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On the Babai and Upper Chromatic Numbers of Graphs of Diameter 2

Peter Johnson and Alexis Krumpelman

Abstract. The Babai numbers and the upper chromatic number are parameters that can be assigned to any metric space. They can, therefore, be assigned to any connected simple graph. In this paper we make progress in the theory of the first Babai number and the upper chromatic number in the simple graph setting, with emphasis on graphs of diameter 2.

1 Introduction

A **distance function** on a non-empty set X is a function $\rho : X \times X \to [0, \infty)$ such that, for all $x, y \in X$, (i) $\rho(x, y) = \rho(y, x)$ and (ii) $\rho(x, y) = 0 \iff x = y$. If, in addition, (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$, then ρ is a **metric**, and the pair (X, ρ) is a **metric space**.

A **coloring** of a set X is a function $\phi : X \to C$ for some set C, called the set of colors. Such a coloring is completely described by the **color sets** $\phi^{-1}(\{c\}) = \{x \in X | \phi(x) = c\}$ for $c \in C$. Some of these sets may be empty. If we allow empty sets in a partition, then the color sets partition X. Conversely, every partition of X constitutes a coloring of X.

Suppose that ρ is a distance function on a set X, $\phi : X \to C$ is a coloring of X, $d > 0$ and $c \in C$. We will say that the distance d **is forbidden for** the color c if $\phi^{-1}(\{c\})$ contains no two points $x, y \in X$ such that $\rho(x, y) = d$. To put it another way, if $x, y \in X$ and $\rho(x, y) = d$, then x and y cannot both be colored c. If the distance d is forbidden for every $c \in C$, then we say that the distance d is forbidden by the coloring ϕ.

Suppose that X and ρ are as above, and $D \subseteq (0, \infty)$. The **distance graph** $G(X, D)$ is the simple graph with vertex set X and $x, y \in X$ adjacent if and only if $\rho(x, y) \in D$. (It is conventional to suppress mention of ρ in the notation. In this paper, it will always be clear what

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Corresponding author: Peter Johnson .

distance function is under discussion.) When $D = \{d\}$, we can use the notation $G(X, d)$ as well as $G(X, \{d\})$.

If P is a graph parameter it is customary to shorten $P(G(X, D))$ to $P(X, D)$. Thus, the chromatic number of $G(X, D)$ is denoted $\chi(X, D)$. Due to the way adjacency is defined in $G(X, D)$, we have $\chi(X, D) = min[|C|$; there is a coloring $\phi : X \to C$ which forbids each $d \in D]$.

Let k be a positive integer. The k^{th} **Babai number** of (X,ρ) is $B_k(X) = sup[\chi(X, D); D \subseteq (0, \infty)$ and $|D| = k]$. Since $\chi(X, D) \leq |X|$ for all $D \subseteq (0, \infty)$, the sup, or least upper bound, in the definition of $B_k(X)$ is valid. However, in most cases, including all of those of interest to us in this paper, $B_k(X)$ is finite and the sup above is a maximum.

The **upper chromatic number** of (X, ρ), denoted $\hat{\chi}(X)$, is the smallest possible integer n, if any exists, such that for every sequence $d_1, ..., d_n \in (0, \infty)$ there is a coloring $\phi : X \to \{1, ..., n\}$, such that the distance d_i is forbidden for the color i, for all $i = 1, ..., n$. If no such positive integer exists, we set $\hat{\chi}(X) = \infty$. (The definition can be refined to allow $\hat{\chi}$ to take on different values among the infinite cardinal numbers, but in this paper the values of $\hat{\chi}$ will always be finite.)

Later in this paper we may use $c_1, ..., c_n$ as color names, rather than the integers $1, ..., n$. It is assumed throughout that $c_1, ..., c_n$ are distinct colors.

For the rest of this section, we continue to assume that ρ is a distance function on some set X.

