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Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions

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Abstract. This paper deals with the existence and uniqueness results for the solution of a Neutral fractional integro-differential problem with nonlocal conditions. Using the Nonlinear alternative for single valued maps, Krasnoselskii's and Banach fixed point theorems to proof our main results. An example is given to illustrate our main results.

Keywords. Neutral fractional integro-differential equation; existence; stability; nonlocal conditions; fixed point theory; single valued maps

1 Introduction and fractional calculus

Fractional calculus is a main mathematical branch investigates the properties of derivatives and integrals of non-integer orders. In particular, this discipline involves the concept and methods of solving of differential equations that include fractional derivatives of the unknown function. The history of fractional calculus began almost at the same time when classical calculus was established. For example in dynamics first derivative is rate or velocity: dx/dt or the second derivative is acceleration: d^2x/dt^2 but in some cases we see fractional differential equations such as $(d^\alpha x/dt^\alpha, \alpha \in (0,1))$. In mathematics there is no problem with this but in physics, it has meaning.
Fractional derivatives are non-local so the 1/2 derivative can not have a local meaning like tangent or curvature but would have to take into account the properties of the curve over a large extent (boundary conditions). The meaning for fractional (in time) derivative may change from one definition to the next. In the case of Riemann-Louiville and Caputo like fractional derivatives, the differential equations that involve them arise as models (in the limit) of a variety of stochastic processes with time delay, but there are also the consequence of some formalism in heat transfer phenomena that was proposed back in the 1960s.
Fractional differential equations (FDEs) has been applied widely in a variety sciences in control of dynamical systems, physical and biological sciences, see for details [1, 2, 13, 16, 19].
Nowadays, many researchers have given attention to the existence and uniqueness theory of nonlinear FDEs of various types for more information, for more information (see[4, 5, 6, 7, 8, 11, 12, 13, 15, 21, 22, 23]). For example In [15], M. S. Abdo et *al.* studied the Cauchy-type problem for a integro-differential equation of fractional order with nonlocal conditions in Banach space.

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There are concerned with the existence and uniqueness results for fractional integro-differential equations of the type

$$ {}^{C}D^{p}{}_{a^{+}}x(t) = h(x(t)) + f(t,x(t)) + \int_0^t K(t,s,x(s))ds, \quad t \in [a,b], \tag{1.1} $$

$$ x(a) = \sum_{j=1}^{m} c_j x(\tau_j), \quad \tau_j \in [a,b]. \tag{1.2} $$

A various classes of Neutral fractional differential equations have been taken into consideration by some authors, (see [10, 6]). For example in [10], the authors introduced and studied a related problem. Precisely the authors studied the existence for the following new problem :

$$ {}^{C}D^{p}\{{}^{C}D^{q}x(t) + f(t,x(t))\} = g(t,x(t)), \tag{1.3} $$

$$ x(0) = \sum_{j=1}^{m} \beta_j x(\sigma_j), \ bx(1) = a\int_0^1 x(s)dH(s) + \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \tag{1.4} $$

where

$$ 0 < \sigma_j < \xi_i < \eta_i < 1, \quad 0 < p,q < 1, $$
$$ \beta_j, \alpha_i \in \mathbb{R}, \quad i = 1,2,...,n, \quad j = 1,2,...,m. $$

${}^{C}D^{p}$ is the Caputo fractional derivative of order p, $f, g,$, are given continuous functions. The stability analysis is a central task in the study of FDEs and the stability analysis has been performed by many authors (see [9, 14, 17, 18, 20]). In [9], the authors studies the existence and stability results for a fractional order differential equation with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions.
In this paper we consider the following BVP of fractional differential problem

$$ {}^{C}D^{p}\{{}^{C}D^{q}x(t) + f(t,x(t))\} = g(t,x(t)) + \int_0^t K(t,s,x(s))ds, \tag{1.5} $$

$$ x(0) = \sum_{j=1}^{m} \beta_j x(\sigma_j), \qquad x(1) = \sum_{i=1}^{n} \alpha_i x(\xi_i), \tag{1.6} $$

where

$$ 0 < \sigma_j < \xi_i < 1, \quad 0 < p,q < 1, $$
$$ \beta_j, \alpha_i \in \mathbb{R}, \quad j = 1,2,...,m, \quad i = 1,2,...,n. $$

${}^{C}D^{p}, {}^{C}D^{q}$ are the Caputo fractional derivatives, f, g, K, are given functions with

$$ f, g \in C([0,1] \times \mathbb{R}, \mathbb{R}) \ \text{ and } \ K \in C(D \times \mathbb{R}, \mathbb{R}), $$

where

$$ D = \{(t,s) | \ \ t \in [0,1], s \in [0,1]\}. $$

We organize the paper as follows: In section 2, we state some basic concepts of fractional calculus, fixed point theorems and we also prove an auxiliary lemma which are used throughout this paper. Section 3, provide the proofs of the existence and uniqueness of solution to the problem (1.5) with nonlocal conditions (1.6) and the generalized Ulam stability is proved in section 4. Finally, an illustrative example is introduced in section 5.

