**-Weyl Curvature Tensor within the Framework of Sasakian and \((\kappa, \mu)\)-Contact Manifolds

Venkatesha Venkatesha and H. Aruna Kumara

**Abstract.** The object of the present paper is to study **-Weyl curvature tensor within the framework of Sasakian and \((\kappa, \mu)\)-contact manifolds.

1 Introduction

Let \( M \) be a \((2n+1)\)-dimensional Riemannian manifold with metric \( g \) and let \( TM \) be the Lie algebra of differentiable vector fields in \( M \). The Ricci operator \( Q \) of \((M, g)\) is defined by \( g(QX, Y) = Ric(X, Y) \), where \( Ric \) denotes the Ricci tensor of type (0,2) on \( M \) and \( X, Y \in TM \). Weyl ([20, 21]) constructed a generalized curvature tensor on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} \{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY\} + \frac{r}{2n(2n-1)} \{g(Y, Z)X - g(X, Z)Y\},
\]

for \( X, Y, Z \in TM \), where \( R \) and \( r \) denotes the Riemannian curvature tensor and the scalar curvature of \( M \) respectively. If \( n = 1 \) then \( C(X, Y)Z = 0 \) and if \( n \geq 1 \) then \( M \) is locally conformal flat if and only if \( C(X, Y)Z = 0 \). The condition of locally conformal flat holds for 3-dimensional Riemannian manifolds if and only if the Cotton tensor of \( M \), which is given by

\[
K(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4} \{(Xr)Y - (Yr)X\},
\]

vanishes identically.

2010 Mathematics Subject Classification. 53C15, 53C25, 53D15.

Key words and phrases. Sasakian manifolds, \((\kappa, \mu)\)-contact manifolds, **-Ricci tensor, **-Weyl curvature tensor.

Corresponding author: Venkatesha Venkatesha.
In [16] Okumura showed that a conformally flat Sasakian manifold of dimension $>3$ is of constant curvature 1 and in [19] Tanno extended this result to the $K$-contact case and for dimensions $\geq 3$. Noting that the Ricci operator $Q$ commutes with the fundamental collineation $\varphi$ for a Sasakian manifold, but this commutativity need not hold for a contact metric manifold, Blair and Koufogiorgos [3] proved that a conformally flat contact metric manifold on which $Q$ commutes with $\varphi$, is of constant curvature 1. This generalized the above mentioned result of Okumura. In [9], the authors classified a classes of conformally flat contact metric manifolds and characterized a conformally flat contact manifolds as a hypersurfaces of 4-dimensional Kaehler Einstein manifolds. Miyazawa and Yamaguchi [15] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere. Chaki and Tarafdar [6] obtained the same result for a Sasakian manifold satisfying the condition $R(X,Y) \cdot C = 0$, for $X,Y \in TM$.

In 1959 Tachibana [17] defined $\ast$-Ricci tensor $Ric^\ast$ on almost Hermitian manifold. In [11] Hamada gave the definition of $\ast$-Ricci tensor $Ric^\ast$ in the following way

$$Ric^\ast(X,Y) = \frac{1}{2}\text{trace}(Z \rightarrow R(X,\varphi Y)\varphi Z), \tag{1.1}$$

for all $X,Y \in TM$. He also presented $\ast$-Einstein, i.e., $g(Q^\ast X, Y) = \lambda g(X, Y)$, where $\lambda$ is a constant multiple of $g(X, Y)$ and provided classification of $\ast$-Einstein hypersurfaces. Ivey and Ryan in [13] extended the Hamada’s work and studied the equivalence of $\ast$-Einstein condition with other geometric conditions such as the pseudo-Einstein and the pseudo-Ryan condition. By using the concept of $\ast$-Ricci tensor, the present authors with Naik [12] studied the some curvature properties on contact metric generalized $(\kappa,\mu)$-space form.

Recently, Kaimakamis and Panagiotidou [14] introduced the notion of $\ast$-Weyl curvature tensor on real hypersurfaces in non-flat complex space forms and it is defined in the following way

$$C^\ast(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{g(Q^\ast X, Z)X - g(Q^\ast X, Z)Y + g(Y, Z)Q^\ast Z \}
- g(X, Z)Q^\ast Y\} + \frac{r^\ast}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{1.2}$$

for all $X,Y,Z \in TM$, where $Q^\ast$ is the $\ast$-Ricci operator and $r^\ast$ is the $\ast$-scalar curvature corresponding to $Q^\ast$ of $M$.

