General Decay of Solutions in One-Dimensional Porous-Elastic with Memory and Distributed Delay Term

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Abstract. As a continuity to the study by T. A. Apalara in [3], we consider a one-dimensional porous-elastic system with the presence of both memory and distributed delay terms in the second equation. Using the well known energy method combined with Lyapunov functionals approach, we prove a general decay result given in Theorem 2.1.

1 Introduction

Researchers from various fields were interested in elasticity problems, and they have been mainly attracted by the qualitative studies of different type of this problems and many results can be found in the literature. In the one-dimensional case, for instance, the combination of the elastic equations with thermal consequences causes a negative exponential to control the decay of solutions. The one-dimensional porous-elastic model is given by

$$\rho_0 u_{tt} = \mu u_{xx} + \beta \varphi_x, \text{ in } (0, l) \times (0, L),$$
$$\rho_0 k \varphi_{tt} = \alpha \varphi_{tt} - \beta u_x - \tau \varphi_t - \xi \varphi, \text{ in } (0, l) \times (0, L),$$

and it has been studied by many authors. The first contribution in this direction was obtained by [21]. To be more precise, which was developed in [14], the authors showed that the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This concept was introduced in the pioneered work in [17], when the authors have advanced nonlinear theory of elastic materials with voids (See [9], [10]). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho u_{tt} = T_x, J\phi_{tt} = H_x + G, \tag{1.1}$$

where T is the stress tensor, H is the equilibrated stress vector and G is the equilibrated body force. The variables u and ϕ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are

$$T = \mu u_x + b\phi, H = \delta\phi_x - \int_0^t g(t-s)\phi_x(s)ds, G = -bu_x - \xi\phi. \tag{1.2}$$

In [21] substituting (1.2) into (1.1) is concerned

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds = 0, \text{ in } (0,1) \times (0,\infty). \end{cases} \tag{1.3}$$

A porous-elastic system with memory term and Neumann-Dirichlet boundary conditions where g is the relaxation function it has been proved a general decay result, for more detail (see [3]). Quintanilla in [21] considered

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \tau\phi_t, x \in (0,L), t > 0, \end{cases} \tag{1.4}$$

with initial and mixed boundary conditions and supposed that the damping in the porous equation $(-\tau\phi_t)$ is not enough to obtain an exponential decay but only a slow decay can be obtained. To improve this decay several other damping mechanisms were considered. In [7] Casas and Quintanilla have considered

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, x \in (0,L), t > 0, \\ c\theta_t = k\theta_{xx} - \beta u_{xt} - m\phi_t, x \in (0,L), t > 0, \end{cases} \tag{1.5}$$

where θ is the temperature difference with initial and Dirichlet-Neumann boundary conditions. The authors applied the semigroup theory and the method proposed and developed in [27] to establish the exponential decay of the solutions. Later, with $\tau = 0$, in [8] the same authors have proposed that the heat effect alone is not strong sufficient to bring an exponential decay but only a slow decay could be established. However, the heat effect together with micro-temperature created an exponential decay result. Similarly, when $\tau = 0$ and γu_{xxt} is added to the first equation in (1.1), in [19], the authors have proved the lack of exponential stability. However, by taking some regular initial data a polynomial stability is obtained. Also, for $\tau = 0$, problem (1.1) was considered in [25] with the following boundary conditions

$$\begin{cases} u(0,t) = \phi(0,t) = \theta(0,t) = \theta(L,t) = 0, t \geq 0, \\ u(L,t) = -\int_0^t g_1(t-s)[\mu u_x(L,s) + b\phi(L,s)]ds, t \geq 0, \\ \phi(L,t) = -\delta\int_0^t g_2(t-s)\phi_x(L,s)ds, t \geq 0, \end{cases}$$

where g_1 and g_2 are positive decreasing functions. They obtained a general decay result in which the usual exponential and polynomial decay rates are just special cases. ([15], [23], [20], [24], [25] and the references therein).

