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$*$-Weyl Curvature Tensor within the Framework of Sasakian and (κ,μ)-Contact Manifolds

Venkatesha Venkatesha and H. Aruna Kumara

Abstract. The object of the present paper is to study $*$-Weyl curvature tensor within the framework of Sasakian and (κ,μ)-contact manifolds.

1 Introduction

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g and let TM be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M,g) is defined by $g(QX,Y) = Ric(X,Y)$, where Ric denotes the Ricci tensor of type (0,2) on M and $X,Y \in TM$. Weyl ([20, 21]) constructed a generalized curvature tensor on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by

$$\begin{aligned}C(X,Y)Z =& R(X,Y)Z - \frac{1}{2n-1}\{g(QY,Z)X - g(QX,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned}$$

for $X,Y,Z \in TM$, where R and r denotes the Riemannian curvature tensor and the scalar curvature of M respectively. If $n = 1$ then $C(X,Y)Z = 0$ and if $n \geq 1$ then M is locally conformal flat if and only if $C(X,Y)Z = 0$. The condition of locally conformal flat holds for 3-dimensional Riemannian manifolds if and only if the Cotton tensor of M, which is given by

$$K(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{(Xr)Y - (Yr)X\},$$

vanishes identically.

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Corresponding author: Venkatesha Venkatesha.

In [16] Okumura showed that a conformally flat Sasakian manifold of dimension >3 is of constant curvature 1 and in [19] Tanno extended this result to the K-contact case and for dimensions ≥3. Noting that the Ricci operator Q commutes with the fundamental collineation φ for a Sasakian manifold, but this commutativity need not hold for a contact metric manifold, Blair and Koufogiorgos [3] proved that a conformally flat contact metric manifold on which Q commutes with φ, is of constant curvature 1. This generalized the above mentioned result of Okumura. In [9], the authors classified a classes of conformally flat contact metric manifolds and characterized a conformally flat contact manifolds as a hypersurfaces of 4-dimensional Kaehler Einstein manifolds. Miyazawa and Yamaguchi [15] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere. Chaki and Tarafdar [6] obtained the same result for a Sasakian manifold satisfying the condition $R(X, Y) \cdot C = 0$, for $X, Y \in TM$.

In 1959 Tachibana [17] defined $*$-Ricci tensor Ric^* on almost Hermitian manifold. In [11] Hamada gave the definition of $*$-Ricci tensor Ric^* in the following way

$$Ric^*(X, Y) = \frac{1}{2} trace(Z \to R(X, \varphi Y)\varphi Z), \tag{1.1}$$

for all $X, Y \in TM$. He also presented $*$-Einstein, i.e., $g(Q^*X, Y) = \lambda g(X, Y)$, where λ is a constant multiple of $g(X, Y)$ and provided classification of $*$-Einstein hypersurfaces. Ivey and Ryan in [13] extended the Hamada's work and studied the equivalence of $*$-Einstein condition with other geometric conditions such as the pseudo-Einstein and the pseudo-Ryan condition. By using the concept of $*$-Ricci tensor, the present authors with Naik [12] studied the some curvature properies on contact metric generalized (κ, μ)-space form.

Recently, Kaimakamis and Panagiotidou [14] introduced the notion of $*$-Weyl curvature tensor on real hypersurfaces in non-flat complex space forms and it is defined in the following way

$$\begin{aligned} C^*(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}\{g(Q^*Y, Z)X - g(Q^*X, Z)Y + g(Y, Z)Q^*X \\ - g(X, Z)Q^*Y\} + \frac{r^*}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \tag{1.2}$$

for all $X, Y, Z \in TM$, where Q^* is the $*$-Ricci operator and r^* is the $*$-scalar curvature corresponding to Q^* of M.