Motivated by the above studies, in this paper we study certain curvature conditions on the $\ast$-Weyl curvature tensor in Sasakian and $(\kappa,\mu)$-contact manifolds. The paper is organized as follows: In section 2, we give brief introduction of contact metric manifolds. In section 3, we study $\ast$-Weyl curvature tensor within the background of Sasakian manifolds. Here, we show that Sasakian manifold with vanishing $\ast$-Weyl curvature tensor is $\psi(F)_{2n+1}$, and also proved that if Sasakian manifold satisfies $R \cdot C^\ast = 0$, then it is $\eta$-Einstein. In section 4, we study $\ast$-Weyl curvature tensor in $(\kappa,\mu)$-contact manifolds. In this section, we show that a non-Sasakian $(\kappa,\mu)$-contact manifold with vanishing $\ast$-Weyl curvature tensor is a flat for $n = 1$ and locally isometric.
to a Riemannian product \( E^{n+1} \times S^n \). Finally, it is proved that if non-Sasakian \((\kappa, \mu)\)-contact manifold satisfies \( \nabla C^* = \pi \otimes C^* \), then it is locally isometric to a Riemannian product \( E^{n+1} \times S^n \).

## 2 Preliminaries

First, we shall review the basic definitions and formulas of contact metric manifolds. A \((2n+1)\)-dimensional smooth manifold \(M\) is said to be contact if it admits a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) on \(M\). This 1-form is called a contact 1-form. For a contact 1-form \(\eta\), there exists a unique vector field \(\xi\) such that \(d\eta(\xi, X) = 0\) and \(\eta(\xi) = 1\). Polarizing \(d\eta\) on the contact sub-bundle \(D\) (defined by \(\eta = 0\)), we obtain a Riemannian metric \(g\) and a \((1,1)\)-tensor field \(\varphi\) such that

\[
d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}
\]

for all \(X, Y \in TM\). From these equations one can also deduce that

\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}
\]

The structure \((\varphi, \xi, \eta, g)\) on \(M\) is known as a contact metric structure and the metric \(g\) is called an associated metric. A Riemannian manifold \(M\) together with the structure \((\varphi, \xi, \eta, g)\) is said to be a contact metric manifold and we denote it by \((M, \varphi, \xi, \eta, g)\). On a contact metric manifold (see [1])

\[
\nabla_X \xi = -\varphi X - \varphi h X, \quad h \varphi + \varphi h = 0, \tag{2.3}
\]

for any vector field \(X, Y\) on \(M\) and \(\nabla\) denotes the operator of covariant differentiation of \(g\). If the vector field \(\xi\) is Killing (equivalently, \(h = 0\)) with respect to \(g\), then the contact metric manifold \(M\) is said to be \(K\)-contact. On a Sasakian manifold, the following formulas are known [1]

\[
\nabla_X \xi = -\varphi X, \tag{2.4}
\]

\[
Q \xi = 2n \xi, \tag{2.5}
\]

where \(Q\) denote the Ricci operator of \(M\). A contact metric manifold is said to be Sasakian if it satisfies

\[
(\nabla_X \varphi) Y = g(X, Y) \xi - \eta(Y) X. \tag{2.6}
\]

On a Sasakian manifold, the curvature tensor satisfies

\[
R(X, Y) \xi = \eta(Y) X - \eta(X) Y. \tag{2.7}
\]

Also, the contact metric structure on \(M\) is said to be Sasakian if the almost Kaehler structure on the metric cone \((M \times \mathbb{R}^+, r^2 g + dr^2)\) over \(M\), is Kaehler [1]. A Sasakian manifold is \(K\)-contact but the converse is true only in dimension 3. For more details see [1] and [5].
3  *-Weyl Curvature Tensor and Sasakian Manifolds

We are now in a position to find the expression of *-Weyl curvature tensor in the background of Sasakian manifolds. In [10], Ghosh and Patra derive the expression of *-Ricci tensor on Sasakian manifold, which is of the form

\[ Ric^*(X, Y) = Ric(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y). \]  

(3.1)

Contracting this over \( X \) yields

\[ r^* = r - 4n^2. \]  

(3.2)

