

SOME MORE RESULTS ON R-NORM INFORMATION MEASURE

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Abstract. In the present communication, I have defined the new information measure called " α -R-norm information measure". It has been characterized using infimum operation in Section 2 and axiomatically in Section 3. Its properties have been studied in Section 4, joint and conditional α -R-norm information measure are studied in Section 5.

1. Introduction

Bockee and Lubbe [5] studied R-norm information measure of a discrete probability distribution

$$P = (p_1, p_2, \dots, p_n), \quad p_i \geq 0, \quad i = 1, 2, \dots, n$$

where

$$\sum_{i=1}^n p_i = 1$$

and

$$R^* = \{R : R > 0, R \neq 1\}$$

given by

$$H_R(P) = \frac{R}{R-1} \left[1 - \left(\sum_{i=1}^n p_i^R \right)^{\frac{1}{R}} \right], \quad (1.1)$$

The R-norm information measure (1.1) is a real function $\Delta_n \rightarrow R^+$, defined on Δ_n where $n \geq 2$ and R^+ is the set of positive real numbers. This measure is different from Shannon's entropy [9], Renyi [8] and Havrda and Charvat [7] and Daroczy [6].

The most interesting property of this measure is that when $R \rightarrow 1$, R-norm information measure (1.1) approaches to Shannon's entropy and in case $R \rightarrow \infty$, $H_R(P) \rightarrow (1 - \max p_i)$, $i = 1, 2, \dots, n$.

The measure (1.1) has been generalized by Hooda and Anant [2] as:

$$H_R^\beta(P) = \frac{R}{R+\beta-2} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{2-\beta}} \right)^{\frac{2-\beta}{R}} \right], \quad 0 < \beta \leq 1, \quad R(>0) \neq 1 \quad (1.2)$$

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(1.2) has been called as the generalized R-norm entropy of degree β which reduces to (1.1) when $\beta = 1$. In case $R = 1$, (1.2) reduces to

$$H_1^\beta(P) = \frac{1}{\beta-1} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{1}{2-\beta}} \right)^{2-\beta} \right], \quad 0 < \beta \leq 1. \quad (1.3)$$

Setting $r = \frac{1}{2-\beta}$ in (1.3), we get

$$H^r(P) = \frac{r}{r-1} \left[1 - \left(\sum_{i=1}^n p_i^r \right)^{\frac{1}{r}} \right], \quad \frac{1}{2} < r \leq 1. \quad (1.4)$$

which is a measure mentioned by Arimoto [3] as an example of a generalized class of information measure. It may be marked that (1.4) also approaches to Shannon's entropy as $r \rightarrow 1$.

Hooda-Anant [2] studied (1.2) as a generalization of Shannon's and Bockee-Lubbe [5] using infimum operation.

In the present communication, our main endeavour is to generalize (1.1) as α -R-norm information measure as:

$$H_R^\alpha(P) = \frac{R}{R-\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} \right], \quad 0 < \alpha \leq 1, R(>0) \neq 1. \quad (1.5)$$

Naturally (1.5) reduces to (1.1).

Measure (1.5) is a non-additive measure which by applying infimum operation is characterized in Section (2) and axiomatically through functional equation in Section (3). Algebraic and analytical properties are studied in Section (4), Section (5) deals with for joint and conditional generalized α -R-norm information measure for joint and conditional distribution.

2. Characterization

Applying Infimum Operation

In this section, we consider the α -R-norm information measure (1.5) as weighted arithmetic mean representation of elementary R-norm entropies of α of occurrences of various single outcomes.

Theorem 1. *Let*

$$f_R^\alpha(*P_i) = \frac{R}{R-\alpha} \left[1 - *p_i^{\frac{R-\alpha}{R}} \right], \quad R > 0, R \neq 1, 0 < \alpha \leq 1, \quad (2.1)$$

$$\text{then} \quad H_R^\alpha(P) = \inf_{*p} \sum_{i=1}^n p_i f_R^\alpha(*p_i) \quad (2.2)$$

where the operation infimum is taken over the probability distribution

$$(*p_1, *p_2, \dots, *p_n) \in \Delta_n.$$

Proof. Let us consider

$$\sum_{i=1}^n p_i f_R^\alpha(*p_i) = \frac{R}{R-\alpha} \left[\sum_{i=1}^n p_i \left(1 - *p_i^{\frac{R-\alpha}{R}} \right) \right]. \quad (2.3)$$

We minimize (2.3) subject to the natural constraint

$$\sum_{i=1}^n *p_i = 1. \quad (2.4)$$

For this we consider Lagrangian

$$L = \frac{R}{R-\alpha} \left[1 - \sum_{i=1}^n p_i *p_i^{\frac{R-\alpha}{R}} \right] + \lambda \left[\sum_{i=1}^n *p_i - 1 \right]. \quad (2.5)$$

Differentiating (2.5) w.r.t. $*p_i$, we have

$$\frac{\partial L}{\partial *p_i} = -p_i *p_i^{\frac{\alpha}{R}} + \lambda. \quad (2.6)$$

For extreme value, we put (2.6) equal to zero which gives

$$*p_i = \frac{\lambda^{-\frac{R}{\alpha}}}{p_i^{\frac{R}{\alpha}}}. \quad (2.7)$$

We mark here that

$$\frac{\partial^2 L}{\partial *p_i^2} > 0,$$

when $*p_i = \frac{\lambda^{-\frac{R}{\alpha}}}{p_i^{\frac{R}{\alpha}}}, \quad \frac{\partial^2 L}{\partial *p_i^2} > 0.$

