TAMKANG JOURNAL OF MATHEMATICS
Volume 53, Number 1, 59-74, March 2022
DOI:10.5556/j.tkjm.53.2022.3491

Certain Laguerre-based Generalized Apostol Type Polynomials

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Abstract. A variety of polynomials, their extensions and variants have been extensively investigated, due mainly to their potential applications in diverse research areas. In this paper, we aim to introduce Laguerre-based generalized Apostol type polynomials and investigate some properties and identities involving them. Among them, some differential-recursive relations for the Hermite-Laguerre polynomials, which are expressed in terms of generalized Apostol type numbers and the Laguerre-based generalized Apostol type polynomials, an implicit summation formula and addition-symmetry identities for the Laguerre-based generalized Apostol type polynomials are presented. The results presented here, being very general, are pointed out to reduce to yield some known or new formulas and identities for relatively simple polynomials and numbers.

1 Introduction and preliminaries

The two variable Laguerre polynomials $L_n(x,y)$ are defined by the generating function (see [7]; see also [15, 16, 17, 18])

$$e^{yt}\, C_0(xt) = \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}, \tag{1.1}$$

where $C_0(x)$ is the 0-th order Tricomi function (see, e.g., [30])

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}. \tag{1.2}$$

From (1.1) and (1.2), it is easy to derive the following finite series

$$L_n(x,y) = \sum_{s=0}^{n} \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!\,(s!)^2} \quad (n \in \mathbb{N}_0)\,. \tag{1.3}$$

2010 *Mathematics Subject Classification*. 05A10, 05A15, 11B68, 33C45, 33C99.
Key words and phrases. Laguerre polynomials, Hermite polynomials, Hermite-based generalized Apostol type polynomials, Laguerre-based generalized Apostol type polynomials, implicit formula, summation formulae, symmetric identities.
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Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ are defined, respectively, by the following generating functions (see, e.g., [8, p. 253 et seq.], [21, Section 2.8], [23], [32, Section 1.7])

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < 2\pi;\ 1^{\alpha} := 1), \tag{1.4}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < \pi;\ 1^{\alpha} := 1), \tag{1.5}$$

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < \pi;\ 1^{\alpha} := 1). \tag{1.6}$$

A large number of interesting properties and relationships involving those polynomials in (1.4), (1.5) and (1.6) have been presented (see, e.g., [1, 3, 5, 8, 9, 28])

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ (see [22, 25]), the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ (see [22]) and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ (see [23, 24, 27]) are defined, respectively, by the following generating functions

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \left(|t+\ln\lambda| < 2\pi;\ 1^{\alpha} := 1\right), \tag{1.7}$$

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \left(|t+\ln\lambda| < \pi;\ 1^{\alpha} := 1\right), \tag{1.8}$$

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \left(|t+\ln\lambda| < \pi;\ 1^{\alpha} := 1\right). \tag{1.9}$$

Setting $x=0$ in (1.7), (1.8) and (1.9), we have

$$B_n^{(\alpha)}(0;\lambda) := B_n^{(\alpha)}(\lambda), \quad E_n^{(\alpha)}(0;\lambda) := E_n^{(\alpha)}(\lambda), \quad G_n^{(\alpha)}(0;\lambda) = G_n^{(\alpha)}(\lambda), \tag{1.10}$$

which are called Apostol-Bernoulli number of order α, Apostol-Euler number of order α and Apostol-Genocchi number of order α, respectively. Clearly,

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1), \quad E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x;1), \quad G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1). \tag{1.11}$$

Srivastava et al. [33, 34] have introduced and investigated certain extensions of generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{C}$, Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{C}$ and Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{C}$ which are defined, respectively, by the following generating functions

$$\left(\frac{t}{\lambda b^t-a^t}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \tag{1.12}$$

$$\left(a,\ b,\ c\in\mathbb{R}^+ \text{ with } a\neq b;\ \left|t\ \ln\left(\frac{b}{a}\right)+\ln\ \lambda\right|<2\pi;\ 1^{\alpha}:=1;\ x\in\mathbb{R}\right),$$

$$\left(\frac{2}{\lambda b^t+a^t}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \tag{1.13}$$