The viscoelastic damping is represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress $\sigma(x,t)$ and the strain $\epsilon(x,t)$ (See [12], [3])

$$\sigma(x,t) = \epsilon(x,t) + \int_0^t g(t-s)\,\epsilon(x,s)\,ds.$$

This type of viscoelastic dissipation could be said to coincide to viscosity with null initial history because it is assumed that the strains have been zero for $-\infty < t < 0$ or, equivalently, if any past strains have occurred sufficiently long ago that the effect is trivial. In other words, there will be a time prior to which all the strains which have previously occurred will have a trivial contribution. Thus, an experiment generally starts at some time $(t=0)$ when the material is free of stresses.

We must mention the pioneer works recently published by [4], the author considered a one-dimensional porous thermo-elastic system with memory effects and proved a general decay result, for which exponential and polynomial decay results are special cases, depending only on the kernel of the memory effects. The obtained result were established irrespective of the wave speeds of the system (See [2], [5]). In [13] the authors investigated a porous thermo-elastic system where the heat conduction is given by Cattaneo's law and where the energy associated with the solution is not necessary positive. They introduced a stability number and proved an exponential and polynomial decay results.

Our purpose in this paper is to give a general decay result of solutions in one dimensional porous-elastic system with memory and distributed delay term, our result is new and improves previous results in the literature.

Let $\mathcal{H} = (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$, in the present paper, we are interested in the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)\,ds + \mu_1\phi_t \\ \qquad \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x,t-s)\,ds = 0, \end{cases}$$

where

$$(x,s,t) \in \mathcal{H}.$$

As in [16], taking the following new variable

$$z(x,\rho,s,t) = \phi_t(x,t-s\rho),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \phi_t(x,t). \end{cases}$$

Consequently, the problem is equivalent to

$$
\begin{cases}
\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g\left(t-s\right)\phi_{xx}\left(x,s\right)ds + \mu_1\phi_t \\
\quad + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t\left(x,t-s\right)ds = 0, \\
sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0.
\end{cases}
\tag{1.6}
$$

where

$$
(x,\rho,s,t) \in (0,1) \times \mathcal{H}.
$$

The system with memory and delay term acting only on the porous equation together with the initial data

$$
\begin{cases}
u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), \\
\phi\left(x,0\right) = \phi_0\left(x\right), \phi_t\left(x,0\right) = \phi_1\left(x\right), x \in (0,1),
\end{cases}
\tag{1.7}
$$

and Neumann-Dirichlet boundary conditions

$$
u_x\left(0,t\right) = u_x\left(1,t\right) = \phi\left(0,t\right) = \phi\left(1,t\right) = 0, t \geq 0.
\tag{1.8}
$$

Here, u is the longitudinal displacement, ϕ is the volume fraction of the solid elastic material and ρ,μ,b,J,δ,ξ are positive constants with μ,ξ,b satisfying $\mu\xi > b^2$. The integral represents the memory and delay term with $\tau_1,\tau_2 > 0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞ function and g is the relaxation function satisfying

(H1) $g \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ is a non-increasing function satisfying

$$
g\left(0\right) > 0, \; b - \int_0^\infty g\left(s\right)ds = l > 0.
\tag{1.9}
$$

(H2) There exists a positive non-increasing differentiable function $\vartheta \in (\mathbb{R}_+,\mathbb{R}_+)$ satisfying

$$
g'\left(t\right) \leq -\vartheta\left(t\right)g\left(t\right), t \geq 0.
\tag{1.10}
$$

(H3) $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$
\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \leq \mu_1.
\tag{1.11}
$$

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [26] and references therein. In many cases it was shown that delay is a source of instability unless additional condition or control terms are used, see [11] Therefore, the stability issue of systems with delay is of theoretical and practical great importance.

It is well know that, in the single wave equation, if $\mu_2 = 0$ that is, in absence of a decay, the energy of system exponentially decays (see [4]) On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system becomes unstable (see [11]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [16], [18]).