Hence the value of $*p_i$ given by (2.7) is minimum and using (2.4) in (2.7), we can find the value of λ and consequently, we have

$$*p_i = \frac{p_i^{\frac{R}{\alpha}}}{\sum_{i=1}^n p_i^{\frac{R}{\alpha}}}, \quad 0 < \alpha \leq 1, R(>0) \neq 1. \quad (2.8)$$

Now we consider RHS of (2.2)

$$\begin{aligned} \inf_{*p_i} \sum_{i=1}^n p_i f_R^\alpha(*p_i) &= \frac{R}{R-\alpha} \left[1 - \sum_{i=1}^n p_i *p_i^{\frac{R-\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} \left[1 - \frac{\sum_{i=1}^n p_i^{\frac{R}{\alpha}}}{\left(\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right)^{\frac{R-\alpha}{R}}} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{R}{R-\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} \right] \\
&= H_R^\alpha(P).
\end{aligned}$$

Further without loss of generality, we may assume that corresponding to the observed probability distribution $P \in \Delta_n$, there is a prior probability distribution $Q \in \Delta_n$ and replacing $f_R^\alpha(* p_i)$ by $f_R^\alpha(q_i)$ in (2.2), we have

$$H_R^\alpha(P) = \inf_{q_i} \sum_{i=1}^n p_i f_R^\alpha(q_i). \quad (2.9)$$

In case we do not apply the operation of infimum to (2.9), then it depends on two probability distributions P and Q . For $R = 1$, $f_R^\alpha(q)$ is an analogue of $\frac{1}{1-\alpha}(1-q^{1-\alpha})$ which reduces to $\log 1/q$ when $\alpha \rightarrow 1$. Thus (2.9) is equal to

$$\frac{1}{\alpha-1} \left(\sum_{i=1}^n p_i q_i^{1-\alpha} - 1 \right) \quad (2.10)$$

which is a generalized inaccuracy measure of degree α .

(2.10) reduces to inaccuracy measure of degree α characterized by Sharma-Taneja [10]. Therefore, there is no harm to represent (2.10) through $f_R^\alpha(q)$.

$$\begin{aligned}
H_R^\alpha(P; Q) &= \sum_{i=1}^n p_i f_R^\alpha(q_i) \\
&= \frac{R}{R-\alpha} \left[\sum_{i=1}^n p_i \left(1 - q_i^{\frac{R-\alpha}{R}} \right) \right], \quad R(>0) \neq 1, 0 < \alpha \leq 1.
\end{aligned} \quad (2.11)$$

In fact (2.11) can also be described as the average of elementary α - R -norm inaccuracies $f_R^\alpha(q_i)$, $i = 1, 2, \dots, n$ and so called α - R -normed inaccuracy measure of degree α . Thus it seems plausible that (2.11) may be characterized and then by taking its infimum we can arrive at (1.5).

In the following theorem, we characterize the elementary information function $f_R^\alpha(q)$ by assuming only two axioms and applying infimum operation.

Theorem 2. *Let f be a real valued continuous self-information function defined on $(0, 1]$ satisfying the following axioms*

Axiom A₁. $f(x, y) = f(x) + f(y) - \frac{R-\alpha}{R} f(x)f(y)$.

Axiom A₂. $f\left(\frac{1}{n}\right) = \frac{R}{R-\alpha} \left(1 - n^{\frac{R-\alpha}{R}} \right)$, $R(>0) \neq 1$, $0 < \alpha \leq 1$

and $n = 2, 3, \dots$ is maximality constant.

Then $f_R^\alpha(q)$ defined in (2.9) holds.

Proof. Considering $f(x) = \frac{R}{R-\alpha} (1 - \phi(x))$ in axiom A₁, we get

$$\frac{R}{R-\alpha} [1 - \phi(xy)] = \frac{R}{R-\alpha} [1 - \phi(x)] + \frac{R}{R-\alpha} [1 - \phi(y)]$$

$$-\frac{R}{R-\alpha}[(1-\phi(x))(1-\phi(y))]$$

or,

$$\phi(xy) = \phi(x)\phi(y). \quad (2.12)$$

But the relation (2.12) is a well known Cauchy's functional equation (refer Aczel [1]). The continuous solution of (2.12) is given by $\phi(x) = x^a$, where $a \neq 0$ is an arbitrary constant.

On using axiom A_2 , we get $a = \frac{R-\alpha}{R}$ and hence

$$f(x) = \frac{R}{R-\alpha} \left[1 - x^{\frac{R-\alpha}{R}} \right]$$

which is exactly of the form of (1.1).

Next, the measure (1.5) can be easily obtained by applying infimum operation on the equation (2.11) on the lines of the Theorem I.

Remarks. For an incomplete probability distribution scheme

$$P = (p_1, p_2, \dots, p_n), \quad p_i \geq 0, \quad \sum_{i=1}^n p_i = 1, \quad f_R^\alpha(q_i), \quad i = 1, 2, \dots, n,$$

associated with individual events may be worked out. Then as in case (2.10), we may define

$$H_R^\alpha(p_1, p_2, \dots, p_n; q_1, q_2, \dots, q_n) = \frac{\sum_{i=1}^n p_i f_R^\alpha(q_i)}{\sum_{i=1}^n p_i}, \quad (2.13)$$

By using infimum operation with respect to q_i 's the equation (2.13) gives

$$H_R^\alpha(P) = \frac{R}{R-\alpha} \left[1 - \left\{ \frac{\sum_{i=1}^n p_i^{\frac{R}{\alpha}}}{\sum_{i=1}^n p_i} \right\}^{\frac{\alpha}{R}} \right] \quad (2.14)$$

which is the α -R-norm entropy of degree α of incomplete probability distribution.