Motivated by the above studies, in this paper we study certain curvature conditions on the $*$-Weyl curvature tensor in Sasakian and (κ, μ)-contact manifolds. The paper is organized as follows: In section 2, we give brief introduction of contact metric manifolds. In section 3, we study $*$-Weyl curvature tensor within the background of Sasakian manifolds. Here, we show that Sasakin manifold with vanishing $*$-Weyl curvature tensor is $\psi(F)_{2n+1}$, and also proved that if Sasakian manifold satisfies $R \cdot C^* = 0$, then it is η-Einstein. In section 4, we study $*$-Weyl curvature tensor in (κ, μ)-contact manifolds. In this section, we show that a non-Sasakian (κ, μ)-contact manifold with vanishing $*$-Weyl curvature tensor is a flat for $n = 1$ and locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$. Finally, it is proved that if non-Sasakian (κ, μ)-contact manifold satisfies $\nabla C^* = \pi \otimes C^*$, then it is locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$.

2 Preliminaries

First, we shall review the basic definitions and formulas of contact metric manifolds. A (2n+1)-dimensional smooth manifold M is said to be contact if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. This 1-form is called a contact 1-form. For a contact 1-form η, there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle D (defined by $\eta = 0$), we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

for all $X, Y \in TM$. From these equations one can also deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

The structure (φ, ξ, η, g) on M is known as a contact metric structure and the metric g is called an associated metric. A Riemannian manifold M together with the structure (φ, ξ, η, g) is said to be a contact metric manifold and we denote it by $(M, \varphi, \xi, \eta, g)$. On a contact metric manifold (see [1])

$$\nabla_X \xi = -\varphi X - \varphi h X, \qquad h\varphi + \varphi h = 0, \tag{2.3}$$

for any vector field X, Y on M and ∇ denotes the operator of covariant differentiation of g. If the vector field ξ is Killing (equivalently, $h = 0$) with respect to g, then the contact metric manifold M is said to be K-contact. On a Sasakian manifold, the following formulas are known [1]

$$\nabla_X \xi = -\varphi X, \tag{2.4}$$

$$Q\xi = 2n\xi, \tag{2.5}$$

where Q denote the Ricci operator of M. A contact metric manifold is said to be Sasakian if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X. \tag{2.6}$$

On a Sasakian manifold, the curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

Also, the contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the metric cone $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler [1]. A Sasakian manifold is K-contact but the converse is true only in dimension 3. For more details see [1] and [5].

3 $*$-Weyl Curvature Tensor and Sasakian Manifolds

We are now in a position to find the expression of $*$-Weyl curvature tensor in the background of Sasakian manifolds. In [10], Ghosh and Patra derive the expression of $*$-Ricci tensor on Sasakian manifold, which is of the form

$$Ric^*(X,Y) = Ric(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y). \tag{3.1}$$

Contracting this over X yields

$$r^* = r - 4n^2. \tag{3.2}$$

Using (3.1) and (3.2) in (1.2), we obtain

$$\begin{aligned} C^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}\{g(QY,Z)X - g(QX,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\} + \frac{1}{2n-1}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi\} + (\frac{r-4n^2}{2n(2n-1)} + 2)\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.3}$$

Let M be a Sasakian manifold with vanishing $*$-Weyl curvature tensor, that is, $C^*(X,Y)Z = 0$. Relation for $C^*(X,Y)Z = 0$ implies that

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2n-1}\{g(QY,Z)X - g(QX,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &- \frac{1}{2n-1}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi\} - (\frac{r-4n^2}{2n(2n-1)} + 2)\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.4}$$

Covariant differentiation of above relation along W and then contracting the resultant equation over W yields

$$\begin{aligned} &\frac{2(n-1)}{2n-1}\{g((\nabla_X Q)Y,Z) - g((\nabla_Y Q)X,Z)\} \\ &= \frac{2(n-1)}{4n(2n-1)}\{(Xr)g(Y,Z) - (Yr)g(X,Z)\} \\ &- \frac{1}{2n-1}\{2g(X,\varphi Y)\eta(Z) + g(X,\varphi Z)\eta(Y) - g(Y,\varphi Z)\eta(X)\}. \end{aligned} \tag{3.5}$$

