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A Common Solution of Equilibrium, Constrained Convex Minimization and Fixed Point Problems

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Abstract. In this paper, we propose a new iterative scheme with the help of the gradient-projection algorithm (GPA) for finding a common solution of an equilibrium problem, a constrained convex minimization problem, and a fixed point problem. Then, we prove some strong convergence theorems which improve and extend some recent results. Moreover, we give a numerical result to show the validity of our main theorem.

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive, if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Also, a contraction on C is a self-mapping f of C such that $\|f(x)-f(y)\| \leq k\|x-y\|$ for all $x, y \in C$, where $k \in (0,1)$ is a constant. Moreover, $F(T)$ denotes the fixed points set of T.
Let $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. We recall an equilibrium problem as follows: The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0 \text{ for all } v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Some authors (such as [10, 11, 12, 18]) proposed some useful methods for solving the equilibrium problem (1.1). The equilibrium problem (1.1) includes, as special cases, numerous problems in physics, optimization, image reconstruction, ecology, transportation, network, finance, and economics. In fact, the equilibrium problem is a generalization of many mathematical models such as variational inequalities, fixed point problems, and optimization problems. Recently, a lot of iterative algorithms have been studied in infinite dimensional spaces (see [13, 14, 19] and the references therein).

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Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0 \text{ for all } v \in C.$$

The set of solutions of the variational inequality is denoted by $VI(C, A)$.

Consider the constrained convex minimization problems as follows:

$$minimize\{g(x) : x \in C\}, \tag{1.2}$$

where $g : C \to \mathbb{R}$ is a real-valued convex function. The set of solutions of the problem (1.2) is denoted by U. It is well known the gradient-projection algorithm (GPA) plays an important role in solving constrained convex minimization problems. If g is (fréchet) differentiable, then the GPA generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda \nabla g(x_n)) \text{ for all } n \geq 0, \tag{1.3}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla g(x_n)) \text{ for all } n \geq 0, \tag{1.4}$$

where in both (1.3) and (1.4) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers satisfying certain conditions. The convergence of the algorithms (1.3) and (1.4) depends on the behavior of the gradient ∇g. As a matter of fact, it is known if ∇g is $\alpha-$strongly monotone and $L-$Lipschitzian with constants $\alpha, L \geq 0$, then the operator

$$W := P_C(I - \lambda \nabla g) \tag{1.5}$$

is a contraction; hence the sequence $\{x_n\}$ defined by algorithm (1.3) converges in norm to the unique minimizer of (1.2). However, if the gradient ∇g fails to be strongly monotone, the operator W defined by (1.5) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (1.3) may fail to converge strongly (see [20]). If ∇g is Lipschitzian, then the algorithms (1.3) and (1.4) can still converge in the weak topology under certain conditions.

In 2011, Xu [20] proposed an explicit operator-oriented approach to the algorithm (1.4); that is, an averaged mapping approach. He gave his averaged mapping approach to the GPA (1.4) and the relaxed gradient-projection algorithm. Moreover, he constructed a counterexample which shows that the algorithm (1.3) does not converge in norm in an infinite- dimensional space and also presented two modifications of GPA which are shown to have strong convergence [21, 22].

On the other hand, in 2007, Takahashi and Takahashi [16] introduced a general iterative method for finding a common element of $EP(\phi)$ and $F(T)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\} \subseteq [0,1]$ and proved strong convergence of the method (1.6) to $z = P_{F(T)\cap EP(\phi)} f(z)$ in the framework of a Hilbert space, under some suitable conditions on $\{\alpha_n\}, \{r_n\}$ and bifunction ϕ.

In 2017, Cheawchan et al. [5] studied the following iterative scheme for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of a variational inequality problem, and the set of fixed points of a nonspreading mapping in Hilbert spaces:

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \text{ for all } y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, \quad n \geq 1, \end{cases}$$

where A is an α-inverse strongly monotone mapping, T is a nonspreading map,

$$VI(C,A) \bigcap EP(\phi) \bigcap F(T) \neq \emptyset,$$

$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, $\{r_n\} \subset (0, 2\alpha)$, and $\lambda \in (0, 2\alpha)$. They proved the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to a point in $VI(C,A) \bigcap EP(\phi) \bigcap F(T)$ under certain conditions.