It is also worth mentioning that if we take arithmetic average with weights as continuous function $w()$, then we get the general expression

$$H_R^\alpha(P; Q) = \frac{\sum_{i=1}^n W(p_i) f_R^\alpha(q_i)}{\sum_{i=1}^n W(p_i)}. \quad (2.15)$$

By considering different weight $w()$ satisfying the condition

$$W(pq) = W(p)W(q), \quad \text{where } W() \neq 0,$$

we can obtain various generalized α -R-normed information measures.

3. Axiomatic characterization of α -R-norm information measure

In this section, we attempt to characterize (2.11) by considering it as inaccuracy measure for two distributors P and $Q \in \Delta_n$ and arrive at (1.5) by applying infimum operation with respect to q_i 's, $i = 1, 2, \dots, n$.

Let $S_n = \Delta_n \times \Delta_n \rightarrow R$, $n = 2, 3, \dots$ and G_n be a sequence of functions of p_i 's and q_i 's, $i = 1, 2, \dots, n$ over S_n satisfying the following axioms.

Axiom 1. $G_n(P, Q) = a_1 + a_2 \sum_{i=1}^n h(p_i q_i)$, where a_1 and a_2 are non-zero constants and

$$p_i q_i \in j = (0, 1) \times (0, 1) \cup \{(0, y) : 0 < y \leq 1\} \cup \{(1, y') : 0 < y' \leq 1\}.$$

Axiom 2. For $P, Q \in \Delta_n$ and $P' Q' \in \Delta_m$, G_n satisfies the following property

$$G_{nm}(PP', QQ') = G_n(P, Q) + G_m(P', Q') - \frac{1}{a_1} G_n(P, Q) G_m(P', Q').$$

Axiom 3. $h(p, q)$ is a continuous function of its arguments p and q .

Axiom 4. Let all p_i 's and q_i 's are equiprobable posterior and prior probabilities of events, respectively then

$$G_n\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}; \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) = \frac{R}{R-\alpha} \left[1 - n^{\frac{\alpha-R}{R}}\right],$$

where $n = 2, 3, \dots$, $R(>0) \neq 1$ and $0 < \alpha \leq 1$.

Theorem 3. The inaccuracy measure (2.11) is uniquely determined by the Axioms 1 to 4.

First of all, we prove the following three lemmas to facilitate the proof of the theorem:

Lemma 1. By Axioms 1–2, we have

$$\sum_{i=1}^n \sum_{j=1}^m h(p_i p'_j; q_i q'_j) = \left(-\frac{a_2}{a_1}\right) \sum_{i=1}^n h(p_i, q_i) \sum_{j=1}^m h(p'_j; q'_j) \quad (3.1)$$

where $(p_i, q_i), (p'_j, q'_j) \in j$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Lemma 2. The continuous solution that satisfies (3.1) is the continuous solution of the functional equation

$$h(pp', qq') = \left(-\frac{a_2}{a_1}\right) h(p, q) h(p', q'). \quad (3.2)$$

Proof. Let a, b, c, d and a', b', c', d' be positive integers such that $1 \leq a' \leq a$, $1 \leq b' \leq b$, $1 \leq c' \leq c$ and $1 \leq d' \leq d$.

Setting

$$n = a - a' + 1 = b - b' + 1$$

and

$$\begin{aligned} m &= c - c' + 1 = d - d' + 1 \\ p_i &= \frac{1}{a} \quad (i = 1, 2, \dots, a - a'), \quad p_{a-a'+1} = \frac{a'}{a}, \\ q_i &= \frac{1}{b} \quad (i = 1, 2, \dots, b - b'), \quad q_{b-b'+1} = \frac{b'}{b}, \\ p'_j &= \frac{1}{c} \quad (j = 1, 2, \dots, c - c'), \quad p_{c-c'+1} = \frac{c'}{c}, \\ q'_j &= \frac{1}{d} \quad (j = 1, 2, \dots, d - d'), \quad q_{d-d'+1} = \frac{d'}{d}, \end{aligned}$$

in equation (3.1), we have

$$\begin{aligned} &(a - a')(c - c')h\left(\frac{1}{ac}, \frac{1}{bd}\right) + (c - c')h\left(\frac{a'}{ac}, \frac{b'}{bd}\right) + (a - a')h\left(\frac{c'}{ac}, \frac{d'}{bd}\right) + h\left(\frac{a'c'}{ac}, \frac{b'd'}{bd}\right) \\ &= \left(-\frac{a_2}{a_1}\right) \left[(a - a')h\left(\frac{1}{a}, \frac{1}{b}\right) + h\left(\frac{a'}{a}, \frac{b'}{b}\right) \right] \times \left[(c - c')h\left(\frac{1}{c}, \frac{1}{d}\right) + h\left(\frac{c'}{c}, \frac{d'}{d}\right) \right] \end{aligned} \quad (3.3)$$

taking $a' = b' = c' = d' = 1$ in (3.3), we get

$$h\left(\frac{1}{ac}, \frac{1}{ba}\right) = \left(-\frac{a_2}{a_1}\right) h\left(\frac{1}{a}, \frac{1}{b}\right) h\left(\frac{1}{c}, \frac{1}{d}\right). \quad (3.4)$$

Taking $a' = b' = 1$ in (3.3) and using (3.4), we have

$$h\left(\frac{c'}{ac}, \frac{d'}{bd}\right) = \left(-\frac{a_2}{a_1}\right) h\left(\frac{1}{a}, \frac{1}{b}\right) h\left(\frac{c'}{c}, \frac{d'}{d}\right). \quad (3.5)$$