Lemma 1.1. *If $m > \hat{\chi}(X)$ is a positive integer and $d_1, ..., d_m \in (0, \infty)$, then there is a coloring $\phi : X \to \{1, ..., m\}$ such that the distance d_i is forbidden for the color i for each $i = 1, ..., m$,*

Proof. Let $n = \hat{\chi}(X)$. Given $d_1, ..., d_n \in (o, \infty)$ there is a coloring $\phi : X \to \{1, ..., n\}$ such that d_i is forbidden for the color i, where $i = 1, ..., n$. We can consider ϕ to be a function into $\{1, ..., m\}$. Then, with reference to this coloring, the distance d_k is forbidden for the color k, for $n < k \leq m$, as well as for $1 \leq k \leq n$, because for $n < k \leq m$, $\phi^{-1}(\{k\}) = \emptyset$ □

Lemma 1.2. *If $\hat{\chi}(X) < \infty$, then $B_1(X) \leq \hat{\chi}(X) \leq |X|$.*

Proof. If $|X| = \infty$, then $\hat{\chi}(X) \leq |X|$. Suppose X is finite. Any one-to-one function $\phi : X \to \{1, ..., |X|\}$ forbids all distances for all colors; therefore, $\hat{\chi}(X) \leq |X|$.

Let $n = \hat{\chi}(X)$, and suppose that $d \in (0, \infty)$. Consider the sequence $d, ..., d$ of length n. Since $n = \hat{\chi}(X)$ there is a coloring $\phi : X \to \{1, ..., n\}$ such that d is forbidden for each color. Therefore, the distance d is forbidden by the coloring ϕ, and so $\chi(X, d) \leq n = \hat{\chi}(X)$. Since this holds for every $d > 0$, $B_1(X) \leq \hat{\chi}(X)$. □

A remark: if ρ is the famous "discrete metric" on X, defined by

$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

then $B_1(X) = \chi(X,d) = |X|$.

The upper chromatic number was introduced in [6], where it was shown that, with the usual absolute-value-of-the-difference metric on the real line $\mathbb{R}$, $\hat{\chi}(X) = 3$. Spectacular work on more exigent versions of the upper chromatic number, in which more than one distance must be forbidden for each color, has appeared in [1], [2] and [4], but mainly in the case $X = \mathbb{R}$, where ρ is the usual metric. It is strongly suspected that for $n > 1$, $\hat{\chi}(\mathbb{R}^n) = \infty$ with respect to the usual Euclidean distance, but the question is open.

The Babai numbers were proposed by Laszlo Babai in conversation with the first author of [3], where the numbers may have first appeared in print. (We hear that Paul Erdős had the same idea, but in a limited context.) The following appeared in [3], under the tacit assumption that all of the numbers $B_k(X)$ are finite, but it holds without that assumption.

Lemma 1.3. *For any positive integers k and t, $B_{k+t}(X) \leq B_k(X)B_t(X)$.*

From this and Lemma 1.2 we obtain the following.

Corollary 1.1. *If $\hat{\chi}(X) < \infty$, then for each positive integer k, $B_k(X) \leq B_1(X)^k \leq \hat{\chi}(X)^k$.*

Since all the single-distance graphs $G(\mathbb{R}^2, d)$ (with respect to the usual Euclidean distance on $\mathbb{R}$) are isomorphic, it follows that $B_1(\mathbb{R}^2) = \chi(\mathbb{R}^2, 1)$, the "chromatic number of the plane", now known to be either 5, 6, or 7 [7].

In [11] it was shown that $B_k(\mathbb{R}) = k + 1$ for each positive integer k, and from this the same result follows with $\mathbb{R}$ replaced by $\mathbb{Z}$. In [9] estimates of $B_1(Q_n)$ and $B_1(\mathbb{Z}^n)$ are obtained. Here Q_n is the n-cube, with either the Euclidean distance in $\mathbb{R}^n$ or distance in Q_n as a graph.