In a Sasakian manifold we have the following relation:

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{3.6}$$

For a Sasakian manifold we know that ξ is Killing and hence $L_\xi Ric = 0$. Therefore, by (2.4) we easily get $\nabla_\xi Q = \varphi Q - Q\varphi$. For a Sasakian manifold Q and φ commute (see [1]) and hence $\nabla_\xi Q = 0$. Now replacing Y by ξ in (3.5), recalling the last relation and (3.6) we find

$$2(n-1)g(Q\varphi X, Z) - (4n(n-1)+1)g(\varphi X, Z) - \frac{2(n-1)}{4n}(Xr)\eta(Z) = 0.$$

Replacing Z by φZ in the foregoing equation and making use of (2.2) we obtain

$$Ric(X, Z) = ag(X, Z) - b\eta(X)\eta(Z), \tag{3.7}$$

where $a = \frac{4n(n-1)+1}{2(n-1)}$ and $b = \frac{1}{2(n-1)}$. Substituting (3.7) in (3.4), we obtain

$$\begin{aligned} R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + +q[g(X, W)\eta(Y)\eta(Z) \\ - g(X, Z)\eta(Y)\eta(W) + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)], \end{aligned} \tag{3.8}$$

where $p = \{\frac{2a}{2n-1} - \frac{r-4n^2}{2n(2n-1)} - 2\}$ and $q = -\frac{b+1}{2n-1}$. The relation (3.8) leads to

$$R(X, Y, Z, W) = F(Y, Z)F(X, W) - F(X, Z)F(Y, W), \tag{3.9}$$

where

$$F(X, Y) = \sqrt{p}g(X, Y) + \frac{q}{\sqrt{p}}\eta(X)\eta(Y). \tag{3.10}$$

An n-dimensional Riemannian manifold whose curvature tensor R of type (0,4) satisfies the condition (3.9), where F is a symmetric tensor of type (0, 2) is called a special manifold with the associated symmetric tensor F, and is denoted by $\psi(F)_n$. Such type of manifolds have been studied by Chern [7] in 1956.

By virtue of (3.9), we have the following;

Theorem 3.1. *A Sasakian manifold with vanishing ∗-Weyl curvature tensor is $\psi(F)_{2n+1}$ with associated symmetric tensor F given by* (3.10).

In view of (3.8), we have following result;

Theorem 3.2. *A Sasakian manifold with vanishing ∗-Weyl curvature tensor is a manifold of quasi constant curvature.*

A Riemannian manifold (M, g) is said to be semisymmetric if $R \cdot R = 0$. Now we study, Sasakian manifold with ∗-Weyl curvature tensor satsifying relation $R \cdot C^* = 0$ and prove that

Theorem 3.3. *If a $(2n+1)$-dimensional Sasakian manifold satisfies $R \cdot C^* = 0$, then it is η-Einstein manifold.*

Proof. Let us consider a $(2n+1)$-dimensional Sasakian manifold satisfying $(R(X,Y)\cdot C^*)(U,V)W = 0$. Then, by definition we have

$$R(X,Y)C^*(U,V)W - C^*(R(X,Y)U,V)W \\ - C^*(U,R(X,Y)V)W - C^*(U,V)R(X,Y)W = 0.$$

Replacing X by ξ in the above equation and then taking inner product of resultant equation, we obtain

$$\eta(R(\xi,Y)C^*(U,V)W) - \eta(C^*(R(\xi,Y)U,V)W) \\ - \eta(C^*(U,R(\xi,Y)V)W) - \eta(C^*(U,V)R(\xi,Y)W) = 0. \tag{3.11}$$