Again taking $c' = d' = 1$ in (3.3), we have

$$h\left(\frac{a'}{ac}, \frac{b'}{bd}\right) = \left(-\frac{a_2}{a_1}\right) h\left(\frac{1}{c}, \frac{1}{d}\right) h\left(\frac{a'}{a}, \frac{b'}{b}\right). \quad (3.6)$$

Now (3.3) together with (3.4), (3.5) and (3.6) reduces to

$$h\left(\frac{a'b'}{ab}, \frac{c'd'}{bd}\right) = \left(-\frac{a_2}{a_1}\right) h\left(\frac{a'}{a}, \frac{b'}{b}\right) h\left(\frac{c'}{c}, \frac{d'}{d}\right). \quad (3.7)$$

Setting $\frac{a'}{a} = p$, $\frac{b'}{b} = q$, $\frac{c'}{c} = p'$, $\frac{d'}{d} = q'$ in (3.7), we get the required result (3.2) for rational numbers which by continuing of h holds for all real $p, q, p', q' \in J$.

In the next Lemma, we get the most general solution of (3.2).

Lemma 3. *The most general continuous solutions of equation (3.2) are given by*

$$h(p, q) = \left(-\frac{a_1}{a_2}\right) p^u q^v, \quad u \neq 0, \quad v \neq 0. \quad (3.8)$$

and

$$h(p, q) = 0. \quad (3.9)$$

Proof. Taking $g(p, q) = \left(-\frac{a_2}{a_1}\right)h(p, q)$ in (3.2), we have

$$g(pp', qq') = g(p, q)g(p', q'). \quad (3.10)$$

The most general continuous solution of (3.10) c.f. Aczel [1] is given by

$$g(p, q) = p^u q^v, \quad u \neq 0, \quad v \neq 0. \quad (3.11)$$

and

$$g(p, q) = 0. \quad (3.12)$$

On substituting $g(p, q) = \left(-\frac{a_1}{a_1}\right)h(p, q)$ in (3.11) and (3.12), we get (3.8) and (3.9) respectively. This proves the Lemma 3.

Proof of theorem. Substituting the solution (3.8) in Axiom 1, we have

$$G_n(P, Q) = a_1 \left(1 - \sum_{i=1}^n p_i^u q_i^v\right), \quad u, v \neq 0. \quad (3.13)$$

Using Axiom 4 in (3.13), we get

$$a_1 = \frac{R}{R-\alpha}, \quad u = 1 \quad \text{and} \quad v = \frac{R-\alpha}{R}.$$

Substituting these values in (3.13), we have

$$G_n(P, Q) = H_R^\alpha(P, Q)$$

and hence this completes the proof of the Theorem 3.

Remark. In this equation (3.8) if $u = 0$ and $v = 0$ then $h(p, q) = \left(-\frac{a_1}{a_2}\right)$ which is a trivial solution and is of no interest. The solution (3.9) does not even contain any variable and hence it is again discarded.

4. Properties of $H_R^\alpha(P)$

This section presents the algebraic and analytical properties of the α -R-norm information measure $H_R^\alpha(P)$ and satisfies the following properties.

1. $H_R^\alpha(P) = H_R^\alpha(p_1, p_2, \dots, p_n)$ is a symmetric function of (p_1, p_2, \dots, p_n) .
2. $H_R^\alpha(P)$ is expansible i.e. $H_R^\alpha(p_1, p_2, \dots, p_n, 0) = H_R^\alpha(p_1, p_2, \dots, p_n)$.
3. $H_R^\alpha(P)$ is decisive i.e. $H_R^\alpha(1, 0) = H_R^\alpha(0, 1) = 0$.

4. $H_R^\alpha(P)$ is non-recursive.
5. $H_R^\alpha(P, Q) = H_R^\alpha(P) + H_R^\alpha(Q) - \frac{R-\alpha}{R} H_R^\alpha(P) H_R^\alpha(Q)$ i.e. $H_R^\alpha(P)$ is non-additive.

Proof. Properties 1 to 3 can be verified easily, now we consider 4.

$$H_R^\alpha\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) = \frac{R}{R-\alpha} \left[1 - \left\{ \frac{p_1^{\frac{R}{\alpha}} + p_2^{\frac{R}{\alpha}}}{(p_1+p_2)^{\frac{R}{\alpha}}} \right\}^{\frac{\alpha}{R}} \right] \quad (4.1)$$

and

$$H_R^\alpha(p_1+p_2, p_3, \dots, p_n) = \frac{R}{R-\alpha} \left[1 - \left\{ (p_1+p_2)^{\frac{R}{\alpha}} + p_3^{\frac{R}{\alpha}} + \dots + p_n^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right]. \quad (4.2)$$

By combining (4.1) and (4.2), we have

$$H_R^\alpha(p_1+p_2, p_3, \dots, p_n) + (p_1+p_2) H_R^\alpha\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) \neq H_R^\alpha(p_1, p_2, \dots, p_n).$$

Thus $H_R^\alpha(p_1, p_2, \dots, p_n)$ is non-recursive.

