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Position Vectors of Curves Generalizing General Helices and Slant Helices in Euclidean 3-Space

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Abstract. In this paper, we give a new characterization of a k-slant helix which is a generalization of general helix and slant helix. Thereafter, we construct a vector differential equation of the third order to determine the parametric representation of a k-slant helix according to standard frame in Euclidean 3-space. Finally, we apply this method to find the position vector of some examples of 2-slant helix by means of intrinsic equations.

1 Introduction

In differential geometry, a curve called general helix is defined by the property that its tangent vector field makes a constant angle with a fixed straight line which is the axis of the general helix in Euclidean $3-$space. A classical result stated by M.A. Lancret in 1802 and first proved by B. Saint Venant in 1845 (see [6, 11] for details) says that: A necessary and sufficient condition that a curve be a general helix is that the ratio

$$\sigma_0 = \frac{\tau}{\kappa},$$

is constant along the curve, where κ and τ denote the curvature and the torsion of the curve, respectively. If both κ and τ are non-zero constants, then the curve is called a circular helix.

Izumiya and Takeuchi [13] have introduced the concept of the slant helix by saying that the principal normal lines make a constant angle with a fixed straight line and they characterize a slant helix if and only if the geodesic curvature

$$\sigma_1 = \frac{\sigma_0'}{\kappa\left(1+\sigma_0^2\right)^{\frac{3}{2}}},$$

of principal image of the principal normal indicatrix is a constant function.

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In [5], the authors investigate a curve whose spherical images (the tangent indicatrix and binormal indicatrix) are slant helices and called it as a $C-$slant helix where $C = \frac{N'}{\|N'\|}$ (N the principal normal of the curve). They characterize a $C-$slant helix if and only if the geodesic curvature

$$\sigma_2 = \frac{\sigma_1'}{\kappa\sqrt{1+\sigma_0^2}\left(1+\sigma_1^2\right)^{\frac{3}{2}}},$$

of the principal image of the vector field C indicatrix is a constant fonction.

The determining of the position vector of some different curves according to the intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ (where κ and τ are the curvature and torsion of the curve) is considered as a one of important subjects. Recently, the parametric representation of general helices and slant helices as an important special curves in Euclidean space E^3 are deduced by Ali [1, 2].

The purpose of this paper is to determine the position vector of k-slant helices (see [3, 4]) which a generalization of general helices and slant helices. Firstly, we give a new characterization of $k-$slant helices and construct a vector differential equation of the third order to determine the parametric representation of $k-$slant helices. By applying this method, we present some examples of $2-$slant helix.

2 Preliminaries

In Euclidean space E^3, we known that each unit speed curve has at least four continuous derivatives, one can associate three orthogonal unit vector fields T, N and B are the tangent, the principal normal and the binormal vector fields respectively [7].

Let $\psi : I \subset \mathbb{R} \longrightarrow E^3, \psi = \psi(s)$, be an arbitrary curve in E^3. Recall that the curve ψ is said to be unit speed or parametrized by the arc-length if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. Thus, we will assume throughout this work that ψ is a unit speed curve, where $\langle , \rangle$ denotes the standard inner product given by :

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2.$$

Let $(T(s), N(s), B(s))$ be the Frenet moving frame along ψ. The Frenet equations for ψ are given by [6] :

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.1}$$

where κ and τ are the curvature and the torsion of the curve ψ in terms of Frenet frame, respectively.

Denote by $\{N, C, W = N \wedge C\}$ the alternative moving frame along the curve ψ in Euclidean 3-space. Note that $N, C = \frac{N'}{\|N'\|}$ and $W = \frac{\tau T + \kappa B}{\sqrt{\tau^2+\kappa^2}}$ are the unit principal normal, the dérivative of principal normal vector and the Darboux vector, respectively. For the derivatives of the alternative moving frame, we have :

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f_1(s) & 0 \\ -f_1(s) & 0 & g_1(s) \\ 0 & -g_1(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix}, \tag{2.2}$$

where $f_1 = \sqrt{\tau^2 + \kappa^2} = \kappa\sqrt{1+\sigma_0^2}$ and $g_1 = \sigma_1 f_1$, are curvatures of the curve ψ in terms of the alternative moving frame.