In view of (2.7), it follows from (3.11) that

$$C^*(U,V,W,Y) - \eta(C^*(U,V)W)\eta(Y) - g(Y,U)\eta(C^*(\xi,V)W) \\ + \eta(U)\eta(C^*(Y,V)W) - g(Y,V)\eta(C^*(U,\xi)W) + \eta(V)\eta(C^*(U,Y)W) \\ - g(Y,W)\eta(C^*(U,V)\xi) + \eta(W)\eta(C^*(U,V)Y) = 0.$$

Replacing Y by U in the above equation, we have

$$C^*(U,V,W,U) - g(U,U)\eta(C^*(\xi,V)W) \\ - g(U,V)\eta(C^*(U,\xi)W) + \eta(W)\eta(C^*(U,V)U) = 0. \tag{3.12}$$

By virtue of (3.3), one can easily see that

$$\eta(C^*(X,Y)Z) = -\frac{1}{2n-1}\{Ric(Y,Z)\eta(X) - Ric(X,Z)\eta(Y)\} \\ + (\frac{r}{2n(2n-1)} - 4)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{3.13}$$

$$\eta(C^*(X,Y)\xi) = 0, \tag{3.14}$$

$$\eta(C^*(X,\xi)Z) = \frac{1}{2n-1}Ric(X,Z) - \frac{2n}{2n-1}\eta(X)\eta(Z) \\ - (\frac{r}{2n(2n-1)} - 4)\{g(X,Z) - \eta(X)\eta(Z)\}, \tag{3.15}$$

$$\sum_{i=1}^{2n+1} C^*(e_i,Y,Z,e_i) = \frac{2n(2n-2)+1}{2n-1}g(Y,Z) + \eta(Y)\eta(Z), \tag{3.16}$$

where $\{e_i\}_{i=1}^{2n+1}$ is an orthonormal basis of the tangent space at any point of the manifold. Taking $U = e_i$ in (3.12) and summing over i and making use of (3.13)-(3.16), we get

$$Ric(V,W) = \alpha g(V,W) - \beta\eta(V)\eta(W),$$

where $\alpha = \frac{20n^2-12n+1-r}{2n}$ and $\frac{\beta=20n^2-8n+1-r}{2n}$. The above relation shows that the manifold is Sasakian. This completes the proof. □

4 ∗-Weyl Curvature Tensor and (κ, μ)-contact Manifolds

In [2], Blair et al. introduced and studied a new type of contact metric manifold known as a (κ, μ)-contact manifold. Later on, Boeckx [4] classified these manifolds completely. A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be (κ, μ)-space if the curvature tensor satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{4.1}$$

for all vector fields X, Y on M and for some real numbers (κ, μ). This type of space arises through a D-homothetic deformation ([18]) to a contact metric manifold which satisfies $R(X, Y)\xi = 0$. The class of (κ, μ)-spaces covers Sasakian manifolds (for $\kappa = 1$) and the trivial sphere bundle $E^{n+1} \times S^n(4)$ (for $\kappa = \mu = 0$). There exist examples of non-Sasakian (κ, μ)-contact metric manifolds. For instance, the unit tangent bundles of Riemannian manifolds of constant curvature $\kappa \neq 1$. Since a D-homothetic deformation preserves (κ, μ)-contact structures, one can construct a lot of examples of (κ, μ)-contact structures (see [2]). The following formulas are also valid for a non-Sasakian (κ, μ)-contact manifolds [2]:

$$\begin{aligned} QX &= [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ &\quad + [2(1-n) + n(2\kappa + \mu)]\eta(X)\xi, \qquad (4.2)\\ (\nabla_X h)Y - (\nabla_Y h)X &= (1-\kappa)[2g(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] \\ &\quad + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX, \qquad (4.3)\\ Q\xi &= 2n\kappa\xi, \qquad (4.4)\\ h^2 &= (\kappa - 1)\varphi^2, \quad \kappa < 1, \qquad (4.5) \end{aligned}$$