Property 5. Let A_1, A_2, \dots, A_n and B_1, B_2, \dots, B_m be the two sets of events associated with probability distributions $P \in \Delta_n$ and $Q \in \Delta_m$. We denote the probability of the joint occurrence of events

$$A_i = (i = 1, 2, \dots, n) \text{ and } B_j = (j = 1, 2, \dots, m) \text{ on } p(A_i \cap B_j).$$

Then the α -R-norm entropy is given by

$$H_R^\alpha(P * Q) = \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m p_i^{\frac{R}{\alpha}}(A_i \cap B_j) \right\}^{\frac{\alpha}{R}} \right].$$

Since the events considered here are stochastically independent therefore, we have

$$\begin{aligned} H_R^\alpha(P * Q) &= \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^n p_i^{\frac{R}{\alpha}}(A_i) \right\}^{\frac{\alpha}{R}} \left\{ \sum_{j=1}^m p_j^{\frac{R}{\alpha}}(B_j) \right\}^{\frac{\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \left\{ \sum_{j=1}^m p_j^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} - \frac{R}{R-\alpha} \left[\left(1 - \frac{R-\alpha}{R} H_R^\alpha(P) \right) \left(1 - \frac{R-\alpha}{R} H_R^\alpha(Q) \right) \right] \\ &= H_R^\alpha(P) + H_R^\alpha(Q) - \frac{R-\alpha}{R} H_R^\alpha(P) H_R^\alpha(Q) \end{aligned} \quad (4.3)$$

Corollary. The Property 5, can also be extended for m stochastically independent distributors P_0, P_1, \dots, P_{m-1} having n_0, n_1, \dots, n_{m-1} elements respectively. Let

$$H_R^\alpha(P_0) = \dots = H_R^\alpha(P_{m-1}) = w.$$

Then

$$H_R^\alpha(P^*) = \frac{R}{R-\alpha} \left[1 - \left\{ 1 - \frac{R-\alpha}{R} w \right\}^m \right] \quad (4.4)$$

where P^* stands for the product space of m stochastically independent distributor P_0, P_1, \dots, P_{m-1} .

Proof. In (4.3), we have proved the results for product of two probability distributors on similar lines, it is very easy to show that

$$\begin{aligned} H_R^\alpha(P_0 P_1 P_2) &= H_R^\alpha(P_0) + H_R^\alpha(P_1) + H_R^\alpha(P_2) \\ &\quad - \frac{R-\alpha}{R} \left[H_R^\alpha(P_0) H_R^\alpha(P_1) + H_R^\alpha(P_0) H_R^\alpha(P_2) + H_R^\alpha(P_1) H_R^\alpha(P_2) \right] \\ &\quad + \left(\frac{R-\alpha}{R} \right) H_R^\alpha(P_0) H_R^\alpha(P_1) H_R^\alpha(P_2). \end{aligned}$$

Further, by mathematical Induction, we arrive at

$$\begin{aligned} H_R^\alpha(P^*) &= H_R^\alpha(P_0) + H_R^\alpha(P_1) + \dots + H_R^\alpha(P_{m-1}) \\ &\quad - \frac{R-\alpha}{R} \left[H_R^\alpha(P_0) H_R^\alpha(P_1) + H_R^\alpha(P_0) H_R^\alpha(P_2) + \dots \right] \\ &\quad + \frac{R-\alpha}{R} \left[H_R^\alpha(P_0) H_R^\alpha(P_1) H_R^\alpha(P_2) + \dots \right] \\ &\quad + (-1)^{m-1} \left(\frac{R-\alpha}{R} \right)^{m-1} H_R^\alpha(P_0) H_R^\alpha(P_1) \dots H_R^\alpha(P_{m-1}). \end{aligned} \quad (4.5)$$

Setting $H_R^\alpha(P_0) = \dots = H_R^\alpha(P_{m-1}) = w$, we have

$$\begin{aligned} H_R^\alpha(P^*) &= mw - \left(\frac{R-\alpha}{R} \right) \binom{m}{2} w^2 + \left(\frac{R-\alpha}{R} \right)^2 \binom{m}{3} w^3 \\ &\quad + (-1)^{m-1} \left(\frac{R-\alpha}{R} \right)^{m-1} \binom{m}{m} w^m \\ &= (-1) \left[- \binom{m}{1} w + \binom{m}{2} \left(\frac{R-\alpha}{R} \right) w^2 - \binom{m}{3} \left(\frac{R-\alpha}{R} \right)^2 w^3 \right. \\ &\quad \left. + \dots + (-1)^m \left(\frac{R-\alpha}{R} \right)^{m-1} \binom{m}{m} w^m \right] \\ &= - \frac{R}{R-\alpha} \left[1 - \binom{m}{1} \frac{R-\alpha}{R} w + \binom{m}{2} \left(\frac{R-\alpha}{R} \right)^2 w^2 \right. \\ &\quad \left. - \binom{m}{3} \left(\frac{R-\alpha}{R} \right)^3 w^3 + \dots + (-1)^m \binom{m}{m} \left(\frac{R-\alpha}{R} \right)^m w^m - 1 \right] \\ &= - \frac{R}{R-\alpha} \left[\left(1 - \frac{R-\alpha}{R} w \right)^m - 1 \right] \\ &= \frac{R}{R-\alpha} \left[1 - \left(1 - \frac{R-\alpha}{R} w \right)^m \right], \quad \text{which is (4.4).} \end{aligned}$$

This completes the proof of the Property 5.