2 Preliminaries and basic concepts

Here, we state some notations, definitions and auxiliary lemmas concerning fractional calculus and fixed point theorems. Let $J = [0, 1]$, X is Banach space equipped with the norm $\|.\|$ and $C(J, X)$, denotes the Banach space of all continuous bounded functions on J.
Some preliminary concepts of fractional calculus are stated here, see [23].

Definition 1. Let $q > 0$ and $\zeta : J \longrightarrow X$. The Riemann-Liouville fractional integral of order q of a function ζ is defined by

$$I_{0+}^{q}\zeta(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\zeta(s)ds, \quad t \in J.$$

Definition 2. Let $n-1 < q < n$, $(n \in \mathbb{N}^*)$ and $g \in C^n(J, X)$. The Caputo fractional derivative of order q of a function ζ is given by

$$\begin{aligned} {}^{C}D_{0+}^{q}\zeta(t) &= \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}\zeta^{(n)}(s)ds \\ &= I_{0+}^{n-q}\frac{d^n}{dt^n}\zeta(t), \quad t \in J. \end{aligned}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of the real number q.

Lemma 2.1. *[23] For real numbers $q, p > 0$ and appropriate function ζ, we have for all $t \in J$:*
1) $I_{0+}^{q}I_{0+}^{p}\zeta(t) = I_{0+}^{p}I_{0+}^{q}\zeta(t) = I_{0+}^{q+p}\zeta(t)$.

2) $I_{0+}^{q}\,{}^{C}D_{0+}^{q}\zeta(t) = \zeta(t) - \zeta(0), \quad 0 < q < 1.$

3) ${}^{C}D_{0+}^{q}I_{0+}^{q}\zeta(t) = \zeta(t)$.

Here, we mean by I^q and ${}^{C}D^q$, the fractional integral I_{0+}^{q} and fractional derivative ${}^{C}D_{0+}^{q}$ respectively.

Lemma 2.2. *[3] (Nonlinear alternative for single valued maps)*
Let E be a Banach space, C a closed, convex subset of E, U an open subset of C with boundary ∂U and $0 \in U$. Suppose that $F : \bar{U} \longrightarrow C$ is a continuous, compact map (that is, $F(\bar{U})$is a relatively compact subse
Then either
i) F has a fixed point in $\bar{U}$, or
ii) There is a $u \in \partial U$ and $\epsilon \in (0, 1)$ with $u = \epsilon F(u)$.

Lemma 2.3. *(Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that:*
(i) $Ax + By \in M$ whenever $x, y \in M$,
(ii) A is compact and continuous,
(iii) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.

The next auxiliary Lemma is useful.

Lemma 2.4. *Let $0 < q, p < 1$, assume that g, f and K are three continuous functions. If $x \in C(J, X)$ then x is solution of (1.5), (1.6) if and only if x satisfies the integral equation*

$$x(t) \;=\; \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s, x(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s, \tau, x(\tau))d\tau ds$$

$$
\begin{aligned}
&- \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \lambda_1(t)\Big[\sum_{i=1}^{n}\alpha_i\Big(\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds \\
&+ \int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau ds \\
&- \int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\Big) - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds \\
&- \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau ds + \int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\Big] \\
&+ \lambda_2(t)\sum_{j=1}^{m}\beta_j\Big[\int_0^{\sigma_j}\Big(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}f(s,x(s)) - \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s)) \\
&- \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau\Big)ds\Big], \qquad (2.1)
\end{aligned}
$$

where

$$\lambda_1(t) = \frac{1}{k}(\rho_1 - \frac{\rho_2 t^q}{\Gamma(q+1)}), \quad \lambda_2(t) = \frac{1}{k}(\rho_3 - \frac{\rho_4 t^q}{\Gamma(q+1)}), \qquad (2.2)$$

and

$$\rho_1 = \sum_{j=1}^{m}\frac{\beta_j\sigma_j^q}{\Gamma(q+1)}, \quad \rho_2 = -1 + \sum_{j=1}^{m}\beta_j, \qquad (2.3)$$

and

$$\rho_3 = \frac{1}{\Gamma(q+1)} - \sum_{i=1}^{n}\frac{\alpha_i\xi_i^q}{\Gamma(q+1)}, \rho_4 = 1 - \sum_{i=1}^{n}\alpha_i, \qquad (2.4)$$

where

$$k = \rho_2\rho_3 - \rho_1\rho_4 \neq 0. \qquad (2.5)$$

Proof. We apply the operator I^p on (1.5), to obtain

$${}^C D^q x(t) + f(t,x(t)) - c_0 = I^p g(t,x(t)) + I^p \int_0^t K(t,s,x(s))ds.$$

By applying the operator I^q on both sides of the last equation,we find

$$x(t) - c_1 + I^q f(t,x(t)) - I^q(c_0) = I^{q+p} g(t,x(t)) + I^{q+p}\int_0^t K(t,s,x(s))ds.$$