In this paper, motivated by the above results, we propose a new composite iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.1), the set of solutions of the constrained convex minimization problem (1.2), and the set of fixed points of a nonexpansive mapping in Hilbert spaces.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and the norm $\|.\|$. Weak and strong convergence are denoted by notation $\rightharpoonup$ and $\rightarrow$, respectively. In a real Hilbert space H, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \beta\gamma\|z-y\|^2 - \alpha\gamma\|z-x\|^2$$

for all $x, y, z \in H$ and $\alpha, \beta, \lambda \subset [0,1]$ with $\alpha + \beta + \lambda = 1$. It is known a Hilbert space H satisfies Opial's property [9], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$liminf_{n\rightarrow\infty}\|x_n - x\| < liminf_{n\rightarrow\infty}\|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \ \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. It is also known P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \ \Leftrightarrow \ \langle x - z, z - y\rangle \geq 0 \ \text{ for all } y \in C.$$

Definition 1. A mapping $T : H \to H$ is called firmly nonexpansive if for any $x, y \in H$,

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle.$$

Lemma 2.1. *[7] Let C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$.*

Lemma 2.2. *[2] Let C be a nonempty closed convex subset of H and $\phi : C\times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(A_1) $\phi(x, x) = 0$ for all $x \in C$;

(A_2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \le 0$ for all $x, y \in C$;

(A_3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1 - t)x, y) \le \phi(x, y)$;

(A_4) for each $x \in C, y \mapsto \phi(x, y)$ is convex and weakly lower semicontinuous.

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge\ 0 \ \textit{for all } y \in C.$$

Lemma 2.3. *[6] Assume $\phi : C \times C \to \mathbb{R}$ satisfies (A_1)-(A_4). For $r > 0$ and $x \in H$, define a mapping $Q_r : H \to C$ as follows:*

$$Q_r x = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \ \textit{for all } y \in C\}$$

for all $x \in H$. Then, the following hold:

(I) Q_r is single-valued;

(II) Q_r is firmly nonexpansive;

(III) $F(Q_r) = EP(\phi)$;

(IV) $EP(\phi)$ is closed and convex.

Remark 1. A mapping $T : H \to H$ is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive. Obviously, projections are firmly nonexpansive.

Definition 2. [17] A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is, $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, we say that T is $\alpha-$averaged.

Clearly, firmly nonexpansive mappings are $\frac{1}{2}-$averaged mappings.

Proposition 2.1. *[4] The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \dots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Definition 3. A nonlinear operator G whose domain $D(G) \subseteq H$ and the range $R(G) \subseteq H$ is said to be ν- inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Gx - Gy\rangle \geq \nu\|Gx - Gy\|^2 \text{ for all } x, y \in D(G).$$

It can be easily seen the projection map P_C is a 1-ism. The inverse strongly monotone (also referred to as co-coercive) operators have been widely used to solve practical problems in various fields, for instance, in traffic assignment problems; see, for example, [3, 8] and reference therein.

Proposition 2.2. *[4] Let $T : H \to H$ be given. We have*

(a) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}-$ism.

(b) If T is $\nu-$ism, then for $\gamma > 0$, $\gamma\, T$ is $\frac{\nu}{\gamma} - ism$.

(c) T is averaged if and only if the complement $I - T$ is $\nu-$ism for some $\nu > \frac{1}{2}$; Indeed, for $\alpha \in (0,1)$, T is $\alpha-$averaged if and only if $I - T$ is $\frac{1}{2\alpha}-$ism.

Lemma 2.4. *[1] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n v_n + \mu_n,$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$, $\{\mu_n\}$ is a sequence of nonnegative real numbers, and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} \gamma_n = \infty, \limsup_{n\to\infty} v_n \leq 0$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main result

In this paper, we always assume $g : C \to \mathbb{R}$ is a real-valued convex function and ∇g is an $L-$Lipschitzian mapping with $L \geq 0$. We observe $x^* \in C$ solves the minimization problem (1.2) if and only if $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \lambda\nabla g)x^*,$$

where $\lambda > 0$ is any fixed positive number. Since ∇g is L-Lipschitzian, ∇g is $\frac{1}{L}$-ism, which then implies $\lambda\nabla g$ is $\frac{1}{\lambda L}$-ism. So, by Proposition 2.2, $I - \lambda\nabla g$ is $\frac{\lambda L}{2}$-averaged. Since the projection P_C

is $\frac{1}{2}$-averaged, we see from Proposition 2.1 that the composite $P_C(I-\lambda\nabla g)$ is $(\frac{2+\lambda L}{4})$-averaged for $0<\lambda<\frac{2}{L}$. Hence $P_C(I-\lambda_n\nabla g)$ is $(\frac{2+\lambda_n L}{4})-$averaged for each $n\in\mathbb{N}$. Therefore, we can write

$$P_C(I-\lambda_n\nabla g)=\frac{2-\lambda_n L}{4}I+\frac{2+\lambda_n L}{4}O_n=s_nI+(1-s_n)O_n,$$

where O_n is nonexpansive and $s_n=\frac{2-\lambda_n L}{4}$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T:C\to C$ be a nonexpansive mapping, $\phi:C\times C\to\mathbb{R}$ be a bifunction satisfying the conditions $(A_1)-(A_4)$ (of Lemma 2.2), f be a contractions of C into itself with coefficient k, $g:C\to\mathbb{R}$ be a real-valued convex function, ∇g be an $L-$Lipschitzian mapping with $L\geq 0$, and $F:=U\bigcap EP(\phi)\bigcap F(T)\neq\emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\}\subset[0,1]$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;