In what follows, we consider (u, ϕ) to be a solution of system (1.6)-(1.8) with the regularity needed to justify the calculations in this paper. We specify Section 2 to the statements and prove of our stability result. We use c throughout this paper to denote a generic positive constant. Meanwhile, from (1.6) and (1.8), it follows that

$$\frac{d^2}{dt^2} \int_0^1 u(x, t)\, dx = 0. \tag{1.12}$$

So, by solving (1.12) and using the initial data of u, we get

$$\int_0^1 u(x, t)\, dx = t \int_0^1 u_1(x)\, dx + \int_0^1 u_0(x)\, dx.$$

Consequently, if we let

$$\bar{u}(x, t) = u(x, t) - t \int_0^1 u_1(x)\, dx - \int_0^1 u_0(x)\, dx, \tag{1.13}$$

we get

$$\int_0^1 \bar{u}(x, t)\, dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}$ is justified. In addition, simple substitution shows that $\left(\bar{u}, \phi\right)$ satisfies system (1.6) with initial data for $\bar{u}$ given as

$$\bar{u_0}(x) = u_0(x) - \int_0^1 u_0(x)\, dx \text{ and } \bar{u_1}(x) = u_1(x) - \int_0^1 u_1(x)\, dx.$$

Henceforth, we work with $\bar{u}$ instead of u but write u for simplicity of notation.

2   Main Result

In this section, we state and prove our decay result for the energy of the system (1.6)-(1.8) using the multiplier technique. We need the following lemmas.

Lemma 2.1. *The energy functional E, defined by*

$$E\left(t\right) \;=\; \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \left(\delta - \int_0^t g\left(s\right)ds\right)\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right]dx$$

$$+\frac{1}{2}g\circ\phi_x + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2\left(x,\rho,s,t\right)dsd\rho dx, \tag{2.1}$$

satisfies

$$E'\left(t\right) \;=\; \frac{1}{2}g'\circ\phi_x - \frac{1}{2}g\left(t\right)\int_0^1\phi_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_0^1\phi_t^2 dx, \tag{2.2}$$

and

$$E'(t) \;\leq\; \frac{1}{2}g'\circ\phi_x - \eta_0\int_0^1\phi_t^2 dx \leq 0, \tag{2.3}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds \geq 0$ and $g\circ v = \int_0^1\int_0^t g\left(t-s\right)\left(v_x\left(t\right)-v_x\left(s\right)\right)^2 dsdx$.

Proof. Multiplying the first equation of (1.6) by u_t and the second equation by ϕ_t, then integration by parts over $(0,1)$, and using (1.8), we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right]dx - \int_0^1\phi_{xt}\int_0^t g\left(t-s\right)\phi_x\left(s\right)dsdx$$

$$+\mu_1\int_0^1\phi_t^2 dx + \int_0^1\phi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z\left(x,1,s,t\right)dsdx = 0. \tag{2.4}$$

The last term in the left hand side of (2.4) is estimated as follows.

$$-\int_0^1\phi_{xt}\int_0^t g\left(t-s\right)\phi_x\left(s\right)dsdx$$

$$= \int_0^1\phi_{xt}\int_0^t g\left(t-s\right)\left(\phi_x\left(t\right)-\phi_x\left(s\right)\right)dsdx - \int_0^t g\left(s\right)ds\int_0^1\phi_{xt}\phi_x dx$$

$$= \frac{1}{2}\frac{d}{dt}g\circ\phi_x - \frac{1}{2}\frac{d}{dt}\int_0^t g\left(s\right)ds\int_0^1\phi_x^2 dx - \frac{1}{2}g'\circ\phi_x + \frac{1}{2}g\left(t\right)\int_0^1\phi_x^2 dx. \tag{2.5}$$

Now, multiplying the last equation in (1.6) by $z|\mu_2(s)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx$$

$$= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|zz_\rho\left(x,\rho,s,t\right)dsd\rho dx$$

$$= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\rho}z^2\left(x,\rho,s,t\right)dsd\rho dx$$

$$= \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|(z^2(x,0,s,t) - z^2(x,1,s,t))dsdx$$

$$= \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \int_0^1 \phi_t^2 dx - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)\,dsdx.$$