That is

$$
\begin{aligned}
x(t) &= -I^q f(t,x(t)) + I^{q+p} g(t,x(t)) \\
&+ I^{q+p}\int_0^t K(t,s,x(s))ds + c_1 + c_0\frac{t^q}{\Gamma(q+1)},
\end{aligned}
$$

where c_0, c_1 are two constants. By (1.6) and (2.6), we get

$$c_0(\sum_{j=1}^{m}\frac{\beta_j\sigma_j^q}{\Gamma(q+1)}) + c_1(\sum_{j=1}^{m}\beta_j - 1) = I_1, \qquad (2.6)$$

and

$$c_0\left(\frac{1}{\Gamma(q+1)}-\sum_{i=1}^{n}\frac{\alpha_i\xi_i^q}{\Gamma(q+1)}\right)+c_1\left(1-\sum_{i=1}^{n}\alpha_i\right)=I_2. \tag{2.7}$$

By using (2.3) and (2.4) in (2.6) and (2.7), we find

$$\begin{cases} \rho_1c_0+\rho_2c_1=I_1, \\ \rho_3c_0+\rho_4c_1=I_2, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} I_1 &= \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j}\Bigg(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}f(s,x(s))-\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s)) \\ &- \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau\Bigg)ds, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} I_2 &= \sum_{i=1}^{n}\alpha_i\int_0^{\xi_i}\Bigg(\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))-\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}f(s,x(s)) \\ &+ \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau\Bigg)ds \\ &+ \int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds \\ &- \int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau ds. \end{aligned} \tag{2.10}$$

Solving the system (2.8) for c_0, c_1 and $k=\rho_2\rho_3-\rho_1\rho_4\neq 0$, we obtain

$$c_0=\frac{\rho_2I_2-I_1\rho_4}{k}, \quad c_1=\frac{\rho_3I_1-\rho_1I_2}{k}.$$

Substituting c_0, c_1 in (2.6), we get

$$\begin{aligned} x(t) &= \int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau ds \\ &- \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+c_1+c_0\frac{t^q}{\Gamma(q+1)} \\ &= \int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,x(s))ds+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,x(\tau))d\tau ds \\ &- \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\lambda_1(t)I_2+\lambda_2(t)I_1. \end{aligned}$$

By the definition of I^q and I^{q+p} we find the solution (2.1).
Conversely, by Lemma 2.1 and by applying the operator ${}^CD^{q+p}$ on both sides of (2.6), we find

$${}^CD^{q+p}x(t) = {}^CD^{q+p}\Big[-I^qf(t,x(t))+I^{q+p}g(t,x(t))+I^{q+p}\int_0^t K(t,s,x(s))ds$$

$$
\begin{aligned}
&+ \quad c_1 + c_0\frac{t^q}{\Gamma(q+1)}\Big]\\
&= \quad -{}^C D^p f(t,x(t)) + g(t,x(t)) + \int_0^t K(t,s,x(s))ds\\
&+ \quad {}^c D^{q+p}\Big(c_1 + c_0\frac{t^q}{\Gamma(q+1)}\Big)\\
&= \quad -{}^C D^p f(t,x(t)) + g(t,x(t)) + \int_0^t K(t,s,x(s))ds.
\end{aligned}
$$

This means that x satisfies (1.5), (1.6). Furthermore, by substituting t by 0 then by 1 in (2.1), we conclude that the boundary conditions in (1.6) hold. Therefore, x is solution of problem (1.5)-(1.6). □

3 Existence and Uniqueness Result via a different appraochs

We are going to prove the existence and uniqueness result of (1.5), (1.6) in $C(J,\mathbb{R})$ by fixed point Theorems. For this end, we transform (1.5)-(1.6) into fixed point problem as $x = Tx$, where the operator

$$T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R}),$$

is defined by

$$
\begin{aligned}
Tx(t) \quad &= \quad \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds\\
&- \quad \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \lambda_1(t)\Big[\sum_{i=1}^n \alpha_i\Big(\int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds\\
&+ \quad \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds\\
&- \quad \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big) - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds\\
&- \quad \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big]\\
&+ \quad \lambda_2(t)\sum_{j=1}^m \beta_j\Big[\int_0^{\sigma_j}\Big(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)} f(s,x(s)) - \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))\\
&- \quad \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau\Big)ds\Big].
\end{aligned}
\tag{3.1}
$$

We set

$$
\begin{aligned}
\Lambda \quad &= \quad \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)}\\
&+ \quad \bar{\lambda}_1\Big[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)}\\
&+ \quad \sum_{i=1}^n |\alpha_i|\Big(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big]
\end{aligned}
$$

$$+\quad \bar{\lambda}_2\sum_{j=1}^{m}|\beta_j|\left(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{p+q}}{\Gamma(q+p+1)}+\frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\right), \tag{3.2}$$

and

$$\Lambda_1=\Lambda-\left\{\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}\right\}, \tag{3.3}$$

where

$$\bar{\lambda}_1=\frac{1}{|\,k\,|}\left(\,|\,\rho_1\,|+\frac{|\,\rho_2\,|}{\Gamma(q+1)}\right),\quad \bar{\lambda}_2=\frac{1}{|\,k\,|}\left(\,|\,\rho_3\,|+\frac{|\,\rho_4\,|}{\Gamma(q+1)}\right). \tag{3.4}$$

In the next, we present the main results.