equality holds when $\kappa = 1$ (equivalently, $h = 0$), i.e., M is Sasakian. For the non-Sasakian case, i.e., $\kappa < 1$, the (κ, μ)-nullity condition determines the curvature of M completely. In view of this, Boeckx [4] proved that a non-Sasakian (κ, μ)-contact manifold is locally homogeneous and hence analytic. Moreover, the constant scalar curvature r of such structures is given by

$$r = 2n(2(n-1) + \kappa - n\mu),$$

which is constant. On a (κ, μ)-contact manifold we have

$$(\nabla_\xi Q)X = \mu(2(n-1) + \mu)h\varphi X, \tag{4.6}$$

for any vector field X on M. In [10], Ghosh and Patra gave a expression of ∗-Ricci tensor on non-Sasakian (κ, μ)-contact manifolds, which is of the form

$$Ric^*(X, Y) = (n\mu + \kappa)\{-g(X, Y) + \eta(X)\eta(Y)\}. \tag{4.7}$$

Contracting this over X provides

$$r^* = -2n(n\mu + \kappa). \tag{4.8}$$

Making use of (4.7) and (4.8) in (1.2), we ultimately have

$$\begin{aligned} C^*(X,Y)Z =& R(X,Y)Z - \frac{n\mu + \kappa}{2n-1}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y,Z)X \\ &+ g(Y,Z)\eta(X)\xi + g(X,Z)Y - g(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{4.9}$$

Let M be a (κ, μ)-contact manifold with vanishing $*$-Weyl curvarure tensor, i.e., $C^*(X,Y)Z = 0$. Relation (4.9) for $C^*(X,Y)Z = 0$ implies

$$\begin{aligned} R(X,Y)Z = \frac{n\mu + \kappa}{2n-1}&\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y,Z)X \\ &+ g(Y,Z)\eta(X)\xi + g(X,Z)Y - g(X,Z)\eta(Y)\xi\}. \end{aligned}$$

Substituting Z by ξ in the above equation, we obtain

$$R(X,Y)\xi = 0.$$

Hence by Blair's theorem (see [1], p.122) M is locally flat in dimension 3, and in higher dimension it is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$. Thus we state the following theorem;

Theorem 4.1. *Let M be a non-Sasakian (κ, μ)-contact manifold with vanishing $*$-Weyl curvature tensor. Then M is flat for $n = 1$ and for $n > 1$, M is locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$.*

A Riemannian manifold (M, g) is said to be recurrent if there exists a 1-form ω such that Riemannian curvature tensor R satisfies $\nabla R = \pi \otimes R$, where ∇ is Levi-Civita connection of g. This type of manifold appears as a generalization of symmetric manifold. In [8] Ghosh studied conformally recurrent (κ, μ)-contact manifold of dimension > 3 and show that it is locally isometric to either (i) unit sphere $S^{2n+1}(1)$ or (ii) $E^{n+1} \times S^n(4)$. Now we study, non-Sasakian (κ, μ)-contact manifold with $*$-Weyl curvature tensor satisfying recurrent relation, i.e., $\nabla C^* = \pi \otimes C^*$ and prove that

Theorem 4.2. *If a non-Sasakian (κ, μ)-contact manifold M, $(n > 1)$ satisfies $\nabla C^* = \pi \otimes C^*$, then M is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$.*