(B_2) $\{\beta_n\}\subset(0,1)$, $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$;

(B_3) $\{\gamma_n\}\subset[0,1]$ *and* $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$;

(B_4) $\{r_n\}\subset(a,\infty)$ $(a>0)$ *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases}\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\ \text{for all } y\in C,\\ x_{n+1}=\alpha_nf(x_n)+\beta_nP_C(I-\lambda_n\nabla g)u_n+\gamma_nTP_C(I-\lambda_n\nabla g)x_n,\quad n\geq 1,\end{cases}\tag{3.1}$$

where $x_1\in C$, $\alpha_n+\beta_n+\gamma_n=1$, $\{\lambda_n\}\subset(0,\frac{2}{L})$, $u_n=Q_{r_n}x_n$, $P_C(I-\lambda_n\nabla g)=s_nI+(1-s_n)O_n$, and $s_n=\frac{2-\lambda_nL}{4}$. Let $\lim_{n\to\infty}s_n=0$ and $\sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.1) converge strongly to $q\in F$, where $q=P_Ff(q)$, which solves the following variational inequality:

$$\langle(I-f)q,q-x\rangle\leq 0\ \text{for all } x\in F.$$

Proof. Since P_Ff is a contraction of C into itself, there exists a unique element $q\in C$ such that $q=P_Ff(q)$. Let $V_n=P_C(I-\lambda_n\nabla g)$. Now, we proceed with the following steps:

Step 1. We claim $\{x_n\}$ and $\{u_n\}$ are bounded. Let $p\in F$. Then, from $u_n=Q_{r_n}x_n$ and $Q_{r_n}p=p$, $\|u_n-p\|\leq\|x_n-p\|$. Thus, from $V_np=p$ and (3.1),

$$\begin{aligned}\|x_{n+1}-p\|\leq&\ \|\alpha_n(f(x_n)-p)+\beta_n(V_nu_n-p)+\gamma_n(TV_nx_n-p)\|\\ \leq&\ \alpha_n(\|f(x_n)-f(p)\|+\|f(p)-p\|)+\beta_n\|u_n-p\|+\gamma_n\|x_n-p\|\\ \leq&\ (1-\alpha_n(1-k))\|x_n-p\|+\alpha_n\|f(p)-p\|\\ \leq&\ \max\{\|x_n-p\|,\tfrac{\|f(p)-p\|}{1-k}\}.\end{aligned}$$

By induction,

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-k}\} \text{ for all } n \geq 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$.

Step 2. We claim $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Set

$$M = \sup\{\|f(x_n)\|, \|\nabla g(x_n)\|, \|\nabla g(u_n)\|, \|V_n u_n\|, \|TV_n x_n\|, \frac{1}{a}\|u_n - x_n\| : n \in \mathbb{N}\}.$$

By the definition of $\{x_n\}$,

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\alpha_n f(x_n) + \beta_n V_n u_n + \gamma_n TV_n x_n - \alpha_{n-1} f(x_{n-1}) - \beta_{n-1} V_{n-1} u_{n-1} \\
&\quad - \gamma_{n-1} TV_{n-1} x_{n-1}\| \\
&= \|\alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) + \beta_n (V_n u_n - V_n u_{n-1}) \\
&\quad + \beta_n V_n u_{n-1} - \beta_{n-1} V_{n-1} u_{n-1} + \gamma_n (TV_n x_n - TV_n x_{n-1}) + \gamma_n TV_n x_{n-1} \\
&\quad - \gamma_{n-1} TV_{n-1} x_{n-1}\| \\
&\leq \alpha_n k \|x_n - x_{n-1}\| + M|\alpha_n - \alpha_{n-1}| + \gamma_n \|x_n - x_{n-1}\| + \beta_n \|u_n - u_{n-1}\| \\
&\quad + \beta_n \|V_n u_{n-1} - V_{n-1} u_{n-1}\| + |\beta_n - \beta_{n-1}| M + |\gamma_n - \gamma_{n-1}| M \\
&\quad + \gamma_n \|TV_n x_{n-1} - TV_{n-1} x_{n-1}\|
\end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. Let $u_n = Q_{r_n} x_n$ and $u_{n-1} = Q_{r_{n-1}} x_{n-1}$. So