$$\left(a,\ b,\ c\in\mathbb{R}^+ \text{ with } a\neq b;\ \left|t\ \ln\left(\frac{b}{a}\right)+\ln\ \lambda\right|<\pi;\ 1^{\alpha}:=1;\ x\in\mathbb{R}\right),$$

$$\left(\frac{2t}{\lambda b^t+a^t}\right)^{\alpha} c^{xt}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \tag{1.14}$$

$$\left(a,\ b,\ c\in\mathbb{R}^+ \text{ with } a\neq b;\ \left|t\ \ln\left(\frac{b}{a}\right)+\ln\ \lambda\right|<\pi;\ 1^{\alpha}:=1;\ x\in\mathbb{R}\right).$$

Clearly, the special cases of (1.12), (1.13) and (1.14) when $a=1, b=c=e$ reduce, respectively, to (1.7), (1.8) and (1.9).

Lu and Luo [20] have introduced and investigated the generalized Apostol type polynomials $F_n^{(\alpha)}(x;\lambda;\mu,\nu)$ $(\alpha\in\mathbb{N}_0,\ \lambda,\ \mu,\ \nu\in\mathbb{C})$ of order α defined by means of the following generating function

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}F_n^{(\alpha)}(x;\lambda;\mu,\nu)\,\frac{t^n}{n!}\quad (|t|<|\log(-\lambda)|), \tag{1.15}$$

where

$$F_n^{(\alpha)}(\lambda;\mu,\nu):=F_n^{(\alpha)}(0;\lambda;\mu,\nu) \tag{1.16}$$

are the Apostol type numbers of order α. Note that the generalized Apostol type polynomials $F_n^{(\alpha)}(x;\lambda;\mu,\nu)$ in (1.15) reduce to yield the polynomials in (1.7), (1.8) and (1.9) as follows:

$$B_n^{(\alpha)}(x;\lambda)=(-1)^{\alpha}F_n^{(\alpha)}(x;-\lambda;0,1), \tag{1.17}$$

$$E_n^{(\alpha)}(x;\lambda)=F_n^{(\alpha)}(x;\lambda;1,0), \tag{1.18}$$

$$G_n^{(\alpha)}(x;\lambda)=F_n^{(\alpha)}(x;\lambda;1,1). \tag{1.19}$$

The 2-variable Kampé de Fériet generalization of the Hermite polynomials $H_n(x,y)$ is given by (see, e.g., [2], [15])

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{1.20}$$

which are defined by the following generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.21}$$

The particular case of (1.21) when $y=-1$ and x is replaced by $2x$ reduces to generate the ordinary Hermite polynomials $H_n(x)$ (see, e.g., [2], [30]).

The Hermite-Laguerre polynomials ${}_HL_n(x,y,z)$ are defined by the following generating function (see [15, Eq. (1.16)])

$$e^{yt+zt^2}\, C_0(xt) = \sum_{n=0}^{\infty} {}_HL_n(x,y,z) \frac{t^n}{n!}. \tag{1.22}$$

Lu and Luo [20, Eq. (4.5)] introduced the Hermite-based generalized Apostol type polynomials of order $\alpha \in \mathbb{N}_0$

$$\left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_HF_n^{(\alpha)}(y,z;\lambda;\mu,\nu) \frac{t^n}{n!} \quad (|t| < |\log(-\lambda)|) \tag{1.23}$$

and

$${}_HF_n^{(1)}(y,z;\lambda;\mu,\nu) := {}_HF_n(y,z;\lambda;\mu,\nu). \tag{1.24}$$

Here, we introduce the Laguerre-based generalized Apostol type polynomials

$${}_L\mathcal{F}_n^{(\alpha)}(\alpha,\ x,y,z;a,b,c,d;\lambda;\mu,\nu)$$

in the following definition.