Now, using Young's inequality, we have

$$
\begin{aligned}
E'(t) &\leq \frac{1}{2}g' \circ \phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\right)\int_0^1 \phi_t^2 dx \\
&\leq \frac{1}{2}g' \circ \phi_x - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\right)\int_0^1 \phi_t^2 dx,
\end{aligned}
\tag{2.6}
$$

then, by $(H3)$, there exists a positive constant η_0 such that

$$E'(t) \leq \frac{1}{2}g' \circ \phi_x - \eta_0 \int_0^1 \phi_t^2 dx, \tag{2.7}$$

then we obtain E is a non-increasing function. □

Lemma 2.2. *The functional*

$$D_1(t) := J \int_0^1 \phi_t \phi dx + \frac{b\rho}{\mu} \int_0^1 \phi \int_0^x u_t(y)\,dydx, \tag{2.8}$$

satisfies

$$
\begin{aligned}
D_1'(t) &\leq -\frac{l}{2} \int_0^1 \phi_x^2 dx - \mu_3 \int_0^1 \phi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\
&\quad + cg \circ \phi_x + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)\,dsdx,
\end{aligned}
\tag{2.9}
$$

where $\mu_3 = \xi - \frac{b^2}{\mu} > 0$.

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$
\begin{aligned}
D_1'(t) &= -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + J\int_0^1 \phi_t^2 dx + \frac{b\rho}{\mu}\int_0^1 \phi_t \int_0^x u_t(y)\,dydx \\
&\quad + \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\,dsdx - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,s,t)dsdx \\
&\leq -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 \left(\int_0^x u_t(y)\,dy\right)^2 dx \\
&\quad + \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\,dsdx - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,s,t)dsdx.
\end{aligned}
\tag{2.10}
$$

By Cauchy-Schwartz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y)\, dy \right)^2 dx \le \int_0^1 \left(\int_0^1 u_t dx \right)^2 dx \le \int_0^1 u_t{}^2 dx.$$

So, estimate (2.10) becomes

$$
\begin{aligned}
D_1'(t) \le{}& -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 u_t{}^2 dx \\
& - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\
& + \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\, ds dx.
\end{aligned}
\tag{2.11}
$$

The last term in the RHS of (2.11) is estimated as follows:

$$
\begin{aligned}
& \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)\, ds dx \\
={}& \int_0^t g(s)\, ds \int_0^1 \phi_x^2 dx - \int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))\, ds dx \\
\le{}& \left(\delta_1 + \int_0^t g(s)\, ds \right) \int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1} \left(\int_0^t g(s)\, ds \right) g \circ \phi_x,
\end{aligned}
\tag{2.12}
$$

where we have used Cauchy-Schwartz, Young's and poincare's inequalities, for $\delta_1, \varepsilon_2, \varepsilon_3 > 0$.
By substituting (2.12) into (2.10), we obtain

$$
\begin{aligned}
D_1'(t) \le{}& -\left(\delta - \int_0^t g(s)\, ds - \delta_1 - \mu_1 c\delta_2 - \mu_1 c\delta_3 \right) \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 dx \\
& + \varepsilon_1 \int_0^1 u_t^2 dx + \left(c(1 + \frac{1}{\varepsilon_1}) + \frac{\mu_1}{4\delta_2} \right) \int_0^1 \phi_t^2 dx + \frac{1}{4\delta_1} \left(\int_0^t g(s)\, ds \right) g \circ \phi_x \\
& + \frac{1}{4\delta_3} \int_0^t \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx.
\end{aligned}
\tag{2.13}
$$

Bearing in mind that $\mu \xi > b^2$ and using the fact that $\delta - \int_0^t g(s)\, ds \ge l$,
and letting $\delta_1 = \dfrac{l}{6}, \delta_2 = \delta_3 = \dfrac{l}{6c\mu_1}$, we obtain estimate (2.9). $\square$

