} > 0, \forall y > 0, f(1) = 0,$$

so f is convex and normalized. Then

$$S_f(A,B) = \sum_{i=1}^{n} \left[\mu_{B_i} f \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \right) + (1 - \mu_{B_i}) f \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right) \right]$$

$$= \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{2\mu_{B_i}(\mu_{A_i} + \mu_{B_i})}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{2(1-\mu_{B_i})(2-\mu_{B_i}-\mu_{A_i})}} \right) \right], \tag{4.18}$$

$$C_f(A,B) = \sum_{i=1}^{n} \left[\mu_{B_i} f \left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (1 - \mu_{B_i}) f \left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right]$$

$$= \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{\mu_{A_i} \mu_{B_i}}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right] = E_1^*(A, B). \qquad (4.19)$$

From equations (3.1), (4.18) and (4.19)

$$\sum_{i=1}^{n} \left[E_1^*(A, B) \geq 2 \left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{2\mu_{B_i}(\mu_{A_i} + \mu_{B_i})}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{2(1 - \mu_{B_i})(2 - \mu_{B_i} - \mu_{A_i})}} \right) \right]$$

$$= \sqrt{2} \left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{\mu_{B_i}(\mu_{A_i} + \mu_{B_i})}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{(1 - \mu_{B_i})(2 - \mu_{B_i} - \mu_{A_i})}} \right) \right],$$

and

$$\triangle(A, B) = \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{(\mu_{A_i} + \mu_{B_i})} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{(2 - \mu_{A_i} - \mu_{B_i})} \right) \right]$$

$$< \sqrt{2} \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{(\mu_{A_i})^2 + (\mu_{B_i})^2 + 2\mu_{A_i} \mu_{B_i}}} \right) + \right.$$

$$\left. \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{(1 - \mu_{A_i})^2 + (1 - \mu_{B_i})^2 + 2(1 - \mu_{A_i})(1 - \mu_{B_i})}} \right) \right]$$

$$< \sqrt{2} \sum_{i=1}^{n} \left[\left(\frac{(\mu_{A_i} - \mu_{B_i})^2}{\sqrt{\mu_{B_i}(\mu_{A_i} + \mu_{B_i})}} \right) + \left(\frac{(\mu_{B_i} - \mu_{A_i})^2}{\sqrt{(1 - \mu_{B_i})(2 - \mu_{A_i} - \mu_{B_i})}} \right) \right]$$

$$\leq E_1^*(A, B).$$

Hence the relation (4.17) is proved. $\square$

Result 5. Let $A, B \in X_n$ then we have the following relations

$$\frac{1}{2}\chi^2(A, B) + \triangle(A, B) \leq \varphi(A, B) \qquad (4.20)$$

where $\chi^2(A, B)$, $\triangle(A, B)$ and $\varphi(A, B)$ are fuzzy Chi-square divergence, Fuzzy triangular discrimination and Fuzzy symmetric Chi-square divergence measure give in (2.3), (2.8) and (2.9) respectively.

Proof. Consider the mapping $f : (0, \infty) \longrightarrow R$

$$f(y) = \frac{(y - 1)^2(y + 1)}{y}, f''(y) = 2 + \frac{2}{y^3} > 0, \forall y > 0, f(1) = 0,$$

so f is convex and normalized. Then

$$S_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \right) + (1 - \mu_{B_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right) \right]$$

$$= \frac{1}{4} \sum_{i=1}^{n} \left[\frac{(\mu_{A_i} - \mu_{B_i})^2}{\mu_{B_i}} + \frac{(\mu_{A_i} - \mu_{B_i})^2}{(1 - \mu_{B_i})} \right] + \frac{1}{2} \sum_{i=1}^{n} \left[\frac{(\mu_{A_i} - \mu_{B_i})^2}{\mu_{A_i} + \mu_{B_i}} + \frac{(\mu_{A_i} - \mu_{B_i})^2}{2 - \mu_{A_i} - \mu_{B_i}} \right]$$

$$= \frac{1}{4} \chi^2(A, B) + \frac{1}{2} \triangle(A, B), \tag{4.21}$$

$$C_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (1 - \mu_{B_i}) f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right] = \varphi(A, B) \tag{4.22}$$