Estimation of the numbers $B_k(\mathbb{R}^n)$, for $n, k > 1$, seems to be a largely unexplored area in combinatorial geometry. Also of interest, and with connection to number theory, are the numbers $B_k(\mathbb{Q}^n)$, with the usual Euclidean distance on the set $\mathbb{Q}^n$ of rational points in $\mathbb{R}^n$. In [3] it was shown that $B_1(\mathbb{Q}^2) = 2$ and that $B_2(\mathbb{Q}^2) = 4$. It was shown in [8] and [10] that $B_1(\mathbb{Q}^3) = B_1(\mathbb{Q}^4) = 4$. In [5] it was shown that $\chi(\mathbb{Q}^n, 1)$, and thus $B_1(\mathbb{Q}^n)$, grows geometrically with n. (If memory serves, the result is something like $c(1.12 + o(1))^n \leq \chi(\mathbb{Q}^n, 1)$ for some constant c.)

2 The Babai and upper chromatic numbers of simple graphs

Suppose H is a connected simple graph. Let $dist_H$ denote the usual distance function in H: for $u, v \in V(H)$, $dist_H(u, v)$ is the length of (number of edges in) a shortest walk in H from u to v, or from v to u. (Such a shortest walk will always be a path.) Then $(V(H), dist_H)$ is a metric space, to which we can assign the Babai numbers and the upper chromatic numbers. These will be denoted $B_k(H)$ and $\hat{\chi}(H)$.

If $V(H)$ is finite then $dist_H$ takes only the values $0, 1, ..., diam(H)$, where

$$diam(H) = max[dist_H(u, v); u, v \in V(H)] = \text{ the \textbf{diameter} of } H.$$

In these cases, let, for $k \in \{1, ..., diam(H)\}$, $H^{(k)}$ be the graph defined by

$$V(H^{(k)}) = V(H) \text{ and } E(H^{(k)}) = \big\{uv | u, v \in V(H) \text{ and } dist_H(u, v) = k\big\}.$$

Then, $H^{(1)} = H$ and $B_1(H) = max\{\chi(H^{(k)}); 1 \leq k \leq diam(H)\}$.

Inspection of the definitions reveals that there is no need to confine the discussion of $B_k(H)$ and $\hat{\chi}(H)$ to connected simple graphs H. In a disconnected graph H, $dist_H(u, v)$ is still defined for vertices u and v in the same component of H. The meanings of "a distance d is forbidden for a color c in a coloring ϕ" and "a distance d is forbidden by a coloring $\phi : V(H) \to C$" are clear. Under these extended definitions, it is evident that if H has components $H_1, .., H_t$, then for any positive integer k, $B_k(H) = max\{B_k(H_j); 1 \leq j \leq t\}$, and furthermore, $\hat{\chi}(H) = max\{\hat{\chi}(H_j); 1 \leq j \leq t)\}$.

From here on, all graphs will be finite and simple, unless otherwise specified. If G and H are graphs, the disjoint union of G and H will be denoted $G + H$, and $G \vee H$ will denote the **join** of G and H; this is obtained by taking disjoint copies of G and H and adding an edge uv for every $u \in V(G), v \in V(H)$. The **complement** of G will be denoted $\overline{G}$: this is obtained by taking $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K) \setminus E(G)$, where K is the complete graph on the vertex set $V(G)$. Note that $\overline{\overline{G}} = G$, $\overline{G + H} = \overline{G} \vee \overline{H}$, and $\overline{G \vee H} = \overline{G} + \overline{H}$. The following is obvious by previous remarks.

Proposition 2.1. *For any graphs G and H, $\hat{\chi}(G+H) = max[\hat{\chi}(G), \hat{\chi}(H)]$, and for each positive integer k, $B_k(G + H) = max[B_k(G), B_k(H)]$.*

Since $G \vee H$ is of diameter 2 unless both G and H are complete graphs, we leave the discussion of $\hat{\chi}(G \vee H)$ and the $B_k(G \vee H)$ to the next section. We finish this section with two easy results that may be useful later.