3.1 Existence Result by using Leray-Schauder Nonlinear Alternative

Theorem 3.1. *Let* $f,g\in C([0,1]\times\mathbb{R},\mathbb{R})$ *and* $K\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ *be continuous functions. Assume that*
(H1): There exist functions $p_1,p_2\in C([0,1],\mathbb{R}^+)$, $p_3\in C([0,1]\times[0,1],\mathbb{R}^+)$, *with* $p=\max\{p_1,p_2,p_3\}$ *and nondecreasing functions*

$$\psi_1,\psi_2,\psi_3:\mathbb{R}^+\to\mathbb{R}^+,$$

with $\psi=\max\{\psi_1,\psi_2,\psi_3\}$ *such that*

$$|f(t,x(t))|\le p_1(t)\psi_1(\|x\|),$$

$$|g(t,x(t))|\le p_2(t)\psi_2(\|x\|),$$

and

$$|K(t,s,x(s))|\le p_3(t,s)\psi_3(\|x\|),$$

for all $t\in[0,1], s\in[0,1], x\in\mathbb{R}$.
(H2): There exist a constant $M>0$ *such that* $\frac{M}{\|p\|\psi(M)\Lambda}>1$.
Then the problem (1.5)-(1.6) admits at least one solution on $[0,1]$.

Proof. For $r>0$, let

$$B_r=\{x\in C([0,1],\mathbb{R}):\|x\|\le r\},$$

be a bounded set in $C([0,1],\mathbb{R})$. We will show that T defined by (3.1) maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. Then, by (H1), we have

$$\begin{aligned}|(Tx)(t)| &\le \|p_1\|\psi_1(\|x\|)\sup_{t\in[0,1]}\Bigg[\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds\\ &+ |\lambda_1(t)|\left(\sum_{i=1}^{n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}ds+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\right)\\ &+ |\lambda_2(t)|\sum_{i=1}^{n}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}ds\Bigg)\Bigg]\\ &+ \|p_2\|\psi_2(\|x\|)\sup_{t\in[0,1]}\Bigg[\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}ds\end{aligned}$$

$$
\begin{aligned}
&+ \quad |\lambda_1(t)|\bigg(\sum_{i=1}^{n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}ds\\
&+ \quad \int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}ds\bigg)+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}ds\bigg]\\
&+ \quad \|p_3\|\psi_3(\|x\|)\sup_{t\in[0,1]}\bigg[|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds\\
&+ \quad |\lambda_1(t)|\bigg(\sum_{i=1}^{n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds\\
&+ \quad \int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds\bigg)+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^t d\tau ds\bigg]. \qquad (3.5)
\end{aligned}
$$

Then by some calculations, we get

$$
\begin{aligned}
|(Tx)(t)| \quad &\leq \quad \|p_1\|\psi_1(\|x\|)\sup_{t\in[0,1]}\bigg[\frac{t^q}{\Gamma(q+1)}+|\lambda_1(t)|\bigg(\sum_{i=1}^{n}|\alpha_i|\frac{\xi_i^q}{\Gamma(q+1)}\\
&+ \quad \frac{1}{\Gamma(q+1)}\bigg)+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\frac{\sigma_j^q}{\Gamma(q+1)}\bigg]\\
&+ \quad \|p_2\|\psi_2(\|x\|)\sup_{t\in[0,1]}\bigg[\frac{t^{q+p}}{\Gamma(q+p+1)}+|\lambda_2(t)|\sum_{i=1}^{n}|\beta_j|\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)}\\
&+ \quad |\lambda_1(t)|\bigg(\sum_{i=1}^{n}|\alpha_i|\frac{\xi_i^{q+p}}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+1)}\bigg)\bigg]\\
&+ \quad \|p_3\|\psi_3(\|x\|)\sup_{t\in[0,1]}\bigg[\frac{t^{q+p+1}}{\Gamma(q+p+2)}+|\lambda_1(t)|\bigg(\sum_{i=1}^{n}|\alpha_i|\frac{\xi_i^{q+p+1}}{\Gamma(q+p+2)}\\
&+ \quad \frac{t^{q+p-1}}{\Gamma(q+p+2)}\bigg)+|\lambda_2(t)|\sum_{j=1}^{m}|\beta_j|\frac{\sigma_j^{q+p+1}}{\Gamma(q+p+2)}\bigg]\\
&\leq \quad \|p\|\psi(\|r\|)\bigg[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}\\
&+ \quad \bar{\lambda}_1\bigg\{\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}\\
&+ \quad \sum_{i=1}^{n}|\alpha_i|\bigg(\frac{\xi_i^q}{\Gamma(q+1)}+\frac{\xi_i^{p+q}}{\Gamma(q+p+1)}+\frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\bigg)\bigg\}\\
&+ \quad \bar{\lambda}_2\sum_{j=1}^{m}|\beta_j|\bigg(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{p+q}}{\Gamma(q+p+1)}+\frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\bigg)\bigg].
\end{aligned}
$$