3 k-slant helix and its characterizations

Let $\psi = \psi(s)$ a natural representation of a unit speed curve in Euclidean 3-space, and let $(T(s), N(s), B(s))$ denotes the Frenet frame of ψ with $\kappa(s), \tau(s)$ the curvature and the torsion of the curve ψ, respectively.

We denote by $C_0 = \psi(s)$,

$$C_k(s) = \frac{C'_{k-1}(s)}{\|C'_{k-1}(s)\|} \quad \text{and} \quad W_{k+1}(s) = C_k(s) \wedge C_{k+1}(s), \quad k \in \{1, 2, ..\}.$$

Therefore, we can see that (C_k, C_{k+1}, W_{k+1}) is the Frenet frame of $s \to C_{k-1}(s)$. Then the derivative formulae of Frenet frame are given by:

$$\begin{bmatrix} C'_k(s) \\ C'_{k+1}(s) \\ W'_{k+1}(s) \end{bmatrix} = \begin{bmatrix} 0 & f_{k-1}(s) & 0 \\ -f_{k-1}(s) & 0 & g_{k-1}(s) \\ 0 & -g_{k-1}(s) & 0 \end{bmatrix} \begin{bmatrix} C_k(s) \\ C_{k+1}(s) \\ W_{k+1}(s) \end{bmatrix}, \tag{3.1}$$

where f_{k-1} and g_{k-1} are the curvatures of the curve C_{k-1} in terms of (C_k, C_{k+1}, W_{k+1}) moving frame. We can easily see that $f_0 = \kappa$ and $g_0 = \tau$.

If we write this curve in the other parametric representation $C_{k-1} = C_{k-1}(t)$ where $t = \int f_{k-1}(s)\, ds$, we have the new Frenet equations as follows:

$$\begin{bmatrix} C'_k(t) \\ C'_{k+1}(t) \\ W'_{k+1}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \sigma_{k-1}(t) \\ 0 & -\sigma_{k-1}(t) & 0 \end{bmatrix} \begin{bmatrix} C_k(t) \\ C_{k+1}(t) \\ W_{k+1}(t) \end{bmatrix}, \tag{3.2}$$

where $\sigma_{k-1} = \dfrac{g_{k-1}}{f_{k-1}}$.

Definition 1. Let $\psi : I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve in Euclidean 3-space. A curve ψ is called a $k-$slant helix if the unit vector field C_{k+1} makes a constant angle V with a fixed direction U, that is,

$$\langle C_{k+1}, U\rangle = \cos V, \ \ V = \text{constant}$$

along the curve ψ.

Lemma 3.1. *Let $\psi : I \subset \mathbb{R} \longrightarrow E^3$ be a unit speed curve in Euclidean $3-$space. Then the curve ψ is a $k-$slant helix if and only if the geodesic curvature*

$$\sigma_k = \frac{\sigma'_{k-1}}{\kappa\sqrt{1+\sigma_0^2}\sqrt{1+\sigma_1^2}...\sqrt{1+\sigma_{k-2}^2}\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}},$$

of the principal image of the vector field C_{k+1} indicatrix is a constant function.

Proof. If ψ is a $k-$slant helix, we can see that the curve C_{k-1} is a slant helix. Then

$$\sigma_k = \frac{\sigma'_{k-1}}{f_{k-1}\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}}, \tag{3.3}$$

is a constant function. Since

$$f_{k-1} = \sqrt{f_{k-2}^2 + g_{k-2}^2}$$

$$f_{k-1} = f_{k-2}\sqrt{1+\sigma_{k-2}^2} \tag{3.4}$$

if we put (6) in (5) , we obtain

$$\sigma_k = \frac{\sigma'_{k-1}}{f_{k-2}\sqrt{1+\sigma_{k-2}^2}\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}}.$$

By continuing this process k-times, we get

$$\sigma_k = \frac{\sigma'_{k-1}}{\kappa\sqrt{1+\sigma_0^2}\sqrt{1+\sigma_1^2}...\sqrt{1+\sigma_{k-2}^2}\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}}.$$

□

The following lemma gives a new characterization for $k-$slant helices in E^3.