Lemma 2.1. *If H has components $H_1, ..., H_t$ and $m = max\{diam(H_j); 1 \leq j \leq t\}$, then for all $m \leq k$, $B_k(H) = B_m(H)$.*

Proof. Suppose that $m \leq k$; then $B_m(H) \leq B_k(H)$. Also, suppose $D \subseteq (0, \infty)$ and $|D| = k$. Since at most m of the distances in D actually occur in H, H can be colored with $B_m(H)$ colors so that all the distances in D are forbidden. Thus, $\chi(H, D) \leq B_m(H)$. This holds for all such D, so $B_k(H) \leq B_m(H)$. □

Let K_n denote the complete graph on n vertices.

Proposition 2.2. *For all positive integers n and k, $B_k(K_n) = n = \hat{\chi}(K_n)$.*

Proof. We may assume that $n > 1$. Since 1 is the only distance that occurs in K_n, we have by Lemma 2.1 and the fact that $\chi(K_n) = n$, for all $1 \leq k$, $n = \chi(K_n) = \chi(K_n, 1) = B_1(K_n) = B_k(K_n) \leq \hat{\chi}(K_n) \leq |V(K_n)| = n$. □

Note: the graphs K_n, where $1 < n$, are the only graphs of diameter 1.

3 Graphs of diameter 2

Lemma 3.1. *If $diam(G) = 2$, then $B_1(G) = max[\chi(G), \chi(\bar{G})]$.*

Proof. If $diam(G) = 2$, then 1 and 2 are the only distances realized in G. Therefore, $G^{(2)} = \bar{G}$, so $B_1(G) = max[\chi(G^{(1)}), \chi(G^{(2)})] = max[\chi(G), \chi(\bar{G})]$. □

Theorem 3.1. *If $diam(G) = 2$, then $\hat{\chi}(G) \leq \chi(G) + \chi(\bar{G}) - 1$.*

Proof. Let $m = \chi(G) + \chi(\bar{G}) - 1$. We aim to show that if $d_1, ..., d_m > 0$, then there is a coloring of $V(G)$ with colors $c_1, ..., c_m$ such that the distance d_i is forbidden for the color c_i, where $i = 1, ..., m$.

If any $d_i \notin \{1, 2\}$ we can color all of $V(G)$ with c_i. So, we need only consider sequences of 1's and 2's. The order of the d_i is irrelevant so we need consider only sequences $2^k 1^{m-k}$ in which 2 appears k times and 1 appears $m - k$ times. Since $m = \chi(G) + \chi(\bar{G}) - 1$, for any $k \in \{0, ..., m\}$, either $\chi(\bar{G}) \leq k$ or $\chi(G) \leq m - k$. If the former is true, color $V(G)$ with $\chi(\bar{G})$ colors so that the distance 1 in $\bar{G}$ is forbidden for all colors. Then, the distance 2 in G is forbidden for all $\chi(\bar{G})$ colors. Similarly, if $\chi(G) \leq m - k$, color $V(G)$ with $\chi(G)$ colors so that the distance 1 is forbidden for each color. □

The following lemma is a corollary of remarks made in the proof of Theorem 3.1.

Lemma 3.2. *Suppose that $diam(G) = 2$. In testing whether or not $\hat{\chi}(G) \leq n$, for any $n < \chi(G) + \chi(\bar{G}) - 1$, it suffices to consider only sequences of distances $2^k 1^{n-k}$ such that both $k < \chi(\bar{G})$ and $n - k < \chi(G)$.*