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \tag{3.3}$$

and

$$\phi(u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0 \text{ for all } y \in C. \tag{3.4}$$

Set $y = u_{n-1}$ in (3.3) and $y = u_n$ in (3.4). Then by adding these two inequalities and using condition (A_2), we have

$$\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\rangle \geq 0$$

and hence $\langle u_n - u_{n-1}, u_{n-1} - u_n + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0$. This implies

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 \leq\ & \langle u_n - u_{n-1}, x_n - x_{n-1} + (1 - \tfrac{r_{n-1}}{r_n})(u_n - x_n)\rangle \\
\leq\ & \|u_n - u_{n-1}\|\{\|x_n - x_{n-1}\| + \tfrac{1}{a}|r_{n-1} - r_n|\|u_n - x_n\|\}.
\end{aligned}$$

Therefore

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| M. \tag{3.5}$$

Also, we have

$$\begin{aligned}\|V_n x_{n-1} - V_{n-1}x_{n-1}\| = &\ \|P_C(I-\lambda_n\nabla g)x_{n-1} - P_C(I-\lambda_{n-1}\nabla g)x_{n-1}\| \\ \leq &\ \|(I-\lambda_n\nabla g)x_{n-1} - (I-\lambda_{n-1}\nabla g)x_{n-1}\| \\ = &\ |\lambda_n-\lambda_{n-1}|\|\nabla g(x_{n-1})\| \leq M|\lambda_n-\lambda_{n-1}|.\end{aligned} \tag{3.6}$$

Similarly, we can prove

$$\|V_n u_{n-1} - V_{n-1}u_{n-1}\| \leq M|\lambda_n-\lambda_{n-1}|. \tag{3.7}$$

Substituting (3.5), (3.6) and (3.7) in (3.2), we have

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\leq \alpha_n k\|x_n-x_{n-1}\| + M|\alpha_n-\alpha_{n-1}| + \gamma_n\|x_n-x_{n-1}\| + \beta_n(\|x_n-x_{n-1}\| \\ &\quad +|r_n-r_{n-1}|M) + (\beta_n+\gamma_n)M|\lambda_n-\lambda_{n-1}| + |\beta_n-\beta_{n-1}|M \\ &\quad +|\gamma_n-\gamma_{n-1}|M \\ &\leq (1-(1-k)\alpha_n)\|x_n-x_{n-1}\| + (|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}| + |\gamma_n-\gamma_{n-1}| \\ &\quad +|\lambda_n-\lambda_{n-1}| + |r_n-r_{n-1}|)M\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, by Lemma 2.4, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 3. We claim $\lim_{n\to\infty}\|x_n-u_n\| = 0$. Let $p \in F$. By Lemma 2.3,

$$\begin{aligned}\|u_n-p\|^2 = &\ \|Q_{r_n}x_n - Q_{r_n}p\|^2 \leq \langle x_n-p, u_n-p\rangle \\ = &\ \tfrac{1}{2}(\|x_n-p\|^2 + \|u_n-p\|^2 - \|u_n-x_n\|^2).\end{aligned}$$

This implies

$$\|u_n-p\|^2 \leq \|x_n-p\|^2 - \|u_n-x_n\|^2.$$

So

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq &\ \|\alpha_n(f(x_n)-p) + \beta_n(V_n u_n - p) + \gamma_n(TV_n x_n - p)\|^2 \\ \leq &\ \alpha_n(\|f(x_n)-f(p)\| + \|f(p)-p\|)^2 + \beta_n\|u_n-p\|^2 \\ &\ +\gamma_n\|x_n-p\|^2 \\ \leq &\ (1-\alpha_n(1-k^2))\|x_n-p\|^2 + \alpha_n\|f(p)-p\|^2 \\ &\ +2\alpha_n k\|x_n-p\|\|f(p)-p\| - \beta_n\|u_n-x_n\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\beta_n\|u_n-x_n\|^2 \leq &\ \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n\|f(p)-p\|^2 \\ &\ +2\alpha_n k\|x_n-p\|\|f(p)-p\| \\ \leq &\ \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + \alpha_n\|f(p)-p\|^2 \\ &\ +2\alpha_n k\|x_n-p\|\|f(p)-p\|.\end{aligned}$$

Hence, by (B_1), (B_2), and Step 2, $\lim_{n\to\infty}\|x_n-u_n\| = 0$.