Using (3.1) and (3.2) in (1.2), we obtain

\[ C^*(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1} \{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \]

\[ \quad - \frac{1}{2n - 1} \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \]

\[ \quad - g(X, Z)\eta(Y)\xi + (\frac{r - 4n^2}{2n(2n - 1)} + 2) \{g(Y, Z)X - g(X, Z)\xi \}. \]  

(3.3)

Let \( M \) be a Sasakian manifold with vanishing *-Weyl curvature tensor, that is, \( C^*(X, Y)Z = 0 \). Relation for \( C^*(X, Y)Z = 0 \) implies that

\[ R(X, Y)Z = \frac{1}{2n - 1} \{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \]

\[ \quad - \frac{1}{2n - 1} \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \]

\[ \quad - g(X, Z)\eta(Y)\xi - (\frac{r - 4n^2}{2n(2n - 1)} + 2) \{g(Y, Z)X - g(X, Z)\xi \}. \]  

(3.4)

Covariant differentiation of above relation along \( W \) and then contracting the resultant equation over \( W \) yields

\[ \frac{2(n - 1)}{2n - 1} \{g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z)\} \]

\[ = \frac{2(n - 1)}{4n(2n - 1)} \{(Xr)g(Y, Z) - (Yr)g(X, Z)\} \]

\[ \quad - \frac{1}{2n - 1} \{2g(X, \varphi Y)\eta(Z) + g(X, \varphi Z)\eta(Y) - g(Y, \varphi Z)\eta(X)\}. \]  

(3.5)

In a Sasakian manifold we have the following relation:

\[ (\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \]  

(3.6)
For a Sasakian manifold we know that $\xi$ is Killing and hence $L_\xi Ric = 0$. Therefore, by (2.4) we easily get $\nabla_\xi Q = \varphi Q - Q\varphi$. For a Sasakian manifold $Q$ and $\varphi$ commute (see [1]) and hence $\nabla_\xi Q = 0$. Now replacing $Y$ by $\xi$ in (3.5), recalling the last relation and (3.6) we find

$$2(n-1)g(Q\varphi X, Z) - (4n(n-1) + 1)g(\varphi X, Z) - \frac{2(n-1)}{4n} (Xr)\eta(Z) = 0.$$  

Replacing $Z$ by $\varphi Z$ in the foregoing equation and making use of (2.2) we obtain

$$Ric(X, Z) = ag(X, Z) - b\eta(X)\eta(Z), \quad (3.7)$$

where $a = \frac{4n(n-1)+1}{2(n-1)}$ and $b = \frac{1}{2(n-1)}$. Substituting (3.7) in (3.4), we obtain

$$R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + q[g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W) + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)]$$

$$= F(Y, Z)F(X, W) - F(X, Z)F(Y, W), \quad (3.8)$$

where $p = \{ \frac{2a}{2n-1} - \frac{r-4n^2}{2n(2n-1)} - 2 \}$ and $q = -\frac{b+1}{2n-1}$. The relation (3.8) leads to

$$F(X, Y) = \sqrt{p}g(X, Y) + \frac{q}{\sqrt{p}}\eta(X)\eta(Y). \quad (3.10)$$

An $n$-dimensional Riemannian manifold whose curvature tensor $R$ of type $(0,4)$ satisfies the condition (3.9), where $F$ is a symmetric tensor of type $(0,2)$ is called a special manifold with the associated symmetric tensor $F$, and is denoted by $\psi(F)_n$. Such type of manifolds have been studied by Chern [7] in 1956.

By virtue of (3.9), we have the following:

**Theorem 3.1.** A Sasakian manifold with vanishing $\ast$-Weyl curvature tensor is $\psi(F)_{2n+1}$ with associated symmetric tensor $F$ given by (3.10).

In view of (3.8), we have following result;

**Theorem 3.2.** A Sasakian manifold with vanishing $\ast$-Weyl curvature tensor is a manifold of quasi constant curvature.