In the following Lemma, we use the essential hypothesis that the wave speeds of the system are equal

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{2.14}$$

Lemma 2.3. *Assume that (H1) and (2.14) hold. Then the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_t u_x dx - \frac{\rho}{\mu J} \int_0^1 u_t \int_0^t g(t-s)\phi_x(s)\, ds dx,$$

satisfies, for any $\varepsilon_2 > 0$

$$D_2'(t) \leq -\frac{b}{2J}\int_0^1 u_x^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \phi_x^2 dx + c\varepsilon_2 \int_0^1 u_t^2 dx \qquad (2.15)$$

$$+c\int_0^1 \phi_t^2 + cg\circ\phi_x - \frac{c}{\varepsilon_2}g'\circ\phi_x + c\mu_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dx.$$

Proof. By differentiating D_2, then using (1.6), integration by parts, and (1.8) we obtain

$$D_2'(t) = -\frac{b}{J}\int_0^1 u_x^2 dx + \left(\frac{\delta}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx}dx + \frac{b}{\rho}\int_0^1 \phi_x^2 dx - \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx$$

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx - \frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx$$

$$-\frac{\rho}{\mu J}\int_0^1 u_t\int_0^t g'(t-s)\phi_x(s)dsdx$$

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx. \qquad (2.16)$$

In what follows, we estimate the last five terms in the right hand side of (2.16), using Young's, Cauchy-Schwartz, and Poincare's inequalities. For $\delta_4, \delta_5, \varepsilon_2 > 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{\xi}{J}\delta_4\int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \dfrac{b}{6\xi}$, using Poincar's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{b}{6J}\int_0^1 u_x^2 dx + c\int_0^1 \phi^2 dx, \qquad (2.17)$$

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx$$

$$= \frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)(\phi_x(t) - \phi_x(s))dsdx - \frac{b}{\mu J}\int_0^t g(s)ds\int_0^1 \phi_x^2 dx$$

$$\leq \left(\delta_5 - \frac{b}{\mu J}\right)\int_0^t g(s)ds\int_0^1 \phi_x^2 dx + \frac{c}{\delta_5}g\circ\phi_x.$$

By letting $\delta_5 = \dfrac{b}{\mu J}$ we get

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \leq cg\circ\phi_x, \qquad (2.18)$$

$$-\frac{\rho}{\mu J}\int_0^1 u_t\int_0^t g'(t-s)\phi_x(s)dsdx$$

$$
\begin{aligned}
&= \frac{b}{\mu J} \int_0^1 u_t \int_0^t g'(t-s)\left(\phi_x(t) - \phi_x(s)\right) ds dx - \frac{b}{\mu J} \int_0^t g'(s) ds \int_0^1 u_t \phi_x dx \\
&\leq \frac{\rho \varepsilon_2}{2\mu J} \int_0^1 u_t^2 dx + \frac{\rho g(0)}{\mu J} \int_0^1 u_t \phi_x dx - \frac{\rho g(t)}{\mu J} \int_0^1 u_t \phi_x dx \\
&\qquad + \frac{\rho}{2\mu J \varepsilon_2} \int_0^1 g'(s) ds \int_0^1 \int_0^t g'(t-s)\left(\phi_x(t) - \phi_x(s)\right)^2 ds dx \\
&\leq \frac{\rho \varepsilon_2}{\mu J} \int_0^1 u_t^2 dx + \frac{\rho}{2\mu J \varepsilon_2}\left(\int_0^1 g'(s) ds\right) g' \circ \phi_x + \frac{\rho g(0)}{\mu J} \int_0^1 u_t \phi_x dx \\
&\qquad + \frac{\rho g(t)}{2\mu J \varepsilon_2} \int_0^1 u_t \phi_x dx \\
&\leq \frac{\rho \varepsilon_2}{\mu J} \int_0^1 u_t^2 dx + \frac{\rho}{2\mu J \varepsilon_2}\left(\int_0^1 g'(s) ds\right) g' \circ \phi_x + \frac{\rho g(0)}{\mu J} \int_0^1 u_t \phi_x dx \\
&\qquad + \frac{\rho (g(t))^2}{2\mu J \varepsilon_2} \int_0^1 \phi_x^2 dx \\
&\leq c \varepsilon_2 \int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2} \int_0^1 \phi_x^2 dx + \frac{\rho g(0)}{\mu J} \int_0^1 u_t \phi_x dx - \frac{c}{\varepsilon_2} g' \circ \phi_x,
\end{aligned}
\tag{2.19}
$$