Step 4. We claim $\lim_{n\to\infty}\|u_n - P_C(I-\frac{2}{L}\nabla g)u_n\| = 0$ and $\lim_{n\to\infty}\|x_n - TV_nx_n\| = 0$. We know

$$\|V_nu_n - x_n\| = \|(1-s_n)u_n + s_nO_nu_n - x_n\| \le (1-s_n)\|u_n - x_n\| + s_n\|O_nu_n - x_n\|.$$

So, from Step 3, $\lim_{n\to\infty}\|V_nu_n - x_n\| = 0$. This implies $\lim_{n\to\infty}\|u_n - V_nu_n\| = 0$. Therefore

$$\begin{aligned}\|u_n - P_C(I-\tfrac{2}{L}\nabla g)u_n\| \le\ & \|P_C(I-\tfrac{2}{L}\nabla g)u_n - P_C(I-\lambda_n\nabla g)u_n\| \\ & +\|P_C(I-\lambda_n\nabla g)u_n - u_n\| \\ \le\ & (\tfrac{2}{L}-\lambda_n)\|\nabla g(u_n)\| + \|V_nu_n - u_n\|.\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n - P_C(I-\frac{2}{L}\nabla g)u_n\| = 0$. From definition of $\{x_n\}$,

$$\begin{aligned}\|x_{n+1} - TV_nx_n\| =\ & \|\alpha_nf(x_n) + \beta_nV_nu_n + (\gamma_n - 1)TV_nx_n\| \\ \le\ & \alpha_n\|f(x_n) - TV_nx_n\| + \beta_n\|V_nu_n - TV_nx_n\| \\ \le\ & \alpha_n\|f(x_n) - TV_nx_n\| + \beta_n(\|V_nu_n - x_n\| + \|x_n - TV_nx_n\|).\end{aligned}$$

So

$$\begin{aligned}\|x_n - TV_nx_n\| \le\ & \|x_{n+1} - x_n\| + \|x_{n+1} - TV_nx_n\| \\ \le\ & \|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - TV_nx_n\| + \beta_n(\|V_nu_n - x_n\| \\ & +\|x_n - TV_nx_n\|).\end{aligned}$$

Therefore

$$(1-\beta_n)\|x_n - TV_nx_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - TV_nx_n\| + \beta_n\|V_nu_n - x_n\|.$$

Hence

$$\lim_{n\to\infty}\|x_n - TV_nx_n\| = 0. \tag{3.8}$$

Step 5. We claim $\limsup_{n\to\infty}\langle (I-f)q, q - x_n\rangle \le 0$, where $q = P_Ff(q)$. To show this, choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-f)q, q-u_n\rangle = \lim_{i\to\infty}\langle (I-f)q, q-u_{n_i}\rangle.$$

Since $\{u_{n_i}\}$ is bounded in C, without loss of generality, we may assume $u_{n_i} \rightharpoonup z \in C$. Now, we show $z \in F$. Since ∇g is $\frac{1}{L}-$ism, $P_C(I-\frac{2}{L}\nabla g)$ is nonexpansive self-mapping on C. Therefore, from Step 4 and Lemma 2.1, we obtain $z = P_C(I-\frac{2}{L}\nabla g)z$. This implies $z \in U$. Next, we show $z \in EP(\phi)$. By $u_n = Q_{r_n}x_n$, one can write

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \ \text{ for all } y \in C.$$

From (A_2), $\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq \phi(y,u_n),$ for all $y\in C$. Replacing n by n_i, we have

$$\frac{1}{r_{n_i}}\langle y-u_{n_i}, u_{n_i}-x_{n_i}\rangle \geq \phi(y,u_{n_i}) \text{ for all } y\in C.$$

Since $u_{n_i} \rightharpoonup z$, it follows from Step 2, $(A4)$, and (B_3) that $\phi(y,z) \leq 0$ for all $y \in C$. Set $y_t = ty+(1-t)z$ for all $t\in(0,1]$ and $y\in C$. Then $y_t \in C$ and hence $\phi(y_t,z)\leq 0$. From (A_1) and (A_2),

$$0=\phi(y_t,y_t)\leq t\phi(y_t,y)+(1-t)\phi(y_t,z)\leq t\phi(y_t,y).$$

Therefore $\phi(y_t,y)\geq 0$. Letting $t\to 0$, we get $\phi(z,y)\geq 0$ for all $y\in C$. This implies $z\in EP(\phi)$. Now, we prove $z\in F(T)$. To show this, we suppose $z\neq Tz$. Since $x_{n_i}\rightharpoonup z$, by using Opial's property and (3.8),

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i}-z\| <& \liminf_{i\to\infty}(\|x_{n_i}-TV_{n_i}x_{n_i}\|+\|TV_{n_i}x_{n_i}-Tz\|)\\ =& \liminf_{i\to\infty}\|TV_{n_i}x_{n_i}-Tz\|\\ \leq& \liminf_{i\to\infty}\|x_{n_i}-z\|.\end{aligned}$$