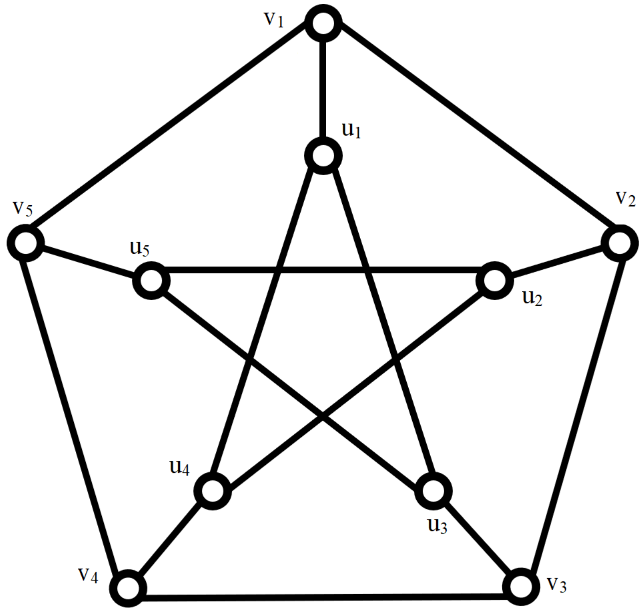

Figure 1: Petersen graph

In the remainder of this section, we determine $B_1(G)$ and $\hat{\chi}(G)$ for two famous graphs of diameter 2, the Petersen graph and the Grötzsch graph, as well as for all nonempty complete multi-partite graphs.

3.1 The Petersen graph

Let P denote the Petersen graph, pictured in Figure 1. Let $\alpha(X)$ denote the vertex independence number of a graph X, which is $max[|S|; S \subseteq V(X)$ and no two vertices of S are adjacent in $X]$. It is well known that for any finite, simple graph X, $|V(X)| \leq \alpha(X)\chi(X)$. It is easy to see that $\alpha(P) = 4$.

For a finite simple graph X, $\omega(X)$ denotes the **clique number** of X, the maximum order of a complete subgraph of X. Clearly, $\omega(X) \leq \chi(X)$. Also, $\alpha(X) = \omega(\bar{X})$, so $\alpha(\bar{X}) = \omega(\bar{\bar{X}}) = \omega(X)$.

Theorem 3.2. $B_1(P) = \hat{\chi}(P) = 5$.

Proof. It is well known, and easily checked, that $\chi(P) = 3$.

P is "triangle free", meaning that it has no complete subgraph on 3 vertices. Therefore, $\omega(P) = \alpha(\bar{P}) = 2$. So $|V(\bar{P})| = |V(P)| = 10 \leq 2 \cdot \chi(\bar{P})$, which implies that $5 \leq \chi(\bar{P})$. On the other hand, the sets $\{u_i, v_i\}$ for $i = 1, ..., 5$, partition $V(\bar{P})$ and are independent in $\bar{P}$. Therefore, $\chi(\bar{P}) = 5$. So $B_1(P) = max[\chi(P), \chi(\bar{P})] = max[3, 5] = 5 \leq \hat{\chi}(P)$.

By Lemma 3.2, to show that $\hat{\chi}(P) = 5$ is suffices to consider distance lists $2^k 1^{5-k}$ in which $k < 5$ and $5 - k < 3$.

Case 1. $2^3 1^2$: Partition $V(P)$ by taking $C_1 = \{v_5\}, C_2 = \{u_1\}, C_3 = \{u_2\}, C_4 = \{v_1, v_3, u_4, u_5\}$ and $C_5 = \{v_2, v_4, u_3\}$. Distance 2 does not occur within C_1, C_2, C_3, and distance 1 does not occur within C_4, C_5.

Case 2. $2^4 1^1$: Set $C_1 = \{u_1\}, C_2 = \{v_2, u_2\}, C_3 = \{u_3\}, C_4 = \{v_4, v_5\}$ and $C_5 = \{v_1, v_3, u_4, u_5\}$. Distance 2 is forbidden within each of C_1, C_2, C_3, C_4, and distance 1 is forbidden within C_5.