Definition 1. The Laguerre-based generalized Apostol type polynomials of order α ${}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,$ are defined by the following generating function

$$\left(\frac{d^\mu t^\nu}{\lambda a^t + b^t}\right)^\alpha c^{yt+zt^2}\, C_0(xt) = \sum_{n=0}^{\infty} {}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c;\lambda;\mu,\nu)\, \frac{t^n}{n!} \tag{1.25}$$

$$\big(\alpha,\ x,\ y,\ z,\ \mu,\ \nu \in \mathbb{C};\ \ \lambda \in \mathbb{C}\setminus\{0\};$$
$$a,\ b,\ c,\ d \in \mathbb{R}^+ \ \text{ with }\ a \neq b;\ \ |t| < \big|\log(-\lambda)\big|/\big|\log(a/b)\big|\big).$$

Here $C_0(x)$ is the same as in (1.2). Here and throughout, for simplicity, all involved multiple-valued functions are assumed to take the principal branches. Also let

$${}_L\mathcal{F}_n^{(\alpha)}(x,y,z;e,1,e,2;\lambda;\mu,\nu) := {}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu). \tag{1.26}$$

In this paper, we aim to investigate certain basic properties and identities of the Laguerre-based generalized Apostol type polynomials ${}_L\mathcal{F}_n^{(\alpha)}(\alpha,\, x, y, z; a, b, c, d; \lambda; \mu, \nu)$ in (1.25). Also, we give some differential-recursive relations for the Hermite-Laguerre polynomials, which are expressed in terms of generalized Apostol type numbers and the Laguerre-based generalized Apostol type polynomials. Further, by mainly using some formal manipulations of double series to the generating function, we present an implicit summation formula and various addition-symmetry identities for the Laguerre-based generalized Apostol type polynomials, which are pointed out to reduce to yield corresponding formulas and identities for a number of relatively simple known polynomials and numbers (see, e.g., [6, 11, 12, 13, 14, 15, 16, 17, 18, 19, 26, 29, 31, 36, 37, 38]).

2 Elementary properties and identities involving ${}_L\mathcal{F}_n^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu)$

We present some properties and identities involving the Laguerre-based generalized Apostol type polynomials ${}_L\mathcal{F}_n^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu)$ in (1.25), which are given in Theorem 2.1.

For easier reference, we recall some formal manipulations of double series in the following lemma (see, e.g., [4], [16], [30, pp. 56-57], and [35, p. 52]).

Lemma 2.1. *The following identities hold:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n-pk} \quad (p \in \mathbb{N}); \tag{2.1}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+pk} \quad (p \in \mathbb{N}); \tag{2.2}$$

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{2.3}$$

Here, the $A_{k,n}$ and $f(N)$ $(k,\, n,\, N \in \mathbb{N}_0)$ are real or complex valued functions indexed by the k, n and N, respectively, and x and y are real or complex numbers. Also, for possible rearrangements of the involved double series, all the associated series should be absolutely convergent.

Theorem 2.1. *Let $\alpha,\, x,\, y,\, z,\, \mu,\, \nu \in \mathbb{C}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Also let $a,\, b,\, c,\, d \in \mathbb{R}^+$ with $a \neq b$. Then*

$$\begin{aligned} &\frac{\partial^k}{\partial y^k}\, {}_L\mathcal{F}_n^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu) \\ &\quad = \frac{n!\, \log^k c}{(n-k)!}\, {}_L\mathcal{F}_{n-k}^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu) \quad (n,\, k \in \mathbb{N}_0;\, k \leq n); \end{aligned} \tag{2.4}$$

$$\begin{aligned}&\frac{\partial^k}{\partial z^k}\,{}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\\&\quad=\frac{n!\,\log^k c}{(n-2k)!}\,{}_L\mathcal{F}_{n-2k}^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\quad(n,\,k\in\mathbb{N}_0;\,2k\le n);\end{aligned}\tag{2.5}$$

$$\begin{aligned}&\frac{\partial^{k+\ell}}{\partial y^k\partial z^\ell}\,{}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)=\frac{\partial^{\ell+k}}{\partial z^\ell\partial y^k}\,{}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\\&=\frac{n!\,\log^{k+\ell} c}{(n-k-2\ell)!}\,{}_L\mathcal{F}_{n-k-2\ell}^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\quad(n,\,k,\,\ell\in\mathbb{N}_0;\,k+2\ell\le n);\end{aligned}\tag{2.6}$$