Theorem 4. Let $H_R^\alpha(P) = H_R^\alpha(p_1, p_2, \dots, p_n)$ be the generalized α -R-norm information measure. Then for $P \in \Delta_n$ and $R \in R^+$ and $0 < \alpha \leq 1$, we have

- (a) $H_R^\alpha(P)$ is Non-negative.
- (b) $H_R^\alpha(P) \geq H_R^\alpha(1, 0, 0, \dots, 0) = 0$.
- (c) $H_R^\alpha(P) \leq H_R^\alpha\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) = \frac{R}{R-\alpha} \left[1 - n^{-\frac{\alpha-R}{R}}\right]$.
- (d) $H_R^\alpha(P)$ is a monotonic function of P .
- (e) $H_R^\alpha(P)$ is continuous at $R \in R^+$.
- (f) $H_R^\alpha(P)$ is stable in p_i , $i = 1, 2, \dots, n$.
- (g) $H_R^\alpha(P)$ is small for small probabilities.
- (h) $H_R^\alpha(P)$ is a concave function for all p_i .
- (i) $\lim_{R \rightarrow \infty} H_R^\alpha(P) = 1 - \max_i p_i$.

Proof. To prove that $H_R^\alpha(P)$ is non-negative, we consider the following cases:

Case I. When $R > \alpha$ or $\frac{R}{\alpha} > 1$ then $p_i^{\frac{R}{\alpha}} \leq p_i \forall i$

$$\begin{aligned} &\Rightarrow \sum_{i=1}^n p_i^{\frac{R}{\alpha}} \leq \sum_{i=1}^n p_i = 1, \\ &\Rightarrow \left[\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \leq 1 \end{aligned} \quad (4.6)$$

Case II. When $0 < R < \alpha$ or $0 < \frac{R}{\alpha} < 1$ then on the same lines, we have

$$\left[\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \geq 1. \quad (4.7)$$

We know that

$$\frac{R}{R-\alpha} > 0 \text{ if } R > \alpha \text{ and } \frac{R}{R-\alpha} < 0 \text{ if } R < \alpha.$$

Hence from (4.6) and (4.7), we conclude that $H_R^\alpha(P) \geq 0$.

Property (b) follows if one of the probabilities is equal to 1 and others are equal to zero. Property (c) is Axiom 2.

Also it is noted that the generalized α -R-norm information measure is maximal if all probabilities are equal and is minimum if one probability is unit and others are zero.

(d) $H_R^\alpha(P)$ is monotonic iff $H_R^\alpha(p, 1-p)$ is non-decreasing on $p \in \left[0, \frac{1}{2}\right]$.

From (1.5), we have

$$H_R^\alpha(p, 1-p) = \frac{R}{R-\alpha} \left[1 - \left\{ (1-p)^{\frac{R}{\alpha}} + p^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right]. \quad (4.8)$$

Let us define the function $G(p)$ by

$$G(p) = 1 - \left[(1-p)^{\frac{R}{\alpha}} + p^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}.$$

Then

$$\frac{dG(p)}{dp} \geq 0 \text{ for } R > \alpha$$

and

$$\frac{dG(p)}{dp} \geq 0 \text{ for } 0 < R < \alpha.$$

From (4.8), we note that

$$\begin{aligned} \frac{d}{dp} H_R^\alpha(p, 1-p) &= \left(\frac{R}{R-\alpha} \right) \frac{dG(p)}{dp} \text{ which gives} \\ \frac{d}{dp} H_R^\alpha(p, 1-p) &\geq 0 \text{ for } R \in R^+, p \in [0, 1], 0 < \alpha \leq 1. \end{aligned}$$

Thus $H_R^\alpha(p, 1-p)$ is a non-decreasing function and hence monotonic.

(e) We know that $\left[\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}$ is continuous for $R \in [0, \infty)$ and $0 < \alpha \leq 1$.

Hence, $H_R^\alpha(P) = \frac{R}{R-\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} \right]$ is also continuous at

$$R \in R^+ \text{ where } 0 < \alpha \leq 1.$$

(f) It is obvious

$$H_R^\alpha(P, 0) = H_R^\alpha(P). \quad (4.9)$$

Also it is easy to see that

$$Lt_{p \rightarrow 1^+} H_R^\alpha(P; 1-p) = H_R^\alpha(P, 0). \quad (4.10)$$

(g) From (4.8) it follows that

$$Lt_{p \rightarrow 1^+} H_R^\alpha(P, 1-p) = Lt_{p \rightarrow 1^+} \frac{R}{R-\alpha} \left[1 - \left\{ p^{\frac{R}{\alpha}} + (1-p)^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right] = 0.$$

This proves that $H_R^\alpha(p)$ is small for small probabilities.

(h) Let us define the concave function.

Definition. A function f over a set S is said to be concave if for all choices of $x_1, x_2, \dots, x_m \in S$ and for all scalars $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$, the following holds

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \geq \sum_{i=1}^m \lambda_i f(x_i). \quad (4.11)$$

Here we consider random variable x taking its values in the set $S = (x_1, x_2, \dots, x_m)$ and r probability distributions over S on follows:

$$P_k(x) = \{p_k(x_1), \dots, p_k(x_m)\} : p_k(x_i) \geq 0, \sum_{i=1}^m p_k(x_i) = 1, \quad k = 1, 2, \dots, r.$$

Let us define another probability distribution over S

$$P_0(x) = \{p_0(x_1), \dots, p_0(x_m)\} \exists \forall i' s.$$

$P_0(x_i) = \sum_{k=1}^r \lambda_k p_k(x_i)$, where λ_k 's are non-negative scalars stisfying $\sum_{k=1}^r \lambda_k = 1$ then we have

$$D = \sum_{k=1}^r \lambda_k H_R^\alpha(P_k) - H_R^\alpha(P_0), \quad R(> 0) \neq 1, \quad 0 < \alpha \leq 1.$$