A Riemannian manifold $(M, g)$ is said to be semisymmetric if $R \cdot R = 0$. Now we study, Sasakian manifold with $\ast$-Weyl curvature tensor satsifying relation $R \cdot C^* = 0$ and prove that

**Theorem 3.3.** If a $(2n+1)$-dimensional Sasakian manifold satisfies $R \cdot C^* = 0$, then it is $\eta$-Einstein manifold.
Proof. Let us consider a \((2n+1)\)-dimensional Sasakian manifold satisfying \((R(X, Y)\cdot C^*)(U, V)W = 0\). Then, by definition we have

\[
\]

Replacing \(X\) by \(\xi\) in the above equation and then taking inner product of resultant equation, we obtain

\[
\eta(R(\xi, Y)C^*(U, V)W) - \eta(C^*(R(\xi, Y)U, V)W) \\
- \eta(C^*(U, R(\xi, Y)V)W) - \eta(C^*(U, V)R(\xi, Y)W) = 0. \tag{3.11}
\]

In view of (2.7), it follows from (3.11) that

\[
C^*(U, V, W, Y) - \eta(C^*(U, V)W)\eta(Y) - g(Y, U)\eta(C^*(\xi, V)W) \\
+ \eta(U)\eta(C^*(Y, V)W) - g(Y, V)\eta(C^*(U, \xi)W) + \eta(V)\eta(C^*(U, Y)W) \\
- g(Y, W)\eta(C^*(U, V)\xi) + \eta(W)\eta(C^*(U, V)Y) = 0.
\]

Replacing \(Y\) by \(U\) in the above equation, we have

\[
C^*(U, V, W, U) - g(U, U)\eta(C^*(\xi, V)W) \\
- g(U, V)\eta(C^*(U, \xi)W) + \eta(W)\eta(C^*(U, V)U) = 0. \tag{3.12}
\]

By virtue of (3.3), one can easily see that

\[
\eta(C^*(X, Y)Z) = -\frac{1}{2n-1}\{Ric(Y, Z)\eta(X) - Ric(X, Z)\eta(Y)\} \\
+ \left(\frac{r}{2n(2n-1)} - 4\right)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{3.13}
\]

\[
\eta(C^*(X, Y)\xi) = 0, \tag{3.14}
\]

\[
\eta(C^*(X, \xi)Z) = \frac{1}{2n-1}Ric(X, Z) - \frac{2n}{2n-1}\eta(X)\eta(Z) \\
- \left(\frac{r}{2n(2n-1)} - 4\right)\{g(X, Z) - \eta(X)\eta(Z)\}, \tag{3.15}
\]

\[
\sum_{i=1}^{2n+1} C^*(e_i, Y, Z, e_i) = \frac{2n(2n-2) + 1}{2n-1}g(Y, Z) + \eta(Y)\eta(Z), \tag{3.16}
\]

where \(\{e_i\}_{i=1}^{2n+1}\) is an orthonormal basis of the tangent space at any point of the manifold. Taking \(U = e_i\) in (3.12) and summing over \(i\) and making use of (3.13)-(3.16), we get

\[
Ric(V, W) = \alpha g(V, W) - \beta\eta(V)\eta(W),
\]

where \(\alpha = \frac{20n^2 - 12n + 1 - r}{2n}\) and \(\beta = \frac{20n^2 - 8n + 1 - r}{2n}\). The above relation shows that the manifold is Sasakian. This completes the proof. \qed
4 -Weyl Curvature Tensor and \((\kappa, \mu)\)-contact Manifolds

In [2], Blair et al. introduced and studied a new type of contact metric manifold known as a \((\kappa, \mu)\)-contact manifold. Later on, Boeckx [4] classified these manifolds completely. A contact metric manifold \((M, \varphi, \xi, \eta, g)\) is said to be \((\kappa, \mu)\)-space if the curvature tensor satisfies

\[
R(X, Y)\xi = \kappa \{\eta(Y)X - \eta(X)Y\} + \mu \{\eta(Y)hX - \eta(X)hY\},
\]

for all vector fields \(X, Y\) on \(M\) and for some real numbers \((\kappa, \mu)\). This type of space arises through a \(D\)-homothetic deformation ([18]) to a contact metric manifold which satisfies \(R(X, Y)\xi = 0\).