This is a contradiction. Therefore $z\in F(T)$. Since $q=P_F f(q)$,

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)q, q-x_n\rangle =& \lim_{i\to\infty}\langle (I-f)q, q-x_{n_i}\rangle\\ =& \lim_{i\to\infty}\langle (I-f)q, q-u_{n_i}\rangle\\ =& \lim_{i\to\infty}\langle (I-f)q, q-z\rangle \leq 0.\end{aligned}$$

Step 6. We claim $\{x_n\}$ and $\{u_n\}$ converge strongly to q. From (3.1),

$$\begin{aligned}\|x_{n+1}-q\|^2 =& \|\alpha_n(f(x_n)-q)+\beta_n(V_nu_n-q)+\gamma_n(TV_nx_n-q)\|^2\\ \leq& \|\alpha_n(f(x_n)-f(q))+\beta_n(V_nu_n-q)+\gamma_n(TV_nx_n-q)\|^2\\ &+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\ \leq& \alpha_nk^2\|x_n-q\|^2+\beta_n\|u_n-q\|^2+\gamma_n\|x_n-q\|^2\\ &+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\ \leq& (1-(1-k^2)\alpha_n)\|x_n-q\|^2+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.\end{aligned}$$

By Step 5 and Lemma 2.4, $\{x_n\}$ converges strongly to q. Consequently, $\{u_n\}$ converges strongly to q. This completes the proof. □

If $A: C\to H$ is α-ism, then it is $\frac{1}{\alpha}-$ Lipschitzian. So, by the same argument in the proof of Theorem 3.1, we can proof the following Theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T: C\to C$ be a nonexpansive mapping, $\phi: C\times C\to\mathbb{R}$ be a bifunction satisfying the conditions $(A_1)-(A_4)$ (of Lemma 2.2), f be a contractions of C into itself with coefficient k, $A: C\to H$ be an α-ism, and $F:=EP(\phi)\bigcap VI(C,A)\bigcap F(T)\neq\emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(B_2) $\{\beta_n\} \subset (0,1)$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(B_3) $\{\gamma_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$;

(B_4) $\{r_n\} \subset (a,\infty)$ $(a > 0)$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n A)u_n + \gamma_n T P_C(I - \lambda_n A)x_n, \quad n \geq 1, \end{cases} \tag{3.9}$$

where $x_1 \in C$, $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_n\} \subset (0, 2\alpha)$, $u_n = Q_{r_n} x_n$, $P_C(I - \lambda_n A) = s_n I + (1 - s_n)O_n$, *and* $s_n = \frac{2\alpha - \lambda_n}{4\alpha}$. *Let* $\lim_{n\to\infty} s_n = 0$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$. *Then, the sequences* $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.9) *converge strongly to* $q \in F$, *where* $q = P_F f(q)$, *which solves the following variational inequality:*

$$\langle (I - f)q, q - x\rangle \leq 0 \ \text{for all } x \in F.$$

Remark 2. If T is a nonexpansive mapping in [5, Theorem 3.1], then Theorem 3.2 is a generalization of [5, Theorem 3.1] .

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, $T : C \to C$ *be a nonexpansive mapping,* f *be a contraction of* C *into itself with coefficient* k, $g : C \to \mathbb{R}$ *be a real-valued convex function,* ∇g *be an* $L-$*Lipschitzian mapping with* $L \geq 0$ *and,* $F := U \bigcap F(T) \neq \emptyset$. *Suppose* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(B_2) $\{\beta_n\} \subset [0,1)$, $0 < \limsup_{n\to\infty} \beta_n < 1$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(B_3) $\{\gamma_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$;

Let $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n \nabla g)x_n + \gamma_n T P_C(I - \lambda_n \nabla g)x_n, \quad n \geq 1,$$

where $x_1 \in C$, $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_n\} \subset (0, \frac{2}{L})$, $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n)O_n$, *and* $s_n = \frac{2 - \lambda_n L}{4}$. *Let* $\lim_{n\to\infty} s_n = 0$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q \in F$, *where* $q = P_F f(q)$, *which solves the following variational inequality:*

$$\langle (I - f)q, q - x\rangle \leq 0 \ \text{for all } x \in F.$$

Proof. Let $\phi = 0$ in Theorem 3.1. Then $u_n = P_C x_n$. Since $x_n \in C$ for all $n \geq 1$, we have $x_n = P_C x_n$. So $u_n = x_n$ and the desired result is directly obtained by Theorem 3.1. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, f be a contraction of C into itself with coefficient k, $g : C \to \mathbb{R}$ be a real-valued convex function and ∇g be an $L-$Lipschitzian mapping with $L \geq 0$, $U \neq \emptyset$, and $x_1 \in C$. Suppose $\{\alpha_n\}$ and $\{\lambda_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset [0,1], \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(B_2) $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C (I - \lambda_n \nabla g) x_n, \quad n \geq 1,$$

converges strongly to $q \in U$, where $q = P_U f(q)$, which solves the following variational inequality:

$$\langle (I - f)q, q - x \rangle \leq 0 \ \textit{for all}\ x \in U.$$

Remark 3. Corollary 3.4 remains true if we replace the condition $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$ with the condition $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$. So, Corollary 3.4 is a generalization of [20, Theorem 5.2] and therefore [20, Corollary 5.3].