$$
-\frac{\mu_1}{J} \int_0^1 \phi_t u_x dx \leq \frac{\mu_1 \delta_6}{2J} \int_0^1 \phi_t^2 dx + \frac{\mu_1}{2J\delta_6} \int_0^1 u_x^2 dx,
\tag{2.20}
$$

$$
\begin{aligned}
\frac{1}{J} \int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\
\leq \frac{\delta_7 \mu_1}{2J} \int_0^1 u_x^2 dx + \frac{1}{2J\delta_7} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds.
\end{aligned}
\tag{2.21}
$$

The replacement of (2.17)-(2.21) into (2.16), and by letting $\delta_6 = \delta_7 = \dfrac{b}{4\mu_1}$, bearing in the mind (2.14), yields (2.15). $\qquad \square$

Lemma 2.4. *The functional*

$$
D_3(t) := -\rho \int_0^1 u_t u dx,
$$

satisfies

$$
D_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + c \int_0^1 \phi_x^2 dx.
\tag{2.22}
$$

Proof. Direct computations give

$$
D_3'(t) = -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b \int_0^1 u_x \phi dx.
$$

Estimat (2.22) easily follows by using Young's and Poincaré inequalities.

$$
\begin{aligned}
D_3'(t) &\leq -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\varepsilon \int_0^1 u_x^2 dx + \frac{b}{4\varepsilon} \int_0^1 \phi^2 dx \\
&\leq -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\varepsilon \int_0^1 u_x^2 dx + \frac{bc}{4\varepsilon} \int_0^1 \phi_x^2 dx,
\end{aligned}
$$

by letting $\varepsilon = \dfrac{\mu}{2b}$, we obtain (2.22).   □

Now, let us introduce the following functional used by

Lemma 2.5. *The functional*

$$
D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx,
$$

satisfies,

$$
\begin{aligned}
D_4'(t) &\leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx + \mu_1 \int_0^1 \phi_t^2 dx \\
&\quad -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t)\, ds\, dx,
\end{aligned} \tag{2.23}
$$

where η_1 is a positive constant.

Proof. By differentiating D_4, with respect to t and using the last equation in $(H3)$, we have

$$
\begin{aligned}
D_4'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z z_\rho(x, \rho, s, t)\, ds\, d\rho\, dx \\
&= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx \\
&\quad -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| [e^{-s} z^2(x, 1, s, t) - z^2(x, 0, s, t)]\, ds\, dx.
\end{aligned}
$$

Using the fact that $z(x, 0, s, t) = \phi_t(x, t)$, and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$
\begin{aligned}
D_4'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t)\, ds\, d\rho\, dx \\
&\quad -\int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t)\, ds\, dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \phi_t^2 dx.
\end{aligned}
$$

□

Because $-e^{-s}$ is a increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (H3), we obtain (2.23). We are now ready to prove the main result.