Proof. By hypothesis we have

$$(\nabla_W C^*)(X,Y)Z = \pi(W)C^*(X,Y)Z. \tag{4.10}$$

Contracting (4.10) over W provide

$$(divC^*)(X,Y)Z = C^*(X,Y,Z,P), \tag{4.11}$$

where P is the recurrence vector metrically associated to the recurrence form π. Taking covariant differentiation of (4.9) along W and then contracting the resultant equation over W and using $Trh = Tr\varphi h = 0$, we entails that

$$\begin{aligned}(divC^*)(X,Y)Z &= g((\nabla_X Q)Y,Z) - g((\nabla_Y Q)X,Z) - \frac{n\mu+\kappa}{2n-1}\{2g(X,\varphi Y)\eta(Z) \\ &+ g(X,\varphi Z + h\varphi Z)\eta(Y) - g(Y,\varphi Z + h\varphi Z)\eta(X)\}.\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we find that

$$\begin{aligned}g((\nabla_X Q)Y,Z) - g((\nabla_Y Q)X,Z) &= \frac{n\mu+\kappa}{2n-1}\{2g(X,\varphi Y)\eta(Z) \\ + g(X,\varphi Z + h\varphi Z)\eta(Y) - g(Y,\varphi Z + h\varphi Z)\eta(X)\} &+ C^*(X,Y,Z,P).\end{aligned} \tag{4.13}$$

Next, differentiating covariantly (4.4) along an arbitrary vector field X and using (2.3) we get

$$(\nabla_X Q)\xi = Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX). \tag{4.14}$$

Substituting Z by ξ in (4.13) and making use of (4.14) we find that

$$\begin{aligned}&g(Q\varphi X + \varphi QX + Q\varphi hX + h\varphi QX - 4n\kappa\varphi X, Y) \\ &= \frac{2(n\mu+\kappa)}{2n-1}g(X,\varphi Y) + C^*(X,Y,\xi,P).\end{aligned} \tag{4.15}$$

Replacing X by φX, Y by φY and Z by ξ in (4.9) and by virtue of (4.1), it follows that $C^*(\varphi X,\varphi Y)\xi = 0$. Thus setting $X = \varphi X, Y = \varphi Y$ in (4.15) and making use of last equality we obtain

$$Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa\varphi X + \frac{2(n\mu+\kappa)}{2n-1}\varphi X = 0.$$

By virtue of (4.2), the foregoing equation reduces to

$$\kappa\mu - n\mu - 2\kappa - \mu + \frac{2(n\mu+\kappa)}{2n-1} = 0. \tag{4.16}$$

Taking the covariant differentiation of (4.2) and making use of (2.3) gives

$$\begin{aligned}(\nabla_X Q)Y = [2(n-1)+\mu](\nabla_X h)Y &- [2(1-n) + n(2\kappa+\mu)] \\ &\{g(\varphi X - \varphi hX, Y)\xi + \eta(Y)(\varphi X + \varphi hX)\}.\end{aligned} \tag{4.17}$$

Interchanging X and Y in (4.17) and substracting the resultant equation with (4.17) and by virtue of (4.3) and (4.16) we find that

$$\begin{aligned} g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) = \frac{2(n\mu+\kappa)}{2n-1}\{2g(X,\varphi Y)\eta(Z) \\ +\eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)\} + (3\mu - \mu^2 - n\mu + 2n\kappa) \\ \{\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)\}. \end{aligned} \tag{4.18}$$

By virtue of (4.13), the foregoing equation reduces to

$$\begin{aligned} C^*(X,Y,Z,P) = \frac{n\mu+\kappa}{2n-1}\{2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z) \\ -\eta(Y)g(\varphi X, Z)\} + \{(3\mu - \mu^2 - n\mu + 2n\kappa) - \frac{n\mu+\kappa}{2n-1}\} \\ \{\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)\}. \end{aligned} \tag{4.19}$$

Replacing X by φX, Y by φY and Z by ξ in (4.19), it follows that

$$n\mu + \kappa = 0, \tag{4.20}$$

where we used $C^*(\varphi X, \varphi Y)\xi = 0$. Setting $Z = P$ and $X = \xi$ in (4.19) and by virtue of (4.20) yields

$$(3\mu - \mu^2 - n\mu + 2n\kappa)g(\varphi hY, P) = 0.$$

Thus we have two possible cases:

$$(i)\ 3\mu - \mu^2 - n\mu + 2n\kappa = 0, \tag{4.21}$$

$$(ii)\ h\varphi P = 0. \tag{4.22}$$

Case(i). Keeping in mind that $n\mu + \kappa = 0$. Solving (4.16) and (4.21) we have the following solutions

$$\kappa = \mu = 0, \quad \kappa = \mu = n+3 \quad \text{or} \quad \kappa = \frac{n^2-1}{n}, \mu = 2(1-n).$$