Proof. We can assume $\lambda_{n_j} \to \lambda \in (0, \frac{2}{L})$. According to the Step 4 and the Step 5 in the proof of Theorem 3.1, it is suffices to show $\lim_{j\to\infty} \|x_{n_j} - V x_{n_j}\| = 0$, where $V := P_C(I - \lambda \nabla g)$. In fact

$$\begin{aligned}
\|V x_{n_j} - x_{n_j}\| \leq\ & \|V_{n_j} x_{n_j} - x_{n_j}\| + \|V_{n_j} x_{n_j} - V x_{n_j}\| \\
\leq\ & \|V_{n_j} x_{n_j} - x_{n_j+1}\| + \|x_{n_j} - x_{n_j+1}\| \\
& + \|P_C(I - \lambda_{n_j} \nabla g) x_{n_j} - P_C(I - \lambda \nabla g) x_{n_j}\| \\
\leq\ & \alpha_{n_j} \|f(x_{n_j}) - V_{n_j} x_{n_j}\| + \|x_{n_j} - x_{n_j+1}\| \\
& + \|(I - \lambda_{n_j} \nabla g) x_{n_j} - (I - \lambda \nabla g) x_{n_j}\| \\
\leq\ & \alpha_{n_j} \|f(x_{n_j}) - V_{n_j} x_{n_j}\| + \|x_{n_j} - x_{n_j+1}\| + M |\lambda_{n_j} - \lambda|.
\end{aligned}$$

Hence $\lim_{j\to\infty} \|x_{n_j} - V x_{n_j}\| = 0$ □

4 Numerical Test

In this section, we give an example to illustrate the scheme (3.1) given in Theorem 3.1.

Example 1. Let $C = [-10, 10] \subset H = \mathbb{R}$ and define $\phi(x, y) = -6x^2 + xy + 5y^2$. First, we verify that ϕ satisfies the conditions $(A_1) - (A_4)$ as follows:
(A_1) $\phi(x, x) = -6x^2 + x^2 + 5x^2 = 0$ for all $x \in [-10, 10]$;
(A_2) $\phi(x, y) + \phi(y, x) = -(y - x)^2 \leq 0$ for all $x, y \in [-10, 10]$;
(A_3) For all $x, y, z \in [-10, 10]$,

$$\limsup_{t\to 0^+} \phi(tz + (1-t)x, y) = \limsup_{t\to 0^+}(-6(tz + (1-t)x)^2 + (tz + (1-t)x)y + 5y^2) = \phi(x, y).$$

(A_4) For all $x \in [-10, 10]$, $\Phi(y) = \phi(x, y) = -6x^2 + xy + 5y^2$ is a lower semicontinuous and convex function.

From Lemma 2.3, Q_r is single-valued for all $r > 0$. Now, we deduce a formula for $Q_r(x)$. For any $y \in [-10, 10]$ and $r > 0$, we have

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Leftrightarrow 5ry^2 + ((r + 1)z - x)y + xz - (6r + 1)z^2 \geq 0.$$

Set $G(y) = 5ry^2 + ((r + 1)z - x)y + xz - (6r + 1)z^2$. Then $G(y)$ is a quadratic function of y with coefficients $a = 5r, b = (r + 1)z - x$ and $c = xz - (6r + 1)z^2$. So its discriminate $\Delta = b^2 - 4ac$ is

$$\begin{aligned} \Delta = & \ [(r + 1)z - x]^2 - 20r(xz - (6r + 1)z^2) \\ = & \ (r + 1)^2z^2 - 2(r + 1)xz + x^2 - 20rxz + (120r^2 + 20r)z^2 \\ = & \ [(11r + 1)z - x]^2. \end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C$, this is true if and only if $\Delta \leq 0$. That is, $[(11r + 1)z - x]^2 \leq 0$. Therefore, $z = \frac{x}{11r+1}$, which yields $Q_r(x) = \frac{x}{11r+1}$. So, from Lemma 2.3, we get $EP(\phi) = \{0\}$. Let $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{7n-2}{15n}$, $\gamma_n = \frac{8n-3}{15n}$, $\lambda_n = \frac{2n-1}{4n}$, $r_n = 1$ for all $n \in \mathbb{N}$, $Tx = \frac{1}{5}x$, $f(x) = \frac{1}{2}x$, and $g(x) = x^2$. Hence $U \bigcap EP(\phi) \bigcap F(T) = \{0\}$, ∇g is $4-$Lipschitzian, and $s_n = \frac{2-\lambda_n L}{4} = \frac{1}{4n}$. Also,