Theorem 2.1. *Assume (H1), (H2), (H3) and (2.14) hold. Then, for any $t_0 > 0$, there exist positive constants α and β such that the energy functional given by (2.1) satisfies*

$$E(t) \leq \alpha e^{-\beta \int_{t_0}^{t} \vartheta(s)ds}, \forall t \geq t_0. \tag{2.24}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{2.25}$$

where N, N_1, N_2, and N_4 are positive constants to be selected later. By differentiating (2.25) and using (2.1), (2.9), (2.15), (2.22), (2.23), we have

$$
\begin{aligned}
\mathcal{L}'(t) \leq\ & -\left[\frac{lN_1}{2} - cN_2(1 + \frac{1}{\varepsilon_2}) - c\right] \int_0^1 \phi_x^2 dx - \left[\rho - N_1\varepsilon_1 - N_2 c\varepsilon_2\right] \int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right] \int_0^1 u_x^2 dx - \left[\eta_0 N - cN_1(1 + \frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4\right] \int_0^1 \phi_t^2 dx \\
& -N_1\mu_3 \int_0^1 \phi^2 dx - \left[N_4\eta_1 - cN_1 - cN_2\right] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x, 1, s, t)\, dsdx \\
& +c\left[N_1 + N_2\right] g \circ \phi_x + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right] g' \circ \phi_x \\
& -N_4\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x, \rho, s, t)\, dsd\rho dx.
\end{aligned}
$$

By setting

$$\varepsilon_1 = \frac{\rho}{4N_1}, \varepsilon_2 = \frac{\rho}{4cN_2},$$

we obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq\ & -\left[\frac{lN_1}{2} - cN_2(1 + N_2) - c\right] \int_0^1 \phi_x^2 dx - \frac{\rho}{2} \int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right] \int_0^1 u_x^2 dx - \left[\eta_0 N - cN_1(1 + N_1) - cN_2 - \mu_1 N_4\right] \int_0^1 \phi_t^2 dx \\
& -N_1\mu_3 \int_0^1 \phi^2 dx - \left[N_4\eta_1 - cN_1 - cN_2\right] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x, 1, s, t)\, dsdx \\
& +c\left[N_1 + N_2\right] g \circ \phi_x + \left[\frac{N}{2} - cN_2^2\right] g' \circ \phi_x \\
& -N_4\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x, \rho, s, t)\, dsd\rho dx.
\end{aligned}
$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose N_2 large enough such that

$$\alpha_1 = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

then We choose N_1 large enough such that

$$\alpha_2 = \frac{lN_1}{4} - cN_2(1 + N_2) - c > 0,$$

then We choose N_4 large enough such that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\alpha_2 \int_0^1 \phi_x^2 dx - \alpha_0 \int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_1 \int_0^1 u_x^2 dx - [\eta_0 N - c]\int_0^1 \phi_t^2 dx \\
& + \left[\frac{N}{2} - c\right] g' \circ \phi_x + cg \circ \phi_x - \alpha_3 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& -\alpha_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\, ds d\rho dx,
\end{aligned}
\tag{2.26}
$$

where $\alpha_0 = \mu_3 N_1 = \left(\xi - \dfrac{b^2}{\mu}\right) N_1$. On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),$$

then

$$
\begin{aligned}
|\mathfrak{L}(t)| \leq & \; JN_1 \int_0^1 |\phi\phi_t|\, dx + N_2 \int_0^1 \left|\phi_x u_t + u_x \phi_t - \frac{\rho}{\mu J} u_t \int_0^t g(t-s)\phi_x(s)\, ds\right| dx \\
& \frac{b\rho N_1}{\mu}\int_0^1 \left|\phi \int_0^x u_t(y)\, dy\right| dx + \rho \int_0^1 |u_t u|\, dx \\
& + N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned}
$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$
\begin{aligned}
|\mathfrak{L}(t)| \leq & \; c\int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2\right) dx + cg \circ \phi_x \\
& + c\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\, ds d\rho \\
\leq & \; cE(t).
\end{aligned}
$$

Consequently, we obtain

$$|\mathfrak{L}(t)| = |\mathcal{L}(t) - NE(t)| \le cE(t),$$

that is

$$(N - c) E(t) \le \mathcal{L}(t) \le (N + c) E(t). \tag{2.27}$$

Now, by choosing N large enough such that

$$\frac{N}{2} - c > 0, N - c > 0, N\eta_0 - c > 0,$$

and exploiting (2.1), estimates (2.26) and (2.27), respectively, give

$$\mathcal{L}'(t) \le -k_1 E(t) + k_2 g \circ \phi_x, \forall t \ge t_0, \tag{2.28}$$

and

$$c_2 E(t) \le \mathcal{L}(t) \le c_3 E(t), \forall t \ge 0, \tag{2.29}$$

for some $k_1, k_2, c_2, c_3 > 0$.