The class of \((\kappa, \mu)\)-spaces covers Sasakian manifolds (for \(\kappa = 1\)) and the trivial sphere bundle \(E^{n+1} \times S^n(4)\) (for \(\kappa = \mu = 0\)). There exist examples of non-Sasakian \((\kappa, \mu)\)-contact metric manifolds. For instance, the unit tangent bundles of Riemannian manifolds of constant curvature \(\kappa \neq 1\). Since a \(D\)-homothetic deformation preserves \((\kappa, \mu)\)-contact structures, one can construct a lot of examples of \((\kappa, \mu)\)-contact structures (see [2]). The following formulas are also valid for a non-Sasakian \((\kappa, \mu)\)-contact manifolds [2]:

\[
QX = [2(n - 1) - n\mu]X + [2(n - 1) + \mu]hX + [2(1 - n) + n(2\kappa + \mu)]\eta(X)\xi,
\]

\[
(\nabla_X h)Y - (\nabla_Y h)X = (1 - \kappa)[2g(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] + (1 - \mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX,
\]

\[
Q\xi = 2n\kappa\xi,
\]

\[
h^2 = (\kappa - 1)\varphi^2, \quad \kappa < 1,
\]

equality holds when \(\kappa = 1\) (equivalently, \(h = 0\)), i.e., \(M\) is Sasakian. For the non-Sasakian case, i.e., \(\kappa < 1\), the \((\kappa, \mu)\)-nullity condition determines the curvature of \(M\) completely. In view of this, Boeckx [4] proved that a non-Sasakian \((\kappa, \mu)\)-contact manifold is locally homogeneous and hence analytic. Moreover, the constant scalar curvature \(r\) of such structures is given by

\[
r = 2n(2(n - 1) + \kappa - n\mu),
\]

which is constant. On a \((\kappa, \mu)\)-contact manifold we have

\[
(\nabla_\xi Q)X = \mu(2(n - 1) + \mu)h\varphi X,
\]

for any vector field \(X\) on \(M\). In [10], Ghosh and Patra gave an expression of \(\ast\)-Ricci tensor on non-Sasakian \((\kappa, \mu)\)-contact manifolds, which is of the form

\[
Ric^\ast(X, Y) = (n\mu + \kappa)\{-g(X, Y) + \eta(X)\eta(Y)\}.
\]
Contracting this over \(X\) provides
\[ r^* = -2n(n\mu + \kappa). \] (4.8)

Making use of (4.7) and (4.8) in (1.2), we ultimately have
\[
C^*(X,Y)Z = R(X,Y)Z - \frac{n\mu + \kappa}{2n - 1} \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y,Z)X \\
+ g(Y,Z)\eta(X)\xi + g(X,Z)Y - g(X,Z)\eta(Y)\xi\}. \] (4.9)

Let \(M\) be a \((\kappa, \mu)\)-contact manifold with vanishing \(\ast\)-Weyl curvature tensor, i.e., \(C^*(X,Y)Z = 0\). Relation (4.9) for \(C^*(X,Y)Z = 0\) implies
\[
R(X,Y)Z = \frac{n\mu + \kappa}{2n - 1} \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y,Z)X \\
+ g(Y,Z)\eta(X)\xi + g(X,Z)Y - g(X,Z)\eta(Y)\xi\}. \]

Substituting \(Z\) by \(\xi\) in the above equation, we obtain
\[
R(X,Y)\xi = 0. \]

Hence by Blair’s theorem (see [1], p.122) \(M\) is locally flat in dimension 3, and in higher dimension it is locally isometric to the trivial bundle \(E^{n+1} \times S^n(4)\). Thus we state the following theorem;

**Theorem 4.1.** Let \(M\) be a non-Sasakian \((\kappa, \mu)\)-contact manifold with vanishing \(\ast\)-Weyl curvature tensor. Then \(M\) is flat for \(n = 1\) and for \(n > 1\), \(M\) is locally isometric to a Riemannian product \(E^{n+1} \times S^n(4)\).

A Riemannian manifold \((M, g)\) is said to be recurrent if there exists a 1-form \(\omega\) such that Riemannian curvature tensor \(R\) satisfies \(\nabla R = \pi \otimes R\), where \(\nabla\) is Levi-Civita connection of \(g\). This type of manifold appears as a generalization of symmetric manifold. In [8] Ghosh studied conformally recurrent \((\kappa, \mu)\)-contact manifold of dimension \(>3\) and show that it is locally isometric to either (i) unit sphere \(S^{2n+1}(1)\) or (ii) \(E^{n+1} \times S^n(4)\). Now we study, non-Sasakian \((\kappa, \mu)\)-contact manifold with \(\ast\)-Weyl curvature tensor satisfying recurrent relation, i.e., \(\nabla C^* = \pi \otimes C^*\) and prove that

**Theorem 4.2.** If a non-Sasakian \((\kappa, \mu)\)-contact manifold \(M, (n > 1)\) satisfies \(\nabla C^* = \pi \otimes C^*\), then \(M\) is locally isometric to the trivial bundle \(E^{n+1} \times S^n(4)\).