$$P_C(I - \lambda_n \nabla g)x = P_{[-10,10]}(x - \frac{2x(2n - 1)}{4n}) = P_{[-10,10]}(\frac{x}{2n}) = \frac{x}{2n}, \text{ for all } x \in [-10, 10].$$

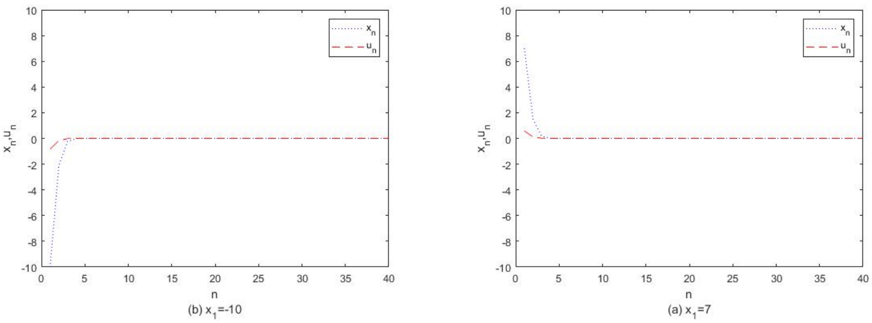

Figure 1: The convergence of $\{x_n\}$ and $\{u_n\}$ with different initial values x_1

Table 1: The values of the sequences $\{x_n\}$ and $\{u_n\}$

Numerical results for $x_1 = 7$ and $x_1 = -10$					
n	x_n	u_n	n	x_n	u_n
1	7	0.58333	1	-10	-0.83333
2	1.4972	0.12477	2	-2.1389	-0.17824
3	0.16969	0.01414	3	-0.24241	-0.020201
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	$5.2131e^{-30}$	$5.2131e^{-30}$	20	$-8.9368e^{-29}$	$-7.4473e^{-30}$
21	$6.208e^{-32}$	$6.208e^{-32}$	21	$-1.0642e^{-30}$	$-8.8685e^{-32}$
22	$7.0424e^{-34}$	$7.0424e^{-34}$	22	$-1.2073e^{-32}$	$-1.0061e^{-33}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
38	$3.4511e^{-66}$	$2.8759e^{-67}$	38	$-4.9301e^{-66}$	$-4.1084e^{-67}$
39	$2.1685e^{-68}$	$1.8071e^{-69}$	39	$-3.0978e^{-68}$	$-2.5815e^{-69}$
40	$1.3277e^{-70}$	$1.1064e^{-71}$	40	$-1.8967e^{-70}$	$-1.5806e^{-71}$

Then, from Lemma 2.4, the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by

$$\begin{cases} u_n = Q_{r_n}x_n = \frac{1}{12}x_n, \\ x_{n+1} = (\frac{1}{6n} + \frac{7n-2}{360n^2} + \frac{8n-3}{150n^2})x_n = \frac{431n-46}{1800n^2}x_n, \end{cases} \tag{4.1}$$

converge strongly to $0 \in U \bigcap EP(\phi) \bigcap F(T)$, where $0 = P_{U \bigcap EP(\phi) \bigcap F(T)}(f)(0)$.

The Table 1 indicates the values of sequences $\{x_n\}$ and $\{u_n\}$ for algorithm (4.1) where $x_1 = 7, x_1 = -10$, and $n = 40$. The Figure 1 presents the behavior of $\{x_n\}$ and $\{u_n\}$ that corresponds to the Table 1 and shows both of the sequences converge to $0 \in F$.

5 Concluding Remarks

The gradient-projection algorithm (GPA) plays an important role in solving constrained convex minimization problems. In this paper, with the help of the GPA and averaged mappings, we introduce a new iterative algorithm for finding a common element of the set of solutions of the equilibrium problem (1.1), the set of solutions of the constrained convex minimization problem (1.2), and the set of fixed points of a nonexpansive mapping. Then, we prove the sequences generated by the algorithm converge strongly to a common element of solution sets of these problems. Also, we derive some consequences from our main result. The results obtained in this paper, improve and extend the corresponding results of [5, 20]. Finally, we give a numerical example to justify the main result.

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