Thus, using (3.2), we get

$$\|Tx\| \leq \|p\|\psi(\|x\|)\Lambda \leq \|p\|\psi(r)\Lambda.$$

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$|Tx(t_2)-Tx(t_1)| \quad \leq \quad \bigg|\int_0^{t_1}\frac{(t_1-s)^{q-1}-(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\bigg|$$

$$
\begin{aligned}
&+ \quad \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds\Big| \\
&+ \quad \Big|\int_0^{t_1} \frac{(t_1-s)^{q+p-1}-(t_2-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds\Big| \\
&+ \quad \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big| \\
&+ \quad \Big|\int_0^{t_1} \frac{(t_1-s)^{q+p-1}-(t_2-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau))|d\tau ds\Big| \\
&+ \quad \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau))|d\tau ds\Big| \qquad (3.6) \\
&+ \quad |\lambda_1(t_2)-\lambda_1(t_1)|\Big[\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds \\
&+ \quad \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&+ \quad \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau))|d\tau ds \\
&+ \quad \sum_{i=1}^{n}|\alpha_i|\Big(\int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&+ \quad \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau))|d\tau ds \\
&+ \quad \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds\Big)\Big] \\
&+ \quad |\lambda_2(t_2)-\lambda_2(t_1)|\sum_{j=1}^{m}|\beta_j|\Big[\int_0^{\sigma_j} \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau))|d\tau ds \\
&+ \quad \int_0^{\sigma_j} \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s))|ds + \int_0^{\sigma_j} \frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big] \\
&\leq \quad \|p\|\psi(\|r\|)\Big[\frac{|t_1^{q+p}-t_2^{q+p}|+2(t_2-t_1)^{q+p}}{\Gamma(q+p+1)} + \frac{|t_1^q-t_2^q|+2(t_2-t_1)^q}{\Gamma(q+1)} \\
&+ \quad \frac{|t_1^{q+p+1}-t_2^{q+p+1}|+2(t_2-t_1)^{q+p+1}}{\Gamma(q+p+2)} + \frac{2t_1(t_2-t_1)^{q+p}}{\Gamma(q+p+1)} \\
&+ \quad \Big|\frac{\rho_4(t_2^q-t_1^q)}{k\Gamma(q+1)}\Big|\sum_{j=1}^{m}|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{p+q}}{\Gamma(q+p+1)}+\frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Big) \\
&+ \quad \Big|\frac{\rho_2(t_2^q-t_1^q)}{k\Gamma(q+1)}\Big| \ \Big\{\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)} \\
&+ \quad \sum_{i=1}^{n}|\alpha_i|\Big(\frac{\xi_i^q}{\Gamma(q+1)}+\frac{\xi_i^{p+q}}{\Gamma(q+p+1)}+\frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big\}\Big].
\end{aligned}
$$

If $(t_2-t_1) \longrightarrow 0$, then the RHS of the above inequality tends to zero independently of $x \in B_r$. That is implies

$$\|Tx(t_2)-Tx(t_1)\| \longrightarrow 0, \quad \text{if} \quad (t_2-t_1) \longrightarrow 0,$$

then T maps bounded sets into equi-continuous sets of C.

By Arzela-Ascoli theorem, we have $T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is completely continuous. We will apply the Leray-schauder nonlinear alternative once we establish the boundedness of the set of all solutions to equation

$$x = \epsilon T x \quad \text{for} \quad \epsilon \in (0,1).$$

Let x be a solution of (1.5)-(1.6), then, we will prove the the boundedness of the operator T. We have

$$\begin{aligned} |x(t)| &\leq \|p\|\psi(\|x\|)\Bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \bar{\lambda}_1\Bigg\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \sum_{i=1}^{n}|\alpha_i|\Bigg(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg\} \\ &+ \bar{\lambda}_2\sum_{j=1}^{m}|\beta_j|\Bigg(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg] \\ &\leq |p\|\psi(\|x\|)\Lambda, \end{aligned} \tag{3.7}$$

which implies

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Lambda} \leq 1.$$

Then by (H2), there exist $M > 0$ such that $M \neq \|x\|$. Let us define a set

$$Y = \{x \in C([0,1],\mathbb{R}) / \ \ \|x\| < M\},$$

and then

$$T : \bar{Y} \longrightarrow C([0,1],\mathbb{R}),$$

is completely continuous. From the choice of Y, there is no $x \in \partial Y$ such that

$$x = \epsilon T x \ \ \text{for} \ \ \epsilon \in (0,1).$$

Then by the nonlinear Leray-Schauder type, we conclude that the operator T has a fixed point $x \in \bar{Y}$ which is solution of the BVP (1.5)-(1.6). □