Lemma 3.2. *Let $\psi : I \longrightarrow E^3$ be a curve that is parameterized by arclenght. The curve is a $k-$slant helix (its vector fields C_{k+1} make a constant angle, with a fixed unit direction U in E^3) if and only if*

$$\sigma_{k-1}(s) = \pm\frac{m\int f_{k-1}ds}{\sqrt{1-m^2\left(\int f_{k-1}ds\right)^2}}, \tag{3.5}$$

where $m = \frac{n}{\sqrt{1-n^2}}$, and $< C_{k+1}, U >= n$.

Proof. $(\Longrightarrow)$ Let U be a unit fixed vector satisfying

$$< C_{k+1}, U >= n. \tag{3.6}$$

Differentiating the Eq.(8) with respect to the variable $t = \int f_{k-1}(s)\, ds$ and using the derivative formulae (4), we get

$$< -C_k(t) + \sigma_{k-1}(t) W_{k+1}(t), U >= 0. \tag{3.7}$$

Therefore,

$$< C_k(t), U >= \sigma_{k-1}(t) < W_{k+1}(t), U > .$$

If we put $< W_{k+1}(t), U >= b$, we can write

$$U = b(t)\sigma_{k-1}(t) C_k(t) + n C_{k+1}(t) + b(t) W_{k+1}(t).$$

From the unitary of the vector U we get

$$b = \pm\sqrt{\tfrac{1-n^2}{1+\sigma_{k-1}{}^2}}. \tag{3.8}$$

Therefore, the vector U can be written as

$$U = \pm\sigma_{k-1}(t)\sqrt{\tfrac{1-n^2}{1+\sigma_{k-1}^2(t)}} C_k(t) + n C_{k+1}(t) \pm \sqrt{\tfrac{1-n^2}{1+\sigma_{k-1}^2(t)}} W_{k+1}(t). \tag{3.9}$$

Differentiating the Eq.(9), we obtain

$$< \sigma'_{k-1}(t) W_{k+1}(t) - \left(1+\sigma_{k-1}^2\right) C_{k+1}(t), U >= 0. \tag{3.10}$$

By Eqs.(12), (10) and (8), we get the following differential equation

$$m = \pm\frac{\sigma'_{k-1}}{\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}},$$

where $m = \frac{n}{\sqrt{1-n^2}}$. Integration the above equation, we obtain

$$\frac{\sigma_{k-1}}{\sqrt{1+\sigma_{k-1}^2}} = \pm m\,(t + c_1), \tag{3.11}$$

where c_1 is an integration constant. The integration constant can disappear with a parameter change $t \longrightarrow t - c_1$. Solving the Eq.$(13)$ with σ_{k-1} as unknown we have

$$\sigma_{k-1} = \pm\frac{mt}{\sqrt{1-m^2t^2}}, \tag{3.12}$$

we obtain the result as desired.

($\Longleftarrow$) Suppose that

$$g_{k-1}(s) = \pm \frac{m f_{k-1}(s) \int f_{k-1}(s) ds}{\sqrt{1-m^2 \left(\int f_{k-1}(s) ds\right)^2}}.$$

The function σ_{k-1} can be written as $\sigma_{k-1}(t) = \pm \frac{mt}{\sqrt{1-m^2t^2}}$ and let us consider the vector

$$U(t) = n\left(tC_k + C_{k+1} \pm \frac{\sqrt{1-m^2t^2}}{m} W_{k+1}\right).$$

Differentiating the vector U bye using the derivative formula (4),

$$\frac{dU}{dt} = n\left(C_k + tC_{k+1} - C_k + \sigma_{k-1} W_{k+1} \mp \frac{mt}{\sqrt{1-m^2t^2}} W_{k+1} \mp \frac{\sigma_{k-1}\sqrt{1-m^2t^2}}{m} C_{k+1}\right) = 0.$$

Therefore, the vector U is constant and $< C_{k+1}, U >= n$, which completes the proof. □

4 Position vectors of $k-$slant helices

To determine the parametric representation of the position vector of a space curve called a $k-$slant helix (its vector fields C_{k+1} make a constant angle with a fixed direction), we firstly establish that for any arbitrary curve, the vector C_{k+1} satisfies a vector differential equation of the third order as follows:

Theorem 4.1. *Let $\psi = \psi(s)$ be a unit speed curve in Euclidean $3-$space. Suppose $\psi = \psi(t)$ is another parametric representation of this curve by the parameter $t = \int f_{k-1} ds$. Then, the vector C_{k+1} satisfies a vector differential equation of the third order as follows:*

$$\frac{1}{\sigma_{k-1}(t)}\left[\frac{1}{\sigma'_{k-1}(t)}\left(C''_{k+1}(t) + \left(1+\sigma^2_{k-1}(t)\right) C_{k+1}(t)\right)\right]' + C_{k+1}(t) = 0, \tag{4.1}$$

where $\sigma_{k-1}(t) = \frac{g_{k-1}(t)}{f_{k-1}(t)}$.

Proof. If we differentiate the second equation of the derivative formulae (4) and using the first and third equations of (4), we get

$$W_{k+1}(t) = \frac{1}{\sigma'_{k-1}}\left[C''_{k+1}(t) + \left(1+\sigma^2_{k-1}(t)\right) C_{k+1}(t)\right]. \tag{4.2}$$

Differentiating the equation (16) and using the third equation from (4), we obtain a vector differential equation of the third order (15) as desired. □

Then Eq.(15) is not easy to solve in the general case. If one solves this equation, we get the following lemma:

Lemma 4.1. *The position vector of an arbitrary space curve can be determined as follows:*

$$\psi(s) = \int \left(\int f_0 \left(\int .. \int f_{k-2} \left(\int f_{k-1} C_{k+1} ds \right) ds..ds \right) ds \right) ds. \tag{4.3}$$

Proof. Let $\psi = \psi(s)$ a natural representation of an arbitrary curve. By using the first equation of formula (3) , we have

$$C_{k+1} = \tfrac{1}{f_{k-1}} \tfrac{dC_k}{ds}. \tag{4.4}$$

For $k \geq 1$, we get

$$C_k = \tfrac{1}{f_{k-2}} \tfrac{dC_{k-1}}{ds}. \tag{4.5}$$

Substituting (19) in (18), we obtain

$$C_{k+1} = \tfrac{1}{f_{k-1}} \tfrac{d}{ds} \left(\tfrac{1}{f_{k-2}} \tfrac{dC_{k-1}}{ds} \right).$$

By continuing this process $k-$times, we get

$$C_{k+1} = \tfrac{1}{f_{k-1}} \tfrac{d}{ds} \left(\tfrac{1}{f_{k-2}} \tfrac{d}{ds} \left(\tfrac{1}{f_{k-3}} \tfrac{d}{ds} \left(... \tfrac{d}{ds} \left(\tfrac{1}{f_0} \tfrac{dC_1}{ds} \right) ... \right) \right) \right),$$

where $\frac{dC_1}{ds} = \frac{dT}{ds} = \frac{d^2\psi}{ds^2}$. □

We can solve the Eq.(15) in the case of a $k-$slant helix.

Lemma 4.2. *Let $\psi = \psi(s)$ a natural representation of a $k-$slant helix (its vector fields C_{k+1} make a constant angle V with a fixed direction). Suppose $\psi = \psi(t)$ is another parametric representation of this curve by the parameter $t = \int f_{k-1} ds$. Then the vector C_{k+1} satisfies a vector differential equation of the third order:*

$$\left(1 - m^2 t^2\right) C'''_{k+1} - 3m^2 t C''_{k+1} + C'_{k+1} = 0,$$

where $m = \frac{n}{\sqrt{1-n^2}}$ and $n = cos(V)$.

Proof. If ψ is a k-slant helix, we can write

$$\sigma_{k-1} = \pm \frac{m \int f_{k-1} ds}{\sqrt{1 - m^2 \left(\int f_{k-1} ds\right)^2}} = \pm \frac{mt}{\sqrt{1-m^2t^2}}.$$

By differentiating the last formula, we obtain

$$\sigma'_{k-1} = \pm m \left(1 - m^2 t^2\right)^{\frac{-3}{2}} \tag{4.6}$$

and

$$\sigma''_{k-1} = \pm 3m^3 t \left(1 - m^2 t^2\right)^{\frac{-5}{2}}. \tag{4.7}$$