**Proof.** By hypothesis we have
\[
(\nabla_W C^*)(X,Y)Z = \pi(W)C^*(X,Y)Z. \] (4.10)
Contracting (4.10) over $W$ provide
\[(\text{div} C^*)(X, Y)Z = C^*(X, Y, Z, P),\] (4.11)
where $P$ is the recurrence vector metrically associated to the recurrence form $\pi$. Taking covariant differentiation of (4.9) along $W$ and then contracting the resultant equation over $W$ and using $Trh = Tr\varphi h = 0$, we entails that
\[(\text{div} C^*)(X, Y)Z = g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) - \frac{n\mu + \kappa}{2n - 1}\{2g(X, \varphi Y)\eta(Z) + g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\}.\] (4.12)
Combining (4.11) and (4.12), we find that
\[g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) = \frac{n\mu + \kappa}{2n - 1}\{2g(X, \varphi Y)\eta(Z) + g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\} + C^*(X, Y, Z, P).\] (4.13)
Next, differentiating covariantly (4.4) along an arbitrary vector field $X$ and using (2.3) we get
\[(\nabla_X Q)\xi = Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX).\] (4.14)
Substituting $Z$ by $\xi$ in (4.13) and making use of (4.14) we find that
\[g(Q\varphi X + \varphi QX + Q\varphi hX + h\varphi QX - 4n\kappa\varphi X, Y) = \frac{2(n\mu + \kappa)}{2n - 1}g(X, \varphi Y) + C^*(X, Y, \xi, P).\] (4.15)
Replacing $X$ by $\varphi X, Y$ by $\varphi Y$ and $Z$ by $\xi$ in (4.9) and by virtue of (4.1), it follows that $C^*(\varphi X, \varphi Y)\xi = 0$. Thus setting $X = \varphi X, Y = \varphi Y$ in (4.15) and making use of last equality we obtain
\[Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa\varphi X + \frac{2(n\mu + \kappa)}{2n - 1}\varphi X = 0.\]
By virtue of (4.2), the foregoing equation reduces to
\[\kappa\mu - n\mu - 2\kappa - \mu + \frac{2(n\mu + \kappa)}{2n - 1} = 0.\] (4.16)
Taking the covariant differentiation of (4.2) and making use of (2.3) gives
\[(\nabla_X Q)Y = [2(n - 1) + \mu](\nabla_X h)Y - [2(1 - n) + n(2\kappa + \mu)]\]
\[\{g(\varphi X - \varphi hX, Y)\xi + \eta(Y)(\varphi X + \varphi hX)\}.\] (4.17)
Interchanging $X$ and $Y$ in (4.17) and subtracting the resultant equation with (4.17) and by virtue of (4.3) and (4.16) we find that
\[
g((\nabla X Q) Y, Z) - g((\nabla Y Q) X, Z) = \frac{2(n\mu + \kappa)}{2n - 1}\{2g(X, \varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z)\} - \eta(Y)g(\varphi X, Z)\}. \tag{4.18}
\]

By virtue of (4.13), the foregoing equation reduces to
\[
C^*(X, Y, Z, P) = \frac{n\mu + \kappa}{2n - 1}\{2g(X, \varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z)
\]
\[-\eta(Y)g(\varphi X, Z)\} + \{(3\mu - \mu^2 - n\mu + 2n\kappa) - \frac{n\mu + \kappa}{2n - 1}\}
\]
\[\{\eta(X)g(\varphi h Y, Z) - \eta(Y)g(\varphi h X, Z)\}. \tag{4.19}
\]

Replacing $X$ by $\varphi X$, $Y$ by $\varphi Y$ and $Z$ by $\xi$ in (4.19), it follows that
\[
n\mu + \kappa = 0, \tag{4.20}
\]

where we used $C^*(\varphi X, \varphi Y)\xi = 0$. Setting $Z = P$ and $X = \xi$ in (4.19) and by virtue of (4.20) yields
\[
(3\mu - \mu^2 - n\mu + 2n\kappa)g(\varphi h Y, P) = 0.
\]