$H_R^\alpha(P)$ will be concave if D is less than zero for $R(> 0) \neq 1$ and $0 < \alpha \leq 1$, so we consider

$$\begin{aligned} D &= \sum_{k=1}^r \lambda_k H_R^\alpha(P_k) - H_R^\alpha(P_0) \\ &= \sum_{k=1}^r \lambda_k \left[1 - \left\{ \sum_{i=1}^m p_k^{\frac{R}{\alpha}}(x_i) \right\}^{\frac{\alpha}{R}} \right] \frac{R}{R-\alpha} - \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^m p_0^{\frac{R}{\alpha}}(x_i) \right\}^{\frac{\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} \left[\left[\sum_{i=1}^m \left(\sum_{k=1}^r \lambda_k p_k(x_i) \right)^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} - \left[\sum_{k=1}^r \lambda_k \left(\sum_{i=1}^m p_k^{\frac{R}{\alpha}}(x_i) \right)^{\frac{\alpha}{R}} \right] \right]. \end{aligned} \quad (4.12)$$

Now using the inequality $\left[\sum_{k=1}^r a_k x_k \right]^t \leq \sum_{k=1}^r a_k x_k^t$ according as $t \leq 1$, we have

$$\left[\sum_{k=1}^r \lambda_k p_k(x_i) \right]^{\frac{R}{\alpha}} \leq \left[\sum_{k=1}^r \lambda_k p_k^{\frac{R}{\alpha}}(x_i) \right] \text{ according as } \frac{R}{\alpha} \leq 1.$$

Therefore

$$\begin{aligned} \left[\sum_{i=1}^m \left(\sum_{k=1}^r \lambda_k p_k(x_i) \right)^{\frac{R}{\alpha}} \right] &\leq \left(\sum_{i=1}^m \sum_{k=1}^r \lambda_k p_k^{\frac{R}{\alpha}}(x_i) \right) \text{ according as } \frac{R}{\alpha} \leq 1. \\ D_1 &= \left[\sum_{i=1}^m \left(\sum_{k=1}^r \lambda_k p_k(x_i) \right)^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \leq \left[\sum_{k=1}^r \lambda_k \left(\sum_{i=1}^m p_k^{\frac{R}{\alpha}}(x_i) \right)^{\frac{\alpha}{R}} \right] \text{ according as } \frac{R}{\alpha} \leq 1. \end{aligned} \quad (4.13)$$

Moreover,

$$\left[\sum_{k=1}^r \lambda_k \left(\sum_{i=1}^m p_k^{\frac{R}{\alpha}}(x_i) \right) \right]^{\frac{\alpha}{R}} \leq \left[\sum_{k=1}^r \lambda_k \left(\sum_{i=1}^m p_k^{\frac{R}{\alpha}}(x_i) \right)^{\frac{\alpha}{R}} \right] = D_2 \text{ according as } \frac{R}{\alpha} \leq 1. \quad (4.14)$$

Thus $D_2 \leq D_1$ according as $\frac{R}{\alpha} \leq 1$, which implies that $D < 0$ in view of the sign of $\frac{R}{R-\alpha}$ according as $\frac{R}{\alpha} \leq 1$.

This proved that $H_R^\alpha(P)$ is concave function P .

(i) We set $\max p_i = p_k$, assuming $m_0 = 1, 2, \dots, R > \alpha$ and $0 < \alpha \leq 1$, we find

$$\left[\sum_{i=1}^m p_i^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \leq \left[m_0 p_k^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} = m_0^{\frac{\alpha}{R}} p_k. \quad (4.15)$$

It is also noted that for $R > \alpha$.

$$\left[\sum_{i=1}^m p_i^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \geq p_k. \quad (4.16)$$

Combing (4.15) and (4.16), we get

$$p_k \leq \left\{ \sum_{i=1}^m p_i^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \leq m_0^{\frac{\alpha}{R}} p_k. \quad (4.17)$$

Taking limits for $R \rightarrow \infty$ in (4.17), we have

$$\lim_{R \rightarrow \infty} \left\{ \sum_{i=1}^{m_0} p_i^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} = p_k = \max_i p_i$$

and finally

$$\lim_{R \rightarrow \infty} H_R^\alpha(P) = \lim_{R \rightarrow \infty} \left[1 - \left\{ \sum_{i=1}^{m_0} p_i^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right] = 1 - \max_i p_i.$$

This completes the proof of Theorem 4.

5. Joint and conditional generalized α -R-nom information measure

The present section depicts the joint and conditional probability distribution of two random variables ξ and η having probability distributors P and Q over the sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ respectively. The generalized α -R-nom information of the random variables $H_R^\alpha(\xi) = H_R^\alpha(P)$ and $H_R^\alpha(\eta) = H_R^\alpha(Q)$, where

$$p_i = P_r(\xi_i = x_i), \quad i = 1, 2, \dots, n$$

$$p_j = P_r(\eta_j = y_j), \quad j = 1, 2, \dots, m$$

are the probabilities of the possible values of the random variables. Similarly, we consider a two-dimensional discrete random variable (ξ, η) with joint probability distribution $\pi = (\pi_{11}, \pi_{12}, \dots, \pi_{1n})$

$$\pi_{ij} = P_r(\xi_i = x_i, \eta_j = y_j), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m$$

is the joint probability for the values (x_i, y_i) of (ξ, η) .