Thus, $\hat{\chi}(P) = 5$. □

3.2 The Grötzsch graph

The Grötzsch graph, depicted in Figure 2, has a couple of claims to fame: it is the smallest triangle free graph with chromatic number 4, and it is the 3^{rd} Mycielski graph. Let us explain the last claim. Suppose that G is a graph with vertices $x_1, \cdots, x_n$. The **Mycielskian** of G, denoted $M(G)$, has vertices $x_1, \cdots, x_n, y_1, \cdots, y_n, z$; $x_1, \cdots, x_n$ induce a copy of G in $M(G)$, and each y_i is adjacent to z and to every x_j to which x_i is adjacent. There are no other adjacencies. The Mycielski construction is mainly famous for:

1. If G is triangle-free, then so is $M(G)$, and
2. $\chi(M(G)) = \chi(G) + 1$.
   For our purposes, it will be useful to add:
3. If $diam(G) = 2$, then $diam(M(G)) = 2$.

If $M_1 = K_2$ and for $k > 1$, $M_k = M(M_{k-1})$, then the M_k are called the **Mycielski graphs**. It is straightforward to see that $M_2 = C_5$ and that M_3 is the Grötzsch graph.

Theorem 3.3. *Let G denote the Grötzsch graph; $B_1(G) = \hat{\chi}(G) = \chi(\bar{G}) = 6$.*

Proof. As in the proof of Theorem 3.2, since G is triangle-free, $\alpha(\bar{G}) = \omega(G) = 2$; therefore, $11/2 = |V(\bar{G})|/2 \leq \chi(\bar{G})$, so $6 \leq \chi(\bar{G})$. On the other hand, the sets $\{x_1, y_2\}$, $\{x_2, y_3\}$, $\{x_3, y_4\}$, $\{x_4, y_5\}$, $\{x_5, y_1\}$, $\{z\}$ partition $V(\bar{G})$, and each set is independent in $\bar{G}$. Therefore, $\chi(\bar{G}) = 6 = max[4, 6] = max[\chi(G), \chi(\bar{G})] = B_1(G) \leq \hat{\chi}(G)$.

By Lemma 3.2, to see that $\hat{\chi}(G) = 6$, it suffices to consider only distance lists $2^k 1^{6-k}$ such that $k < \chi(\bar{G}) = 6$ and $6 - k < \chi(G) = 4$.

Case 1. $2^3 1^3$: Let the color sets for colors $c_1, ..., c_6$ be $C_1 = \{x_1, x_2\}, C_2 = \{x_3, x_4\}, C_3 = \{x_5, y_1\}, C_4 = \{y_2, y_3, y_4, y_5\}, C_5 = \{z\}, C_6 = \emptyset$. Then the distance 2 (in G) is forbidden for c_1, c_2, c_3 and distance 1 for c_4, c_5, c_6.

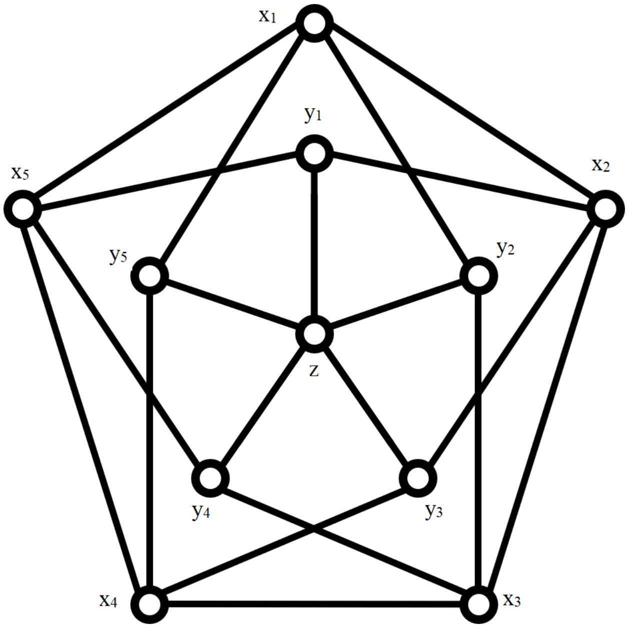

Figure 2: Grötzsch graph

Case 2. $2^4 1^2$: Let C_1, C_2, C_3 be as in Case 1, $C_4 = \emptyset, C_5 = \{y_2, y_3, y_4, y_5\}$, and $C_6 = \{z\}$.