3.2 Existence result by Krasnoselskii's Fixed Point

Theorem 3.2. *Let f, g, K be continuous functions satisfying*
(H3) The inequqlities

$$|f(t,x(t)) - f(t,y(t))| \leq L_1|x-y|,$$
$$|g(t,x(t)) - g(t,y(t))| \leq L_2|x-y|,$$

and

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq L_3|x-y|,$$

with $L = \max\{L_1, L_2, L_3\}$ and $L \leq \frac{1}{\Lambda_1}$, where Λ_1 is given by (3.3).
(H4) The inequqlities

$$|f(t,x(t))| \leq \mu_1(t),$$
$$|g(t,x(t))| \leq \mu_2(t),$$

$$|K(t,s,x(s))| \leq \mu_3(t,s),$$
$$\forall (t,s,x) \in [0,1]\times[0,1]\times\mathbb{R}, \quad \mu_1,\mu_2 \in C([0,1],\mathbb{R}^+), \mu_3 \in C([0,1],\mathbb{R}^+),$$

and

$$\mu = \max\{\mu_1,\mu_2,\mu_3\}.$$

Then the BVP (1.5)-(1.6) has at a least one solution on $[0,1]$.

Proof. We fix $\bar{r} \geq \Lambda\|\mu\|$ and consider the closed ball

$$B_{\bar{r}} = \{x \in C, \|x\| \leq \bar{r}\}.$$

Next, let us define the operators T_1, T_2 on $B_{\bar{r}}$ as follows

$$\begin{aligned} T_1x(t) &= \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+ \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds, \end{aligned}$$

and

$$\begin{aligned} T_2x(t) &= -\lambda_1(t)\Bigg[\sum_{i=1}^n \alpha_i \Bigg(\int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds \\ &+ \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds - \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \Bigg) \\ &- \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s))ds - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau ds \\ &+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \Bigg] + \lambda_2(t) \sum_{j=1}^m \beta_j \int_0^{\sigma_j} \Bigg(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)} f(s,x(s)) \\ &- \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} g(s,x(s)) - \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,x(\tau))d\tau \Bigg) ds. \end{aligned}$$

For $x, y \in B_{\bar{r}}$ and $t \in [0,1]$, and by the assumption (H4) we find

$$\begin{aligned} \|T_1x + T_2y\| &= \sup_{t\in[0,1]} |T_1x + T_2y| \\ &\leq \|\mu\| \Bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \bar{\lambda}_1 \Bigg\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \sum_{i=1}^n |\alpha_i| \Bigg(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)} \Bigg) \Bigg\} \\ &+ \bar{\lambda}_2 \sum_{j=1}^m |\beta_j| \Bigg(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)} \Bigg) \Bigg]. \end{aligned}$$

Then, we obtain,

$$\|T_1x + T_2y\| \leq \|\mu\|\Lambda \leq \bar{r}.$$

This implies that $(T_1x + T_2y) \in B_{\bar{r}}$.
We establish now that T_2 is a contraction for $x, y \in C([0,1], \mathbb{R})$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|T_2x - T_2y\| &= \sup_{t\in[0,1]} |T_2x - T_2y| \\
&\le L\|x-y\| \Big[\bar{\lambda}_1 \Big\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
&+ \sum_{i=1}^{n} |\alpha_i| \Big(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)} \Big) \Big\} \\
&+ \bar{\lambda}_2 \sum_{j=1}^{m} |\beta_j| \Big(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)} \Big) \Big] \\
&= L\Lambda_1 \|x-y\|.
\end{aligned}$$

Then since $L\Lambda_1 \le 1, T_2$ is a contraction mapping. By the continuity of g, f, k we imply that T_1 is continuous. Also, T_1 is uniformly bounded on $B_{\bar{r}}$ as

$$\begin{aligned}
\|(T_1x)(t)\| &= \sup_{t\in[0,1]} |T_1x(t)| \\
&\le \sup_{t\in[0,1]} \int_0^t \Big(- \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| + \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} |g(s,x(s))| \Big) ds \\
&+ \sup_{(t,s)\in D} \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s |K(\tau,s,x(s))| d\tau ds \\
&\le \|\mu\| \Big[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \Big].
\end{aligned}$$

Finally, by (H3), the compactness of the operator T_1 is proved, we define

$$\bar{f} = \sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |f(t,x)|, \bar{g} = \sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |g(t,x)|, \bar{K} = \sup_{((t,s),x)\in D\times B_{\bar{r}}} |K(t,s,x)|.$$

Then, for $0 \le \tau_1 \le \tau_2 \le 1$, we obtain

$$\begin{aligned}
|(T_1x)(\tau_1) - (T_1x)(\tau_2)| &\le \frac{|\tau_1^q - \tau_2^q| + 2(\tau_1-\tau_2)^q}{\Gamma(q+1)} \bar{f} + \frac{|\tau_1^{q+p} - \tau_2^{q+p}| + 2(\tau_1-\tau_2)^{q+p}}{\Gamma(q+p+1)} \bar{g} \\
&+ \frac{|\tau_1^{q+p+1} - \tau_2^{q+p+1}| + 2(\tau_1-\tau_2)^{q+p+1}}{\Gamma(q+p+2)} \bar{K},
\end{aligned}$$

as $\tau_1 - \tau_2 \longrightarrow 0$, independent of x, thus T_1 is relatively compact on $B_{\bar{r}}$.
Hence, by the Arzela-Ascoli theorem, T_1 is compact on $B_{\bar{r}}$. Thus the hypothesis of Lemma 2.3 hold, that is the problem (1.5)-(1.6) has at least one solution on $[0,1]$. □