Therfore the equation (15) becomes

$$\frac{-\sigma_{k-1}''C_{k+1}'''}{\sigma_{k-1}\sigma_{k-1}'^2}+\frac{C_{k+1}''}{\sigma_{k-1}\sigma_{k-1}'}+\frac{\left(1+\sigma_{k-1}^2\right)C_{k+1}'}{\sigma_{k-1}\sigma_{k-1}'}-\frac{\sigma_{k-1}''\left(1+\sigma_{k-1}^2\right)C_{k+1}}{\sigma_{k-1}\sigma_{k-1}'^2}+3C_{k+1}=0. \tag{4.8}$$

Substituting (20) and (21) in (22), we obtain the formula as desired. □

Theorem 4.2. *The position vector $\psi=(\psi_1,\psi_2,\psi_3)$ of a $k-$salnt helix is computed in the natural representation form as follows*

$$\begin{cases} \psi_1(s)=\frac{n}{m}\int\left(\int f_0\left(\int f_1\left(\int ..\left(\int f_{k-1}\cos\left[\frac{1}{n}\arcsin\left(m\int f_{k-1}(s)ds\right)\right]ds\right)..ds\right)ds\right)ds\right)ds, \\ \\ \psi_2(s)=\frac{n}{m}\int\left(\int f_0\left(\int f_1\left(\int ..\left(\int f_{k-1}\sin\left[\frac{1}{n}\arcsin\left(m\int f_{k-1}(s)ds\right)\right]ds\right)..ds\right)ds\right)ds\right)ds, \\ \\ \psi_3(s)=n\int\left(\int f_0\left(\int f_1\left(\int ..\left(\int f_{k-1}ds\right)..ds\right)ds\right)ds\right)ds, \end{cases} \tag{4.9}$$

where $m=\frac{n}{\sqrt{1-n^2}}$, $n=cos\left(V\right)$ and V is the angle between the fixed straight line (axis of a $k-$slant helix) and the vector C_{k+1} of the curve.

Proof. The curve ψ is a $k-$slant helix, i.e. the vector C_{k+1} makes a constant angle, $V=\arccos(n)$, with the constant vector called the axis of the $k-$slant helix. Then the vector C_{k+1} satisfies a vector differential equation:

$$\left(1-m^2t^2\right)C_{k+1}'''-3m^2tC_{k+1}''+C_{k+1}'=0. \tag{4.10}$$

So, without loss of generality, we can take the axis of the $k-$slant helix parallel to e_3, where (e_1,e_2,e_3) is an orthonormal frame in E_3, then

$$C_{k+1}\left(t\right)=C_{k+1_1}\left(t\right)e_1+C_{k+1_2}\left(t\right)e_2+ne_3. \tag{4.11}$$

From the unitary of the vector C_{k+1}, we get

$$C_{k+1_1}^2+C_{k+1_2}^2=1-n^2=\frac{n^2}{m^2}. \tag{4.12}$$

The solution of Eq.(26) is given as follows:

$$\begin{cases} C_{k+1_1}(t)=\frac{n}{m}\cos\left(\lambda(t)\right), \\ \\ C_{k+1_2}(t)=\frac{n}{m}\sin\left(\lambda(t)\right), \end{cases} \tag{4.13}$$

where λ is an arbitrary function of t. Every component of the vector C_{k+1} satisfied the Eq.(24). So, substituting the components $C_{k+1_1}(t)$ and $C_{k+1_2}(t)$ in the Eq.(24), we have the following differential equations of the function $\lambda\left(t\right)$

$$\begin{aligned} &3\lambda'(t)\left[m^2t\lambda'(t)-\left(1-m^2t^2\right)\right]\cos(\lambda(t)) \\ &-\left[\lambda'(t)-3m^2t\lambda''(t)-\left(1-m^2t^2\right)\left(\lambda'^3(t)-\lambda'''(t)\right)\right]\sin(\lambda(t))=0, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &3\lambda'(t)\left[m^2t\lambda'(t)-\left(1-m^2t^2\right)\right]\sin(\lambda(t)) \\ &+\left[\lambda'(t)-3m^2t\lambda''(t)-\left(1-m^2t^2\right)\left(\lambda'^3(t)-\lambda'''(t)\right)\right]\cos(\lambda(t))=0. \end{aligned} \tag{4.15}$$