The equation (3.1) using (4.21) and (4.22) gives the result (4.20). □

Result 6. Let $A, B \in X_n$ then we have the following relations

$$MG_n(B, A) \leq \frac{1}{2} D_n(A, B) \tag{4.23}$$

where $MG_n(A, B)$ and $D_n(A, B)$ are Unified fuzzy relative Jensen-Shannon and Arithmetic-Geometric Divergence of type n and Fuzzy relative information of type n give in (2.6) and (2.2), respectively.

Proof. Consider the mapping $f : (0, \infty) \longrightarrow R$

$$f_n(y) = \begin{cases} [n(n - 1)]^{-1}[y^n - 1] & \text{if} \neq 0, 1, \\ -\log y & \text{if } n = 0, \\ y \log y & \text{if } n = 1. \end{cases} \tag{4.24}$$

Since $f''(y) \geq 0$ and $f(1) = 0$, so the function f is convex and normalized. Then

$$S_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \right) + (1 - \mu_{B_i}) f\left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right) \right]$$

$$= [n(n - 1)]^{-1} \left[\left[\sum_{i=1}^{n} \mu_{B_i} \left(\frac{\mu_{A_i} + \mu_{B_i}}{2\mu_{B_i}} \right)^n + (1 - \mu_{B_i}) \left(\frac{2 - \mu_{A_i} - \mu_{B_i}}{2(1 - \mu_{B_i})} \right)^n \right] - 1 \right] \tag{4.25}$$

$$= MG_n(B, A)$$

$$C_f(A, B) = \sum_{i=1}^{n} \left[\mu_{B_i} f\left(\frac{\mu_{A_i}}{\mu_{B_i}} \right) + (1 - \mu_{B_i}) f\left(\frac{1 - \mu_{A_i}}{1 - \mu_{B_i}} \right) \right]$$

$$= \left(n(n - 1) \right)^{-1} \left[\sum_{i=1}^{n} \left((\mu_{A_i})^n (\mu_{B_i})^{1-n} + (1 - \mu_{A_i})^n (1 - \mu_{B_i})^{1-n} \right) - 1 \right] \tag{4.26}$$

$$= D_n(A, B)$$

The quation (3.1) using (4.24), (4.25) and (4.26) gives the result (4.23).   □

Particular Cases of Result 4.6:-

(i) If $n = 0, 1$ then we have proved in equation (4.1) and (4.7) of results in (4.1) and (4.2) respectively.

(ii) If $n = -1$ then we get the following relation

$$\triangle(A, B) \leq \chi^2(B, A). \tag{4.27}$$

Proof. Put $n = -1$ in equation (4.23), then

$$MG_{-1}(B, A) \leq \frac{1}{2} D_{-1}(A, B), \tag{4.28}$$

$$MG_{-1}(B, A) = \frac{1}{2} \Big(\sum_{i=1}^{n} \Big[\frac{2(\mu_{B_i})^2}{\mu_{A_i} + \mu_{B_i}} + \frac{2(1 - \mu_{B_i})^2}{2 - \mu_{A_i} - \mu_{B_i}} \Big] - 1 \Big)$$

$$= \frac{1}{2} \Big(\sum_{i=1}^{n} \Big[2\mu_{B_i} - \frac{2\mu_{A_i}\mu_{B_i}}{\mu_{A_i} + \mu_{B_i}} + 2(1 - \mu_{B_i}) - \frac{2(1 - \mu_{A_i})(1 - \mu_{B_i})}{2 - \mu_{A_i} - \mu_{B_i}} \Big] - 1 \Big)$$

$$= \frac{1}{2} \Big(2 - \sum_{i=1}^{n} \Big[\frac{2\mu_{A_i}\mu_{B_i}}{\mu_{A_i} + \mu_{B_i}} + \frac{2(1 - \mu_{A_i})(1 - \mu_{B_i})}{2 - \mu_{A_i} - \mu_{B_i}} \Big] - 1 \Big)$$