$${}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)=\sum_{m=0}^{n}\binom{n}{m}\frac{(-x)^{n-m}}{(n-m)!}\,{}_HF_m^{(\alpha)}(y,z;\lambda;\mu,\nu)\quad(n\in\mathbb{N}_0)\,;\tag{2.7}$$

$$\begin{aligned}&{}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\\&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}\frac{(-x)^{n-m}}{(n-m)!}\,H_{m-\ell}(y,z)\,F_\ell^{(\alpha)}(\lambda;\mu,\nu)\quad(n\in\mathbb{N}_0)\,;\end{aligned}\tag{2.8}$$

$$\begin{aligned}&{}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\\&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{[\frac{m}{2}]}\frac{n!\,z^\ell}{(n-m)!\,(m-2\ell!)\,\ell!}\,F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\,L_{m-2\ell}(x,y)\quad(n\in\mathbb{N}_0)\,;\end{aligned}\tag{2.9}$$

$${}_HL_n(x,y,z)=\sum_{m=0}^{n}\binom{n}{m}F_{n-m}^{(-\alpha)}(\lambda;\mu,\nu)\,{}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\quad(n\in\mathbb{N}_0)\,.\tag{2.10}$$

$$\begin{aligned}&\frac{(-1)^n\,x^n}{n!}\\&=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}H_{n-m}(-y,-z)\,F_{m-\ell}^{(-\alpha)}(\lambda;\mu,\nu)\,{}_LF_\ell^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\\&=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}F_{n-m}^{(-\alpha)}(\lambda;\mu,\nu)\,H_{m-\ell}(-y,-z)\,{}_LF_\ell^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\\&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\binom{m}{\ell}F_{n-m}^{(-\alpha)}(\lambda;\mu,\nu)\,{}_LF_{m-\ell}^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\,H_\ell(-y,-z).\end{aligned}\tag{2.11}$$

Proof. Differentiating both sides of (1.25) with respect to the variables y and z, k times, and equating the coefficients of t^n, we obtain (2.4) and (2.5).

Differentiating both sides of (1.25) with respect to the variables y and z, respectively, k and ℓ times, and equating the coefficients of t^n, we get (2.6).

Applying the left members of (1.2) and (1.23) in (1.26), with the aid of (2.1) for $p = 1$, we get (2.7).

Employing the left members of (1.2), (1.16) and (1.21) in (1.26), by a successive use of (2.1) for $p = 1$, we have (2.8).

Applying the left members of (1.1) and (1.21) in (1.26), with the aid of (2.1) for $p = 1$ and $p = 2$, we derive (2.9).

From (1.26), we have

$$e^{yt+zt^2}\, C_0(xt) = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^{-\alpha} \sum_{n=0}^{\infty} {}_LF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)\frac{t^n}{n!} \tag{2.12}$$

and

$$C_0(xt) = e^{-yt-zt^2} \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^{-\alpha} \sum_{n=0}^{\infty} {}_LF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)\frac{t^n}{n!}. \tag{2.13}$$

Using (1.22) and (1.16) in (2.12), and (1.2), (1.22) and (1.16) in (2.13), similarly as above, respectively, we get (2.10) and (2.11).

□

3 Differential-recursive relations

We establish some differential-recursive relations involving the generalized Apostol type numbers $F_n^{(\alpha)}(\lambda; \mu, \nu)$ in (1.16), the Hermite-Laguerre polynomials ${}_HL_n(x, y, z)$ in (1.22), and the Laguerre-based generalized Apostol type polynomials ${}_LF_n^{(\alpha)}(x, y, z; \lambda; \mu, \nu)$ in (1.26), asserted in the following theorem.