It is easy to prove that the above two equations lead to the following two equations:

$$m^2t\lambda'(t)-\left(1-m^2t^2\right)\lambda''(t)=0, \tag{4.16}$$

$$\lambda'(t)-3m^2t\lambda''(t)-\left(1-m^2t^2\right)\left(\lambda'^3(t)-\lambda'''(t)\right)=0. \tag{4.17}$$

The general solution of Eq.(30) is

$$\lambda(t)=c_1\arcsin(mt)+c_2, \tag{4.18}$$

where c_1 and c_2 are constants of integration. The constant c_2 can be disappear if we change the parameter $\lambda\longrightarrow\lambda+c_2$. Substituting the solution (32) in the Eq.(31) , we obtain the following condition:

$$c_1m\left(1+m^2\left(1-c_1\right)\right)=0,$$

which leads to $c_1=\frac{\sqrt{1+m^2}}{m}=\dfrac{1}{n}$, where $m\neq 0$ and $c_1\neq 0$.

Now, the vector C_{k+1} take the following form:

$$\begin{cases} C_{k+1_1}(t)=\frac{n}{m}\cos\left(\frac{1}{n}\arcsin mt\right), \\ \\ C_{k+1_2}(t)=\frac{n}{m}\sin\left(\frac{1}{n}\arcsin mt\right), \\ \\ C_{k+1_3}(t)=n. \end{cases} \tag{4.19}$$

If we substitute the Eq.(33) in the Eq.(17) , we have the Eq.(23) , which completes the proof. □

5 Applications

In this section, we introduce the position vectors of some *2*−slant helices, by using new parametric representations.

Corollary 5.1. *The position vector* $\psi=(\psi_1,\psi_2,\psi_3)$ *of a 2−slant helix whose the vector* $C_3=\frac{C_2'}{\|C_2''\|}=\frac{N'}{\|N'\|}$ *makes a constant angle with a fixed straight line in the space is expressed in the natural representation form as follows :*

$$\begin{cases} \psi_1(s)=\frac{n}{m}\int\left[\int f_0(s)\left[\int f_1(s)\cos\left[\frac{1}{n}\arcsin\left(m\int f_1(s)ds\right)\right]ds\right]ds\right]ds, \\ \\ \psi_2(s)=\frac{n}{m}\int\left[\int f_0(s)\left[\int f_1(s)\sin\left[\frac{1}{n}\arcsin\left(m\int f_1(s)ds\right)\right]ds\right]ds\right]ds, \\ \\ \psi_3(s)=n\int\left[\int f_0(s)\left[\int f_1(s)ds\right]ds\right]ds, \end{cases} \tag{5.1}$$

with $f_1(s) = \sqrt{{g_0}^2(s) + {f_0}^2(s)}$, $m = \frac{n}{\sqrt{1-n^2}}$ *and* $n = cos(V)$, *or in the parametric form*

$$\begin{cases} \psi_1(\theta) = \frac{n}{m}\int \frac{1}{f_1(\theta)}\left[\int \frac{f_0(\theta)}{f_1(\theta)}\left[\int \cos\left[\frac{1}{n}\arcsin(m\theta)\right]d\theta\right]d\theta\right]d\theta, \\ \psi_2(\theta) = \frac{n}{m}\int \frac{1}{f_1(\theta)}\left[\int \frac{f_0(\theta)}{f_1(\theta)}\left[\int \sin\left[\frac{1}{n}\arcsin(m\theta)\right]d\theta\right]d\theta\right]d\theta, \\ \psi_3(\theta) = n\int \frac{1}{f_1(\theta)}\left[\int \frac{f_0(\theta)}{f_1(\theta)}\theta d\theta\right]d\theta, \end{cases} \tag{5.2}$$