When $\kappa = \mu = 0$, we obtain from (4.1) that $R(X,Y)\xi = 0$ and applying Blair's theorem we see that M is locally isometric to the product $E^{n+1} \times S^n(4)$. Since $n > 1$, the last two solutions leads to a contradiction as $\kappa < 1$.

Case(ii). Operating (4.22) by h and making use of (4.5) it follows that $P = \pi(\xi)\xi$. Together this with the condition (4.10) gives $(\nabla_W C^*)(X,Y)Z = \pi(\xi)\eta(W)C^*(X,Y)Z$. Substituting W by $\varphi^2 W$ in the last equality and contracting the resultant equation over W gives

$$(divC^*)(X,Y)Z = g((\nabla_\xi C^*)(X,Y)Z, \xi). \tag{4.23}$$

Taking covariant differentiation of (4.1) along ξ provides

$$(\nabla_\xi R)(X,Y)\xi = \mu^2\{\eta(Y)h\varphi X - \eta(X)h\varphi Y\}. \tag{4.24}$$

On the other hand from (4.9) and together with the help of (4.24) we have

$$g((\nabla_\xi C^*)(X,Y)Z,\xi) = -\mu^2\{\eta(Y)g(h\varphi X,Z) - \eta(X)g(h\varphi Y,Z)\}. \tag{4.25}$$

In view of (4.23) and (4.25) it follows that

$$(divC^*)(X,Y)Z = -\mu^2\{\eta(Y)g(h\varphi X,Z) - \eta(X)g(h\varphi Y,Z)\}.$$

Making use of (4.12) in the foregoing equation yields

$$\begin{aligned} g((\nabla_X Q)Y,Z) - g((\nabla_Y Q)X,Z) &= \frac{n\mu+\kappa}{2n-1}\{2g(X,\varphi Y)\eta(Z) \\ &+g(X,\varphi Z + h\varphi Z)\eta(Y) - g(Y,\varphi Z + h\varphi Z)\eta(X)\} \\ &-\mu^2\{\eta(Y)g(h\varphi X,Z) - \eta(X)g(h\varphi Y,Z)\}. \end{aligned} \tag{4.26}$$

Setting $Y = \xi$ in the above equation and making use of (2.2), (4.6) and (4.14), we find that

$$\begin{aligned} Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX) + \frac{n\mu+\kappa}{2n-1}\varphi X \\ -[\mu(2(n-1) - n\mu) + \frac{n\mu+\kappa}{2n-1} - \mu^2]h\varphi X = 0. \end{aligned} \tag{4.27}$$

By virtue of (4.2), the foregoing equation reduces to

$$\begin{aligned} \{\kappa\mu - n\mu - 2\kappa - \frac{n\mu+\kappa}{2n-1}\}g(\varphi X,Y) + \{(3\mu + 2n\kappa - n\mu) \\ -\frac{n\mu+\kappa}{2n-1}\}g(h\varphi X,Y) = 0. \end{aligned} \tag{4.28}$$

Interchanging X and Y in (4.28) and adding the resultant equation with (4.28) and by virtue of (2.2) we find that

$$(3\mu + 2n\kappa - n\mu) - \frac{n\mu+\kappa}{2n-1} = 0. \tag{4.29}$$

Solving (4.29) and (4.16) it follows that
$\kappa = \mu = 0$ or $\kappa = \frac{(n-1)(n+3)}{n}$, $\mu = \frac{2(n-1)(n+3)}{n-3}$, where we used $n\mu + \kappa = 0$, (in the last solution $n \neq 3$, because if $n = 3$, then from (4.29) it follows that $\kappa = 0$ and hence $\mu = 0$). The first solution shows that M is locally isometric to the product $E^{n+1} \times S^n(4)$. The last solution leads to a contradiction as $\kappa < 1$. This completes the proof. □

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References

[1] D. E. Blair, Riemannian geometry of contact and symplectic manifolds. Birkhauser, Boston, 2010.