Thus we have two possible cases:
\[
(i) \ 3\mu - \mu^2 - n\mu + 2n\kappa = 0, \tag{4.21}
\]
\[
(ii) \ h\varphi P = 0. \tag{4.22}
\]

**Case(i).** Keeping in mind that $n\mu + \kappa = 0$. Solving (4.16) and (4.21) we have the following solutions
\[
\kappa = \mu = 0, \quad \kappa = \mu = n + 3 \quad \text{or} \quad \kappa = \frac{n^2 - 1}{n}, \mu = 2(1 - n).
\]

When $\kappa = \mu = 0$, we obtain from (4.1) that $R(X, Y)\xi = 0$ and applying Blair’s theorem we see that $M$ is locally isometric to the product $E^{n+1}_+ \times S^n(4)$. Since $n > 1$, the last two solutions leads to a contradiction as $\kappa < 1$.

**Case(ii).** Operating (4.22) by $h$ and making use of (4.5) it follows that $P = \pi(\xi)\xi$. Together this with the condition (4.10) gives $(\nabla_W C^*)(X, Y)\xi = \pi(\xi)\eta(W)C^*(X, Y)Z$. Substituting $W$ by $\varphi^2 W$ in the last equality and contracting the resultant equation over $W$ gives
\[
(div C^*)(X, Y)Z = g((\nabla \xi C^*)(X, Y)Z, \xi). \tag{4.23}
\]
Taking covariant differentiation of (4.1) along $\xi$ provides

$$(\nabla_\xi R)(X, Y)\xi = \mu^2 \{\eta(Y)h\varphi X - \eta(X)h\varphi Y\}. \quad (4.24)$$

On the other hand from (4.9) and together with the help of (4.24) we have

$$g((\nabla_\xi C^*)(X, Y)Z, \xi) = -\mu^2 \{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \quad (4.25)$$

In view of (4.23) and (4.25) it follows that

$$(\text{div}C^*)(X, Y)Z = -\mu^2 \{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \quad (4.26)$$

Making use of (4.12) in the foregoing equation yields

$$g((\nabla XQ)Y, Z) - g((\nabla YQ)X, Z) = \frac{n\mu + \kappa}{2n - 1} \{2g(X, \varphi Y)\eta(Z) + g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\} - \mu^2 \{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \quad (4.27)$$

Setting $Y = \xi$ in the above equation and making use of (2.2), (4.6) and (4.14), we find that

$$(Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX) + \frac{n\mu + \kappa}{2n - 1}\varphi X - [\mu(2(n - 1) - n\mu) + \frac{n\mu + \kappa}{2n - 1} - \mu^2]h\varphi X = 0. \quad (4.28)$$

By virtue of (4.2), the foregoing equation reduces to

$$\{\kappa\mu - n\mu + \kappa - \frac{n\mu + \kappa}{2n - 1}\}g(\varphi X, Y) + \{(3\mu + 2n\kappa - n\mu) - \frac{n\mu + \kappa}{2n - 1}\}g(h\varphi X, Y) = 0. \quad (4.28)$$

Interchanging $X$ and $Y$ in (4.28) and adding the resultant equation with (4.28) and by virtue of (2.2) we find that

$$(3\mu + 2n\kappa - n\mu) - \frac{n\mu + \kappa}{2n - 1} = 0. \quad (4.29)$$

Solving (4.29) and (4.16) it follows that

$$\kappa = \mu = 0 \text{ or } \kappa = \frac{(n-1)(n+3)}{n}, \mu = \frac{2(n-1)(n+3)}{n-3},$$

where we used $n\mu + \kappa = 0$, (in the last solution $n \neq 3$, because if $n = 3$, then from (4.29) it follows that $\kappa = 0$ and hence $\mu = 0$). The first solution shows that $M$ is locally isometric to the product $E^{n+1} \times S^n(4)$. The last solution leads to a contradiction as $\kappa < 1$. This completes the proof.
Acknowledgement

The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

References


[8] A. Ghosh, Conformally recurrent \((\kappa, \mu)\)-contact manifolds. Note Mat. 28(2) (2008), 207-212.


Venkatesha Venkatesha Department of Mathematics Kuvempu University Shankaraghatta Karnataka 577451 India
E-mail: vensmath@gmail.com

H. Aruna Kumara Department of Mathematics, Kuvempu University, Shankaraghatta, Karnataka 577451, India
E-mail: arunmathsku@gmail.com