Theorem 3.1. *Let* $\alpha,\ x,\ y,\ z,\ \mu,\ \nu \in \mathbb{C}$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$. *Then*

$$\begin{aligned} &n\, y\, {}_HL_{n-1}(x, y, z) + 2n(n-1)\, z\, {}_HL_{n-2}(x, y, z) + x\, \frac{\partial}{\partial x}\, {}_HL_n(x, y, z) \\ &\quad = n \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha)}(\lambda; \mu, \nu)\, {}_LF_m^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \quad (n \in \mathbb{N} \setminus \{1\}); \end{aligned} \tag{3.1}$$

$${}_HL_{n-k}(x, y, z) = \sum_{m=0}^{n} \frac{(n-k)!}{m!\,(n-m)!} F_{n-m}^{(\alpha)}(\lambda; \mu, \nu)\, \frac{\partial^k}{\partial y^k}\, {}_LF_m^{(\alpha)}(x, y, z; \lambda; \mu, \nu) \tag{3.2}$$

$$(n,\ k \in \mathbb{N}_0,\ \ k \leq n)\,;$$

$$ {}_HL_{n-2k}(x,y,z) = \sum_{m=0}^{n} \frac{(n-2k)!}{m!\,(n-m)!}\, F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\, \frac{\partial^k}{\partial z^k}\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu) \tag{3.3}$$

$$(n,\ k \in \mathbb{N}_0,\ \ 2k \leq n)\,.$$

Proof. From (1.16) and (1.26), using (2.1) with $p = 1$, we have

$$\begin{aligned} e^{yt+zt^2}\, C_0(xt) &= \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^{-\alpha} \sum_{n=0}^{\infty} {}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} F_n^{(\alpha)}(\lambda;\mu,\nu)\, \frac{t^n}{n!}\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m}\, F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

Differentiating both sides of (3.4) with respect to t, we obtain

$$\begin{aligned} &y\, e^{yt+zt^2}\, C_0(xt) + 2zt\, e^{yt+zt^2}\, C_0(xt) + e^{yt+zt^2}\, \frac{\partial}{\partial t} C_0(xt) \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \binom{n}{m}\, F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^{n-1}}{(n-1)!}. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by t and using

$$t\, \frac{\partial}{\partial t} C_0(xt) = x\, \frac{\partial}{\partial x} C_0(xt),$$

we get

$$\begin{aligned} &yt\, e^{yt+zt^2}\, C_0(xt) + 2zt^2\, e^{yt+zt^2}\, C_0(xt) + x\, \frac{\partial}{\partial x} \left\{ e^{yt+zt^2}\, C_0(xt) \right\} \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \binom{n}{m}\, F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^n}{(n-1)!}. \end{aligned} \tag{3.6}$$

Applying (1.22) to the left member of (3.6), we find

$$\begin{aligned} &y \sum_{n=0}^{\infty} {}_HL_n(x,y,z) \frac{t^{n+1}}{n!} + 2z \sum_{n=0}^{\infty} {}_HL_n(x,y,z) \frac{t^{n+2}}{n!} + x \sum_{n=0}^{\infty} \frac{\partial}{\partial x} {}_HL_n(x,y,z) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \binom{n}{m}\, F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\, {}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\, \frac{t^n}{(n-1)!}. \end{aligned} \tag{3.7}$$

Equating the coefficients of t^n in both sides of (3.7), we derive (3.1).

Differentiating both sides of (3.4) with respect to the variable z, $k \in \mathbb{N}_0$ times, and using (1.22), we obtain

$$\begin{aligned}
&\sum_{n=2k}^{\infty} {}_HL_{n-2k}(x,y,z)\frac{t^n}{(n-2k)!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)\,\frac{\partial^k}{\partial z^k}\,{}_LF_m^{(\alpha)}(x,y,z;\lambda;\mu,\nu)\,\frac{t^n}{n!}.
\end{aligned} \tag{3.8}$$

Equating the coefficients of t^n on both sides of (3.8), we get (3.3).

Similarly as in getting (3.3), we derive (3.2). □

4 Addition-symmetry identities

We establish some addition-symmetry identities involving the Laguerre-based generalized Apostol type polynomials ${}_LF_n^{(\alpha)}(x,y,z;\lambda;\mu,\nu)$ in (1.26) with respect to the order α, and the variables y and z, which are given in Theorem 4.1.