3.3 Existence and Uniqueness Result via Banach fixed point

Theorem 3.3. *Assume that f, g, K are continuous functions satisfy the assumption (H3). Then the BVP (1.5)-(1.6) has a unique solutions on $[0,1]$ if $L\Lambda < 1$.*

Proof. Define $M = \max\{M_1, M_2, M_3\}$, where M_1, M_2, M_3 are positive numbers such that

$$\sup_{t\in[0,1]} |f(t,0)| = M_1,$$

$$\sup_{t\in[0,1]} |g(t,0)| = M_2,$$

$$\sup_{(t,s)\in D} |K(t,s,0)| = M_3.$$

Fixing $r \geq \frac{M\Lambda}{1-L\Lambda}$, we consider

$$B_r = \{x \in C; \|x\| \leq r\}.$$

Then, by (H3), we get

$$\begin{aligned} |f(t,x(t))| &= |f(t,x(t)) - f(t,0) + f(t,0)| \\ &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| \\ &\leq L_1\|x\| + M_1, \end{aligned}$$

and

$$\begin{aligned} |g(t,x(t))| &= |g(t,x(t)) - g(t,0) + g(t,0)| \\ &\leq |g(t,x(t)) - g(t,0)| + |g(t,0)| \\ &\leq L_2\|x\| + M_2, \end{aligned}$$

and

$$\begin{aligned} |K(t,s,x(s))| &\leq |K(t,s,x(s)) - K(t,s,x(0))| + |K(t,s,x(0))| \\ &\leq L_3\|x\| + M_3. \end{aligned}$$

We will show that $TB_r \subset B_r$. For any $x \in B_r$, we have

$$\begin{aligned} \|Tx\| &= \sup_{t\in[0,1]} |Tx| \\ &\leq (Lr+M)\Bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} + \bar{\lambda}_1\Bigg\{\frac{1}{\Gamma(q+1)} \\ &+ \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} + \sum_{i=1}^{n}|\alpha_i|\Bigg(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} \\ &+ \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg\} + \bar{\lambda}_2\sum_{j=1}^{m}|\beta_j|\Bigg(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg] \\ &= (Lr+M)\Lambda \ \leq \ r. \end{aligned}$$

This implies that $TB_r \subset B_r$.
Now, for $x, y \in C([0,1],\mathbb{R})$ and for all $t \in [0,1]$, we have

$$\begin{aligned} \|Tx(t) - Ty(t)\| &= \sup\Bigg[\int_s^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau)) - K(s,\tau,y(\tau))|d\tau ds \\ &+ \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}|g(s,x(s)) - g(s,y(s))|ds \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds \\ &+ |\lambda_1(t)|\Bigg\{\sum_{i=1}^{n}\alpha_i\int_0^{\xi_i}\Bigg(\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,x(\tau)) - K(s,\tau,y(\tau))|d\tau \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{(\xi_i - s)^{q+p-1}}{\Gamma(q+p)} |g(s, x(s)) - g(s, y(s))| \\
&+ \quad \frac{(\xi_i - s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, y(s))| \Big) ds \\
&+ \quad \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, y(s))| ds \\
&+ \quad \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} |g(s, x(s)) - g(s, y(s))| ds \\
&+ \quad \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s |K(s, \tau, x(\tau)) - K(s, \tau, y(\tau))| d\tau ds \Big\} \\
&+ \quad |\lambda_2(t)| \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} \Big(\frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s |K(s, \tau, x(\tau)) - K(s, \tau, y(\tau))| d\tau \\
&+ \quad \frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} |f(s, x(s)) - f(s, x(s))| \\
&+ \quad \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} |g(s, x(s)) - g(s, y(s))| \Big) ds \Big] \\
&\leq \quad L\Lambda \|x - y\|. \qquad (3.8)
\end{aligned}
$$

For $L\Lambda < 1$, it follows by (3.8) and Banach fixed point Theorem that the operator T is a contraction, then there exists one solution of (1.5)-(1.6). □

4 Generalized Ulam Stabilities

We will discuss the Ulam stability for (1.5), (1.6) by using the integration.

$$
\begin{aligned}
y(t) \quad = \quad & \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s, x(s)) ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s, \tau, x(\tau)) d\tau ds \\
- \quad & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \lambda_1(t) \Big[\sum_{i=1}^{n} \alpha_i \Big(\int_0^{\xi_i} \frac{(\xi_i - s)^{q+p-1}}{\Gamma(q+p)} g(s, x(s)) ds \\
+ \quad & \int_0^{\xi_i} \frac{(\xi_i - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s, \tau, x(\tau)) d\tau ds - \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \Big) \\
- \quad & \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s, x(s)) ds - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s, \tau, x(\tau)) d\tau ds \\
+ \quad & \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \Big] + \lambda_2(t) \sum_{j=1}^{m} \beta_j \Big[\int_0^{\sigma_j} \Big(\frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f(s, x(s)) \\
- \quad & \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g(s, x(s)) - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s, \tau, x(\tau)) d\tau \Big) ds \Big]. \qquad (4.1)
\end{aligned}
$$