Case 3. $2^5 1^1$: Let C_1, C_2, C_3 be as above, $C_4 = \{z\}, C_5 = \emptyset$, and $C_6 = \{y_2, y_3, y_4, y_5\}$.

Thus, $\hat{\chi}(G) = 6$. □

It is easy to see that $B_1(C_5) = \hat{\chi}(C_5) = 3 = \chi(\bar{C_5}) = \chi(C_5)$. This and Theorem 3.3 raise the question: is it true that for every $k > 1$, $B_1(M_k) = \hat{\chi}(M_k) = \chi(\bar{M_k})$?

3.3 Complete multi-partite graphs

For integers $2 \leq r$ and $1 \leq p_1 \leq ... \leq p_r$, the complete r-partite graph with part sizes $p_1, ..., p_r$ will be denoted $K_{p_1,...,p_r}$. Another way of writing this graph is $\bar{K}_{p_1} \vee ... \vee \bar{K}_{p_r}$. From this we have $\bar{K}_{p_1,...,p_r} = K_{p_1} + ... + K_{p_r}$. Therefore, if $G = K_{p_1,...,p_r}$, $2 \leq r$, $1 \leq p_1 \leq ... \leq p_r$, then $\chi(G) = r$ (as is well known) and $\chi(\bar{G}) = p_r$. From Lemma 3.1 and Theorem 3.1, we conclude the following.

Theorem 3.4. *If* $2 \leq r$, $p_1 \leq ... \leq p_r$ *are positive integers, and* $G = K_{p_1,...,p_r}$, *then* $B_1(G) = max[r, p_r] \leq \hat{\chi}(G) \leq r + p_r - 1$.

Note that when $p_1, ..., p_r = 1$, and only in that case, $G = K_{p_1,...,p_r} = K_r$ does not have diameter 2. However, the conclusion of Theorem 3.4 holds in that case nonetheless.

Theorem 3.5. *If* $2 \leq r$, $1 \leq p$ *and* $G = K_{p,\ldots,p}$ *is complete* r*-partite, then* $\hat{\chi}(G) = r + p - 1$.

Proof. We already have $\hat{\chi}(G) \leq r + p - 1$ by Theorem 3.4 and $\hat{\chi}(G) = r = r + p - 1$ when $p = 1$, by Proposition 2.2. Suppose that $p > 1$. We shall finish the proof by showing that $\hat{\chi}(G) > r + p - 2$.

Consider the sequence of distances $2^{p-1}1^{r-1}$. If there were a coloring of $V(G)$ with colors $c_1, \ldots, c_{r+p-2}$ so that the j^{th} distance in the sequence of distances is forbidden for c_j, $1 \leq j \leq r + p - 2$, then there would be a partition of $V(G)$ into sets $C_1, \ldots, C_{p-1}, C'_1, \ldots, C'_{r-1}$, such that each of $C_1, \ldots, C_{p-1}$ induces a clique in G and each of $C'_1, \ldots, C'_{r-1}$ is an independent set of vertices in G.

Each set C'_j must be completely contained in one of the r parts of G. Since there are only $r - 1$ of the C'_j, it must be that one of the parts of G is devoid of representatives of $\cup C'_j$ for $j = 1, \ldots, r - 1$. Therefore the p vertices of that part must be covered by the C_i, $i = 1, \ldots, p - 1$. However, no C_i can contain more than one vertex from each part. Therefore, because there are only $p - 1$ of the C_i, that part that is devoid of vertices in $\cup C'_j$ for $j = 1, \ldots, r - 1$ cannot be covered by $C_1, \ldots, C_{p-1}$. Thus, there is no such partition of $V(G)$. □

Theorem 3.6. *Suppose that* $2 \leq r$, $1 \leq p_1 \leq \ldots \leq p_r$, *and* $p_1 < p_r$. *Let* $G = K_{p_1,\ldots,p_r}$. *For* $p_1 \leq k \leq p_r - 1$, *define* $g(k) = max[j; p_j \leq k]$. *Then* $\hat{\chi}(G) = max[p_r, r + p_1 - 1, max[r + k - g(k); p_1 \leq k \leq p_r - 1]]$.