Theorem 4.1. *Let $n \in \mathbb{N}_0$, α_1, α_2, x, y_1, y_2, z_1, z_2, μ, $\nu \in \mathbb{C}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Then*

$$\begin{aligned}
&{}_LF_n^{(\alpha_1+\alpha_2)}(x,y,z;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{n}\binom{n}{m} F_{n-m}^{(\alpha_1)}(\lambda;\mu,\nu)\,{}_LF_m^{(\alpha_2)}(x,y,z;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{n}\binom{n}{m} F_{n-m}^{(\alpha_2)}(\lambda;\mu,\nu)\,{}_LF_m^{(\alpha_1)}(x,y,z;\lambda;\mu,\nu);
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha)}(x,y_1+y_2,z;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{n}\binom{n}{m} y_2^{n-m}\,{}_LF_m^{(\alpha)}(x,y_1,z;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{n}\binom{n}{m} y_1^{n-m}\,{}_LF_m^{(\alpha)}(x,y_2,z;\lambda;\mu,\nu);
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha)}(x,y,z_1+z_2;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{[\frac{n}{2}]} m!\binom{n}{2m}\binom{2m}{m} z_2^m\,{}_LF_{n-2m}^{(\alpha)}(x,y,z_1;\lambda;\mu,\nu) \\
&\quad= \sum_{m=0}^{[\frac{n}{2}]} m!\binom{n}{2m}\binom{2m}{m} z_1^m\,{}_LF_{n-2m}^{(\alpha)}(x,y,z_2;\lambda;\mu,\nu);
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha_1+\alpha_2)}(x, y_1+y_2, z; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha_2)}(y_2; \lambda; \mu, \nu)\, {}_LF_m^{(\alpha_1)}(x, y_1, z; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha_1)}(y_2; \lambda; \mu, \nu)\, {}_LF_m^{(\alpha_2)}(x, y_1, z; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha_2)}(y_1; \lambda; \mu, \nu)\, {}_LF_m^{(\alpha_1)}(x, y_2, z; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha_1)}(y_1; \lambda; \mu, \nu)\, {}_LF_m^{(\alpha_2)}(x, y_2, z; \lambda; \mu, \nu);
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha_1+\alpha_2)}(x, y, z_1+z_2; \lambda; \mu, \nu) \\
&= \sum_{m=0}^{n} \sum_{\ell=0}^{[\frac{m}{2}]} \ell! \binom{n}{m} \binom{m}{2\ell} \binom{2\ell}{\ell} z_2^{\ell}\, {}_LF_{n-m}^{(\alpha_1)}(x, y, z_1; \lambda; \mu, \nu) F_{m-2\ell}^{(\alpha_2)}(\lambda; \mu, \nu) \\
&= \sum_{m=0}^{n} \sum_{\ell=0}^{[\frac{m}{2}]} \ell! \binom{n}{m} \binom{m}{2\ell} \binom{2\ell}{\ell} z_2^{\ell}\, {}_LF_{n-m}^{(\alpha_2)}(x, y, z_1; \lambda; \mu, \nu) F_{m-2\ell}^{(\alpha_1)}(\lambda; \mu, \nu);
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha)}(x, y_1+y_2, z_1+z_2; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha)}(x, y_1, z_1; \lambda; \mu, \nu)\, H_m(y_2, z_2) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha)}(x, y_2, z_1; \lambda; \mu, \nu)\, H_m(y_1, z_2) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha)}(x, y_1, z_2; \lambda; \mu, \nu)\, H_m(y_2, z_1) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha)}(x, y_2, z_2; \lambda; \mu, \nu)\, H_m(y_1, z_1);
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&{}_LF_n^{(\alpha_1+\alpha_2)}(x, y_1+y_2, z_1+z_2; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha_1)}(x, y_1, z_1; \lambda; \mu, \nu)\, {}_HF_m^{(\alpha_2)}(y_2, z_2; \lambda; \mu, \nu) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha_2)}(x, y_1, z_1; \lambda; \mu, \nu)\, {}_HF_m^{(\alpha_1)}(y_2, z_2; \lambda; \mu, \nu)
\end{aligned} \tag{4.7}$$

$$= \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha_1)}(x, y_2, z_1; \lambda; \mu, \nu)\, {}_HF_m^{(\alpha_2)}(y_1, z_2; \lambda; \mu, \nu)$$
$$= \sum_{m=0}^{n} \binom{n}{m} {}_LF_{n-m}^{(\alpha_2)}(x, y_2, z_1; \lambda; \mu, \nu)\, {}_HF_m^{(\alpha_1)}(y_2, z_1; \lambda; \mu, \nu).$$

Proof. Choosing to use the involved polynomials, similarly as in the proof of Theorem 2.1, we can prove the identities here. We omit the details.