[2] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition. Isr. J. Math. **91** (1995), 189-214.

[3] D. E. Blair and T. Koufogiorgos, When is the tangent sphere bundle conformally flat. J. Geom. **49** (1994), 55-66.

[4] E. Boeckx, A full classification of contact metric (κ, μ)-spaces. Ill. J. Math. **44** (2000), 212-219.

[5] C. P. Boyer and K. Galicki, Sasakian Geometry. Oxford University Press, Oxford (2008).

[6] M. C. Chaki and M. Tarafdar, On a type of Sasakian manifold. Soochow J. Math. **16**(1) (1990), 23-28.

[7] S. S. Chern, On the curvature and characteristic classes of a Riemannian manifold. Abh. Math. Semin. Univ. Hambg. **20** (1956), 117–126.

[8] A. Ghosh, Conformally recurrent (κ, μ)-contact manifolds. Note Mat. **28**(2) (2008), 207-212.

[9] A. Ghosh, T. Koufogiorgos and R. Sharma, *Conformally flat contact metric manifolds*. J. Geom. **70** (2001), 66-76,

[10] A. Ghosh and D. S. Patra, $*$-Ricci Soliton within the frame-work of Sasakian and (κ, μ)-contact manifold. Int. J. Geom. Methods Mod. Phys. **15**(7) (2018), 1850120

[11] T. Hamada, Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor. Tokyo J. Math. **25** (2002), 473-483.

[12] A. K. Huchchappa, D. M. Naik and V. Venkatesha, Certain results on contact metric generalized (κ, μ)-space forms, Commun. Korean Math. Soc. **34** (4) (2019), 1315-1328.

[13] T. A. Ivey and P. J. Ryan, The $*$-Ricci tensor for hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. Tokyo J. Math. **34** (2011), 445-471.

[14] G. Kaimakamis and K. Panagiotidou, On a new type of tensor on real hypersurfaces in non-flat complex space forms. Symmetry 2019, **11**(4), 559; https://doi.org/10.3390/sym11040559.

[15] T. Miyazawa and S. Yamaguchi, Some theorems on K-contact metric manifolds and Sasakian manifolds. TRU Math. **2** (1966), 46-52.

[16] M. Okumura, Some remarks on spaces with certain contact structures. Tohoku Math. J. **14** (1962), 135-145.

[17] S. Tachibana, On almost-analytic vectors in almost Kahlerian manifolds. Tohoku Math. J. **11** (1959), 247-265.

[18] S. Tanno, The topology of contact Riemannian manifolds. Illinois J. Math. **12** (1968), 700-717.

[19] S. Tanno, Locally symmetric K-contact Riemannian manifolds. Proc. Japan Acad. **43** (1967), 581-583.

[20] H. Weyl, Reine Infinitesimalgeometrie. Math. Z. **2** (3-4) (1918), 384-411.

[21] H. Weyl, Zur Infinitesimalgeometrie, Einordnung der projektiven und der konformen Auffassung. Gottingen Nachrichten. (1921), 99–112.

Venkatesha Venkatesha Department of Mathematics Kuvempu University Shankaraghatta Karnataka 577451 India

E-mail: vensmath@gmail.com

H. Aruna Kumara Department of Mathematics, Kuvempu University, Shankaraghatta, Karnataka 577451, India

E-mail: arunmathsku@gmail.com