By multiplying (2.28) by $\vartheta(t)$, we obtain

$$\vartheta(t) \mathcal{L}'(t) \le -k_1 \vartheta(t) E(t) + k_2 \vartheta(t) g \circ \phi_x, \forall t \ge t_0. \tag{2.30}$$

The final term in (2.30) is estimated as following, using (1.10), we have

$$
\begin{aligned}
\vartheta(t) g \circ \phi_x &= \vartheta(t) \int_0^1 \int_0^t g(t-s) (\phi_x(t) - \phi_x(s))^2 \, ds dx \\
&\le \int_0^1 \int_0^t \vartheta(t-s) g(t-s) (\phi_x(t) - \phi_x(s))^2 \, ds dx \\
&\le -\int_0^1 \int_0^t g'(t-s) (\phi_x(t) - \phi_x(s))^2 \, ds dx = -g' \circ \phi_x \\
&\le -2E'(t).
\end{aligned}
$$

Thus, (2.30) becomes

$$\vartheta(t) \mathcal{L}'(t) \le -k_1 \vartheta(t) E(t) - 2k_2 E'(t), \forall t \ge t_0,$$

which can be rewritten as

$$(\vartheta(t) \mathcal{L}(t) + 2k_2 E(t))' - \vartheta'(t) \mathcal{L}(t) \le -k_1 \vartheta(t) E(t), \forall t \ge t_0,$$

using the fact that $\vartheta'(t) \le 0, \forall t \ge 0$, we have

$$(\vartheta(t) \mathcal{L}(t) + 2k_2 E(t))' \le -k_1 \vartheta(t) E(t), \forall t \ge t_0.$$

By exploiting (2.29), we notice that

$$\mathcal{R}\left(t\right) = \vartheta\left(t\right)\mathcal{L}\left(t\right) + 2k_2 E\left(t\right) \sim E\left(t\right). \tag{2.31}$$

Consequently, for some positive constant λ, we obtain

$$\mathcal{R}'\left(t\right) \leq -\lambda\mathcal{R}\left(t\right)\vartheta\left(t\right), \forall t \geq t_0. \tag{2.32}$$

A simple integration of (2.32) over (t_0, t) leads to

$$\mathcal{R}\left(t\right) \leq \mathcal{R}\left(t_0\right)e^{-\lambda\int_{t_0}^{t}\vartheta(s)ds}, \forall t \geq t_0. \tag{2.33}$$

Consequently, (2.24) is established by virtue of (2.29) and (2.33). □

Remark 1. We give some examples to illustrate the energy decay rates obtained by Theorem 2.1. We consider the three different examples

$$\text{If } g(t) = \beta_1 e^{-\beta_2 t}, \text{ then } g'(t) = -\vartheta(t)g(t), \text{ where } \vartheta(t) = \beta_2,$$

then

$$E(t) \leq c_0 e^{-\beta_2 c_1 t}, \forall t \geq 0,$$

$$\text{If } g(t) = \frac{\beta_1}{(1+t)^{\beta_2+1}}, \text{ then } g'(t) = -\vartheta(t)g(t), \text{ where } \vartheta(t) = \frac{\beta_2+1}{1+t},$$

then

$$E(t) \leq \frac{c_0}{(1+t)^{(\beta_2+1)c_1}}, \forall t \geq 0,$$

$$\text{If } g(t) = \frac{\beta_1}{(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}, \text{ then } g'(t) = -\vartheta(t)g(t), \text{where } \vartheta(t) = \beta_2(\frac{\pi}{2} - arctgt),$$

then

$$E(t) \leq \frac{c_0}{c_1(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}, \forall t \geq 0.$$

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