□

5 Implicit summation formula

Implicit formulas involving various polynomials have been presented (e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19]). Similarly as in the above-cited works, we provide an implicit formula for the Laguerre-based generalized Apostol type polynomials in (1.25), asserted in the following theorem.

Theorem 5.1. *Let $n,\ m \in \mathbb{N}_0$. Also let $\alpha,\ v,\ x,\ y,\ z,\ \mu,\ \nu \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Further let $a,\ b,\ c,\ d \in \mathbb{R}^+$ with $a \neq b$. Then*

$$\begin{aligned} &{}_L\mathcal{F}_{m+n}^{(\alpha)}(x, v, z; a, b, c, d; \lambda; \mu, \nu) \\ &= \sum_{k=0}^{n} \sum_{s=0}^{m} \binom{n}{k} \binom{m}{s} (v-y)^{s+k} \log^{s+k} c\, {}_L\mathcal{F}_{m+n-s-k}^{(\alpha)}(x, v, z; a, b, c, d; \lambda; \mu, \nu). \end{aligned} \tag{5.1}$$

Proof. Replacing t by $t+u$ in (1.25), we have

$$\begin{aligned} &\left(\frac{d^{\mu}(t+u)^{\nu}}{\lambda a^{t+u} + b^t}\right)^{\alpha} c^{z(t+u)^2} C_0(x(t+u)) \\ &\quad = c^{-y(t+u)} \sum_{n=0}^{\infty} {}_L\mathcal{F}_n^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu) \frac{(t+u)^n}{n!}. \end{aligned} \tag{5.2}$$

Using binomial formula to expand $(t+u)^n$ in the right member of (5.2), with the aid of (2.2) for $p = 1$, we obtain

$$\begin{aligned} &\left(\frac{d^{\mu}(t+u)^{\nu}}{\lambda a^{t+u} + b^t}\right)^{\alpha} c^{z(t+u)^2} C_0(x(t+u)) \\ &\quad = c^{-y(t+u)} \sum_{m,n=0}^{\infty} {}_L\mathcal{F}_{m+n}^{(\alpha)}(x, y, z; a, b, c, d; \lambda; \mu, \nu) \frac{t^n}{n!} \frac{u^m}{m!}. \end{aligned} \tag{5.3}$$

We find that the left member of (5.3) is *independent* of the variable y. In this regard, replacing y by another variable v in the right member of (5.3) and equating the right members of the resulting identity and (5.3), we get

$$\begin{aligned} c^{(v-y)(t+u)} &\sum_{m,n=0}^{\infty} {}_L\mathcal{F}_{m+n}^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!} \\ &= \sum_{m,n=0}^{\infty} {}_L\mathcal{F}_{m+n}^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned} \tag{5.4}$$

Expanding the exponential function and using (2.3), we find

$$c^{(v-y)(t+u)} = \sum_{N=0}^{\infty} (v-y)^N \log^N c\,\frac{(t+u)^N}{N!} = \sum_{k,s=0}^{\infty} \frac{(v-y)^{k+s}\log^{k+s} c\; t^k u^s}{k!s!}. \tag{5.5}$$

Setting the double series of (5.5) for $c^{(v-y)(t+u)}$ in the left member of (5.4), we get a quadruple series. Pairing the quadruple series into two double series with the indices (n,k) and (m,s), and using (2.1) with $p=1$ in the respective double series, we have

$$\begin{aligned} &\sum_{m,n=0}^{\infty}\sum_{k=0}^{n}\sum_{s=0}^{m} \frac{(v-y)^{k+s}\log^{k+s} c}{k!s!}\,{}_L\mathcal{F}_{m+n-k-s}^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu) \\ &\times \frac{t^n}{(n-k)!}\frac{u^m}{(m-s)!} = \sum_{m,n=0}^{\infty} {}_L\mathcal{F}_{m+n}^{(\alpha)}(x,v,z;a,b,c,d;\lambda;\mu,\nu)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned} \tag{5.6}$$

Finally, comparing the coefficients of $t^n\,u^m$ on both sides of (5.6), we obtain (5.1).