$$= \frac{1}{2} \Big[1 - \sum_{i=1}^{n} \Big(\frac{(2\mu_{A_i}\mu_{B_i})}{(\mu_{A_i} + \mu_{B_i})} + \frac{2(1 - \mu_{A_i})(1 - \mu_{B_i})}{(2 - \mu_{A_i} - \mu_{B_i})} \Big) \Big],$$

$$MG_{-1}(B, A) = \frac{1}{4} \triangle (A, B)[\text{ using } (2.8) \text{ and } (4.25)], \tag{4.29}$$

and

$$D_{-1}(A, B) = \frac{1}{2} \chi^2(B, A)[\text{ using } (2.3)(2.3) \text{ and } (4.26)], \tag{4.30}$$

using equations (4.28), (4.29) and (4.30) gives the result (4.27).   □

(iii) if $n = \frac{1}{2}$, then we have the following relation

$$4\Big[1 - T\Big(\frac{A+B}{2}, B\Big)\Big] \leq M(A,B). \tag{4.31}$$

Proof. If $n = \frac{1}{2}$ put in (4.23)

$$MG_{\frac{1}{2}}(B,A) \leq \frac{1}{2} D_{\frac{1}{2}}(A,B) \tag{4.32}$$

where

$$MG_{\frac{1}{2}}(B,A) = 4\Big[1 - \sum_{i=1}^{n} \Big(\frac{\sqrt{\mu_{A_i}\mu_{B_i} + (\mu_{B_i})^2}}{2} + \frac{\sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i}) + (1 - \mu_{B_i})^2}}{2}\Big)\Big]$$

$$= 4\Big[1 - T\Big(\frac{A+B}{2}, B\Big)\Big] \tag{4.33}$$

(using (2.7) and (4.25)) and

$$D_{\frac{1}{2}}(A,B) = 4\Big[1 - \sum_{i=1}^{n} \Big(\sqrt{\mu_{A_i}}\mu_{B_i} + \sqrt{(1 - \mu_{A_i})(1 - \mu_{B_i})}\Big)\Big] = 2M(A,B). \tag{4.34}$$

using equations (4.32), (4.33) and (4.34) gives the result (4.31). $\qquad\square$

5   Case Study

We consider the problem having four known patterns A_1, A_2, A_3 and A_4 which have classifications R_1, R_2, R_3 and R_4 respectively. These are represented by the following fuzzy sets in the universe of discourse $X = \{x_1, x_2, x_3\}$.

$$A_1 = \{< x_1, .7 >, < x_2, .3 >, < x_3, .1 >\},$$
$$A_2 = \{< x_1, .4 >, < x_2, .2 >, < x_3, .5 >\},$$
$$A_3 = \{< x_1, .6 >, < x_2, .3 >, < x_3, .8 >\},$$
$$A_4 = \{< x_1, .7 >, < x_2, .5 >, < x_3, .8 >\}.$$

Given an unknown pattern B, represented by the fuzzy set

$$B = \{< x_1, .5 >, < x_2, .4 >, < x_3, .9 >\}$$

From the calculated numerical values of fuzzy divergence measure given in Table 1, the inequalities proposed in result (4.1) are verified

$$S_t(A,B) \leq \frac{1}{2} K(A,B).$$

Table 1: **Calculated numerical values of fuzzy divergence measures.**

$S_t(A, B)$	$K(A, B)$
.1717	.8085
.0512	.2703
.0085	**.0374**
.0154	.0639

6   Conclusions

In this current paper, we have achieved some series of fuzzy divergence measure by using fuzzy Csiszar's f-divergence and fuzzy new f-divergence measures properties with proof of validity, We have recommended generalized series of combination of kullback-Leibler, Arithmetic divergence measure etc. we have also derived Inequalities relating with new and well known fuzzy divergence measures.

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