We shall denote conditional probabilities by p_{ij} and q_{ij} such that $\pi_{ij} = p_{ij}q_j = q_{ji}p_i$

$$p_i = \sum_{j=1}^m \pi_{ij} \quad \text{and} \quad q_i = \sum_{j=1}^n \pi_{ij}.$$

Definition. The joint α -R-norm information measure for $R \in R^+$ and $0 < \alpha \leq 1$ is given by

$$H_R^\alpha(\xi, \eta) = \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^n \sum_{j=1}^m \pi_{ij} \right\}^{\frac{\alpha}{R}} \right]. \quad (5.1)$$

It may be seen that $H_R^\alpha(\xi, \eta)$ is symmetric in ξ and η . Due to non-additivity property if ξ and η are stochastically independent. Then the following holds

$$H_R^\alpha(\xi, \eta) = H_R^\alpha(\xi) + H_R^\alpha(\eta) - \frac{R-\alpha}{R} H_R^\alpha(\xi) H_R^\alpha(\eta). \quad (5.2)$$

Definition. The average conditional α -R-norm information of η given ξ for $R \in R^+$ and $0 < \alpha \leq 1$ is defined as

$$*H_R^\alpha(\eta/\xi) = \frac{R}{R-\alpha} \left[1 - \sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right]. \quad (5.3)$$

Or alternatively

$$**H_R^\alpha(\eta/\xi) = \frac{R}{R-\alpha} \left[1 - \left\{ \sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right]. \quad (5.4)$$

The two conditional measures (5.3) and (5.4) differ by the way, the probability p_i have been taken. The expression (5.3) is a true mathematical expression over ξ , whereas the expression (5.4) is not.

The next theorem, proves three results for conditional α -R-norm information measures given by (5.3) and (5.4).

Theorem 5. If ξ and η are discrete random variables then for

$$R \in R^+ \quad \text{and} \quad 0 < \alpha \leq 1,$$

then the following results hold.

$$(i) \quad *H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta), \quad (5.5)$$

$$(ii) \quad **H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta), \quad (5.6)$$

$$(iii) \quad **H_R^\alpha(\eta/\xi) \leq *H_R^\alpha(\eta/\xi), \quad (5.7)$$

$$(iv) \quad **H_R^\alpha(\eta/\xi) \leq *H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta). \quad (5.8)$$

Proof. We know by [4] that for $\frac{R}{\alpha} > 1$.

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n x_{ij} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \leq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m x_{ij}^{\frac{R}{\alpha}} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}. \quad (5.9)$$

Setting $x_{ij} = \pi_{ij} \geq 0$ in (5.9), we have

$$\left[\sum_{j=1}^m \left\{ \sum_{i=1}^n x_{ij} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \leq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m x_{ij}^{\frac{R}{\alpha}} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}. \quad (5.10)$$

or

$$\begin{aligned} \left[\sum_{j=1}^m q_j^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} &\leq \left[\sum_{i=1}^n \left\{ \sum_{j=1}^m (q_{ji} p_i)^{\frac{R}{\alpha}} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \\ \Rightarrow 1 - \left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right\}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} &\leq 1 - \left[\sum_{j=1}^m q_j^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}. \end{aligned} \quad (5.11)$$

Using $\frac{R}{R-\alpha} > 0$ as $R > \alpha$ and $0 < \alpha \leq 1$.

We find that $* H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta)$. (5.12)

On the same line, we can prove that (5.12) holds for

$$0 < R < \alpha \text{ and } 0 < \alpha \leq 1.$$

Hence (5.5) holds for all $R \in R^+$ and $0 < \alpha \leq 1$. The equality sign holds iff π_{ij} is separable in the sense $\pi_{ij} = p_i q_j$.

From Jensen's inequality for $R > \alpha$ and $0 < \alpha \leq 1$ we find

$$\sum_{i=1}^n p_i q_{ji}^{\frac{R}{\alpha}} \geq \left[\sum_{i=1}^n p_i q_{ji} \right]^{\frac{R}{\alpha}} = q_j^{\frac{R}{\alpha}}. \quad (5.13)$$

After summation over j and raising both sides to power $\frac{\alpha}{R}$, we have

$$\left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \geq \left[\sum_{j=1}^m q_j^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}}. \quad (5.14)$$

Using $\frac{R}{R-\alpha} > 0$ as $R > \alpha$, we get

$$** H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta). \quad (5.15)$$

Equality holds for all i , $q_{ji} = q_i$ which is equivalent to the independent property. For $0 < R < \alpha$, the inequality (5.14) reverses. However, in view of $\frac{R}{R-\alpha} < 0$ as $0 < R < \alpha$, $0 < \alpha \leq 1$ (5.15) still holds. Hence (5.6) is proved.

Next for the proof of (5.7), we apply Jensen's inequality and

$$\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_j^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right] \leq \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \quad (5.16)$$

for $R > \alpha$, $0 < \alpha \leq 1$ and

$$\left[\sum_{i=1}^n p_i \left\{ \sum_{j=1}^m q_j^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right] \geq \left[\sum_{i=1}^n p_i \sum_{j=1}^m q_{ji}^{\frac{R}{\alpha}} \right]^{\frac{\alpha}{R}} \quad (5.17)$$

for $0 < R < \alpha$, $0 < \alpha \leq 1$.

Using the fact $\frac{R}{R-\alpha} > 0$ and (5.16),

We have $H_R^\alpha(\eta/\xi) \leq H_R^\alpha(\eta/\xi)$.

From (5.17) for $\frac{R}{R-\alpha} < 0$, we get the same result. Hence (5.7) is proved for all $R \in R^+$ and $0 < \alpha \leq 1$ (5.8) is a combination of (5.7) and (5.5). Hence the Theorem 5 is proved.

Note. In this paper, I have define a new function depending on the parameters α and R . My motivation for studing this new function is that it generalize R-norm information measure of a discrete probability distribution introduced by Bockee and Lubbe (1980).

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