Proof. Let $m = max[p_r, r + p_1 - 1, max[r + k - g(k); p_1 \leq k \leq p_r - 1]]$. First we will see that $\hat{\chi}(G) \leq m$. By Lemma 3.2 and the fact that $diam(G) = 2$, it suffices to consider distance lists $2^t 1^{m-t}$ in which $t < \chi(\bar{G}) = p_r$ and $m - t < \chi(G) = r$.

Suppose that $m - r + 1 \leq t \leq p_r - 1$. Since $r + p_1 - 1 \leq m$, it follows that $p_1 \leq t \leq p_r - 1$. Since $r + t - g(t) \leq m$ we have $r - g(t) \leq m - t$.

The first $g(t)$ parts of G, i.e. the vertex set of $K_{p_1,\ldots,p_{g(t)}}$, can be colored with $p_{g(t)} = \chi(\bar{K}_{p_1,\ldots,p_{g(t)}})$ colors so that the distance 2 in G is forbidden for each color. Then we can color the last $r - g(t)$ parts of G with $r - g(t) = \chi(K_{p_{g(t)+1},\ldots,p_r})$ new colors so that the distance 1 in G is forbidden for each color. Since $p_{g(t)} \leq t$ and $r - g(t) \leq m - t$, the resulting coloring of $V(G)$ disposes of the challenge list $2^t 1^{m-t}$.

We finish the proof by showing that $\hat{\chi}(G) > m - 1$. First note that $p_r \leq max(r, p_r) \leq \hat{\chi}(G)$. Next, notice that $H = K_{p_1,\ldots,p_1}$, the complete r-partite graph with all parts of cardinality p_1, is a subgraph of G such that the distance in H between any pair of vertices in H is the same as the distance between them in G. Therefore, by Theorem 3.5, $r + p_1 - 1 = \hat{\chi}(H) \leq \hat{\chi}(G)$.

Finally, let $t \in \{p_1, \ldots, p_r - 1\}$ be such that $z = r + t - g(t) = max[r + k - g(k); p_1 \leq k \leq p_r - 1]$. Note that g takes values in $\{1, \ldots, r - 1\}$, and that, since $t - g(t)$ is a maximum, it must be the case that $t = p_{g(t)+1} - 1$, since this is the greatest value of k such that $g(k) = g(t)$. To see that $\hat{\chi}(G) > z - 1$, consider the distance list $2^t 1^{z-t-1}$. Since $z - t - 1 = r - g(t) - 1$, and a color set within which the distance 1 is forbidden must be a subset of one of the parts, after coloring vertices with up to $z-t-1$ colors so that the distance 1 is forbidden for each color, there are at least $g(t) + 1$ parts with no color appearing.

The largest of these parts must have at least $p_{g(t)+1}$ vertices, and $p_{g(t)+1} > t$ by the definition of g. Therefore, no t color sets, each containing at most one vertex per part, can cover these $g(t)+1$ parts. Thus $\hat{\chi}(G) > z - 1$. □

Corollary 3.7. *If* $1 \leq p_1 < p_2$, *then* $\hat{\chi}(K_{p_1,p_2}) = p_2$.

Proof. We look at the conclusion of Theorem 3.6 in the case of $r = 2$ and notice that $r + p_1 - 1 = 2 + p_1 - 1 = p_1 + 1 \leq p_2$, and, for each $k \in \{p_1, ..., p_2 - 1\}$, $r + k - g(k) = 2 + k - 1 = k + 1 \leq p_2 - 1 + 1 = p_2$. □

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Peter Johnson Auburn University

E-mail: johnspd@auburn.edu

Alexis Krumpelman Morehead State University

E-mail: akkrumpelman@moreheadstate.edu