or in the useful parametric form

$$\begin{cases} \psi_1(t) = \frac{n^4}{m^4}\int \frac{\cos(nt)}{f_1(\theta)}\left[\int \frac{f_0(\theta)}{f_1(\theta)}\cos(nt)\left[\int \cos(t)\cos(nt)\,dt\right]dt\right]dt, \\ \psi_2(t) = \frac{n^4}{m^4}\int \frac{\cos(nt)}{f_1(\theta)}\left[\int \frac{f_0(\theta)}{f_1(\theta)}\cos(nt)\left[\int \sin(t)\cos(nt)\,dt\right]dt\right]dt, \\ \psi_3(t) = \frac{n^3}{m^3}\int \frac{\cos(nt)}{f_1(\theta)}\left(\int \frac{f_0(\theta)}{f_1(\theta)}\cos(nt)\sin(nt)\,dt\right)dt, \end{cases} \tag{5.3}$$

where $\theta = \int f_1(s)\,ds$, $t = \frac{1}{n}\arcsin(m\theta)$, $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = cos(V)$ *and* V *is the angle between the fixed straight line (axis of a* $2-$*slant helix) and the vector* C_3 *of the curve.*

Now, we take several choices for the curvature f_0 and torsion g_0 of a regular curve. We check that the curve is a *2*$-$slant hélix, and next, we apply corollary 5.1.

Example 1. The case of a *2*$-$slant helix with

$$f_0 = \frac{\mu}{m}\cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right) \text{ and } g_0 = \frac{-\mu}{m}\cos(\mu s)\sin\left(\frac{1}{m}\cos(\mu s)\right).$$

Therefore $f_1 = \frac{\mu}{m}\cos(\mu s)$ and $g_1 = \frac{\mu}{m}\sin(\mu s)$,we have $\sigma_2 = m$. Substituting f_0 and f_1 in the Eq.(34) , we have the explicit parametric representation of such curve as follows:

$$\begin{cases} \psi_1(s) = \frac{n^2\mu}{2m^3}\int\left[\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left[\frac{n}{n+1}\sin(\frac{n+1}{n}\mu s) + \frac{n}{n-1}\sin\left(\frac{n-1}{n}\mu s\right)\right]ds\right]ds, \\ \psi_2(s) = \frac{-n^2\mu}{2m^3}\int\left(\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left(\frac{n}{n+1}\cos(\frac{n+1}{n}\mu s) + \frac{n}{1-n}\cos\left(\frac{1-n}{n}\mu s\right)\right)ds\right)ds, \\ \psi_3(s) = \frac{-n}{m}\int \cos(\mu s)\sin\left(\frac{1}{m}\cos(\mu s)\right) + m\cos\left(\frac{1}{m}\cos(\mu s)\right)ds. \end{cases}$$

Example 2. The case of a slant-slant helix with

$$f_0 = \frac{ms}{\sqrt{1-m^2s^2}}\cos(s) \text{ and } g_0 = \frac{ms}{\sqrt{1-m^2s^2}}\sin(s).$$

Therefore $f_1 = \frac{ms}{\sqrt{1-m^2s^2}}$ and $g_1 = 1$,we have $\sigma_2 = -m$. Substituting $f_0 = \frac{ms}{\sqrt{1-m^2s^2}} \cos(s) = \frac{\cos(nt)}{\sin(nt)} \cos\left(\frac{1}{m}\cos(nt)\right)$, and $f_1 = \frac{ms}{\sqrt{1-m^2s^2}} = \frac{\cos(nt)}{\sin(nt)}$ in the Eq.(36), we have the explicit parametric representation of such curve as follows:

$$\begin{cases} \psi_1(t) = \frac{n^4}{2m^4} \int \sin(nt) \left[\int \cos(nt) \cos\left(\frac{1}{m}\cos(nt)\right) \left[\frac{\sin((n+1)t)}{n+1} + \frac{\sin((n-1)t)}{n-1}\right] dt\right] dt, \\ \psi_2(t) = \frac{-n^4}{2m^4} \int \sin(nt) \left[\int \cos(nt) \cos\left(\frac{1}{m}\cos(nt)\right) \left[\frac{\cos((n+1)t)}{n+1} + \frac{\cos((1-n)t)}{1-n}\right] dt\right] dt, \\ \psi_3(t) = \frac{-n}{m} \left(\cos(nt) \cos\left(\frac{1}{m}\cos(nt)\right) - 2m \sin\left(\frac{1}{m}\cos(nt)\right)\right), \end{cases}$$

where $\theta = \frac{-1}{m}\sqrt{1-m^2s^2}$ and $t = \frac{1}{n} arcsin\,(m\theta)$.

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