□

6 Remarks and special cases

The Laguerre-based generalized Apostol type polynomials ${}_L\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,c,d;\lambda;\mu,\nu)$ in (1.25) being very general, they can reduce to yield a number of relatively simple polynomials and numbers. For example,

$${}_L\mathcal{F}_n^{(\alpha)}(0,y,z;e,1,e,2;\lambda;\mu,\nu) = {}_HF_n^{(\alpha)}(y,z;\lambda;\mu,\nu);$$

$${}_L\mathcal{F}_n^{(0)}(x,y,z;e,1,e,2;\lambda;\mu,\nu) = {}_HL_n(x,y,z);$$

$${}_L\mathcal{F}_n^{(0)}(0,y,z;e,1,e,2;\lambda;\mu,\nu) = H_n(y,z);$$

$${}_L\mathcal{F}_n^{(0)}(0,y,0;e,1,e,2;\lambda;\mu,\nu) = F_n^{(\alpha)}(y;\lambda;\mu,\nu);$$

$${}_L\mathcal{F}_n^{(0)}(0,y,0;e,1,e,2;-\lambda;\mu,\nu) = (-1)^{\alpha}\,B_n^{(\alpha)}(y;\lambda);$$

$${}_L\mathcal{F}_n^{(0)}(0,y,0;e,1,e,2;\lambda;1,0) = E_n^{(\alpha)}(y;\lambda);$$

$$
\begin{aligned}
{}_L\mathcal{F}_n^{(0)}(0, y, 0; e, 1, e, 2; \lambda; 1, 1) &= G_n^{(\alpha)}(y; \lambda)\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; e, 1, e, 2; -1; 0, 1) &= (-1)^{\alpha}\, B_n^{(\alpha)}(y);\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; e, 1, e, 2; 1; 1, 0) &= E_n^{(\alpha)}(y);\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; e, 1, e, 2; 1; 1, 1) &= G_n^{(\alpha)}(y);\\
{}_L\mathcal{F}_n^{(0)}(x, y, 0; e, 1, e, 2; \lambda; \mu, \nu) &= L_n(x, y);\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; a, b, c, 2; -\lambda; 0, 1) &= (-1)^{\alpha}\, B_n^{(\alpha)}(y; \lambda; b, a, c);\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; b, a, c, 2; \lambda; 1, 0) &= E_n^{(\alpha)}(y; \lambda; a, b, c);\\
{}_L\mathcal{F}_n^{(\alpha)}(0, y, 0; b, a, c, 2; \lambda; 1, 1) &= G_n^{(\alpha)}(x; \lambda; a, b, c).
\end{aligned}
$$

Using these particular cases to the identities in the previous sections, we can get those corresponding identities involving various polynomials and numbers. For example, we give an implicit formula for the Hermite-based generalized Apostol type polynomials ${}_HF_n^{(\alpha)}(y, z; \lambda; \mu, \nu)$ in (1.23), asserted in the following corollary.

Corollary 6.1. *Let* $n,\ m \in \mathbb{N}_0$ *and* $\alpha,\ v,\ y,\ z,\ \lambda,\ \mu,\ \nu \in \mathbb{C}$. *Then*

$$
\begin{aligned}
&{}_HF_{m+n}^{(\alpha)}(v, z; \lambda; \mu, \nu)\\
&\quad = \sum_{k=0}^{n} \sum_{s=0}^{m} \binom{n}{k} \binom{m}{s} (v-y)^{s+k}\, {}_HF_{m+n-s-k}^{(\alpha)}(v, z; \lambda; \mu, \nu).
\end{aligned} \tag{6.1}
$$

Acknowledgments

The authors would like to express their deep-felt gratitude for the reviewer's useful comments to improve this paper as it stands.

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