Here $y \in C([0,1], \mathbb{R})$ possesses a fractional derivative of order $p + q$, where $0 < p, q < 1$ and

$$
f, g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},
$$

and

$$
K : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},
$$

are continuous functions. Then, we define the nonlinear continuous operator

$$G : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R}),$$

as follows

$$Gy(t) = {}^C D^{p+q}y(t) + {}^C D^p f(t,x(t)) - g(t,x(t)) - \int_0^t K(t,s,x(s))ds.$$

Definition 3. For each $\epsilon > 0$ and for each solution y of (1.5), (1.6) such that

$$\|Gy\| \leq \epsilon, \tag{4.2}$$

the problems (1.5), (1.6) is said to be Ulam-Hyers stable if we can find a positive real number ν and a solution $x \in C([0,1],R)$ of (1.5), (1.6) satisfying the inequality:

$$\|x - y\| \leq \nu\epsilon^*, \tag{4.3}$$

where ϵ^* is a positive real number depending on ϵ.

Definition 4. Let $m \in C(\mathbb{R}^+,\mathbb{R}^+)$ such that for each solution y of (1.5), (1.6), we can find a solution $x \in C([0,1],\mathbb{R})$ of (1.5), (1.6) such that

$$|x(t) - y(t)| \leq m(\epsilon), t \in [0,1]. \tag{4.4}$$

Then the problems (1.5), (1.6) is said to be generalized Ulam-Hyers stable

Definition 5. For each $\epsilon > 0$ and for each solution y of (1.5), (1.6), the problems (1.5), (1.6) is called Ulam-Hyers-Rassias stable with respect to $\eta \in C([0,1],\mathbb{R}^+)$ if

$$|Gy(t)| \leq \epsilon\eta(t), t \in [0,1], \tag{4.5}$$

and there exist a real number $\nu > 0$ and a solution $x \in C([0,1],\mathbb{R})$ of (1.5), (1.6) such that

$$|x(t) - y(t)| \leq \nu\epsilon_*\eta(t), t \in [0,1], \tag{4.6}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Under assumption (H3) in Theorem 3.2, with $L\Lambda < 1$. The problems (1.5), (1.6) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $x \in C([0,1],\mathbb{R})$ be a solution of (1.5), (1.6) satisfying (3.6) in the sens of Theorem 3.3. Let y be any solution satisfying (4.2). Furthermore, the equivalence in Lemma 2.4 implies the equivalence between the operators G and $T - Id$ (where Id is the identity operator) for every solution $y \in C([0,1],\mathbb{R})$ of (1.5) and (1.6) satisfying $L\Lambda < 1$. Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned} |y(t) - x(t)| &= |y(t) - Ty(t) + Ty(t) - x(t)| \\ &= |y(t) - Ty(t) + Ty(t) - Tx(t)| \\ &\leq |Ty(t) - Tx(t)| + |Ty(t) - y(t)| \\ &\leq L\Lambda\|x - y\| + \epsilon, \end{aligned} \tag{4.7}$$

because $L\Lambda < 1$ and $\epsilon > 0$, we find

$$\|x - y\| < \frac{\epsilon}{1 - L\Lambda}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-L\Lambda}$ and $\nu = 1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m(\epsilon) = \frac{\epsilon}{1-L\Lambda}$. ☐

Theorem 4.2. *Assume that (H3) holds with $L < \lambda^{-1}$, and there exists a function $\eta \in C([0,1],\mathbb{R}^+)$ satisfying the condition (4.5). Then the problems (1.5), (1.6) is Ulam-Hyers-Rassias stable with respect to η.*

Proof. We have from the proof of Theorem 4.1,

$$|x(t) - y(t)| \leq \epsilon_* \eta(t), t \in [0,1],$$

where $\epsilon_* = \frac{\epsilon}{1-L\Lambda}$, this completes the proof. □

5 Example

Let us consider

$$^C D^{\frac{2}{3}}\{^C D^{\frac{2}{3}} x(t) + f(t,x(t))\} = g(t,x(t)) + \int_0^t K(t,s,x(s))ds, \tag{5.1}$$

$$x(0) = \sum_{j=1}^{j=2} \beta_j x(\sigma_j),\ x(1) = \sum_{i=1}^{i=2} \alpha_i x(\xi_i), \tag{5.2}$$

where

$$p = q = \frac{2}{3};\quad \sigma_1 = \frac{1}{2};\quad \sigma_2 = \frac{1}{3};\quad \xi_1 = \frac{2}{3};\quad \xi_1 = \frac{7}{9},$$
$$\alpha_1 = 2,\qquad \alpha_2 = -5;\quad \beta_1 = 3;\quad \beta_2 = 4.$$

The functions $f(t,x(t))$, $g(t,x(t))$ and $K(t,s,x(t))$ will be fixed later.
We then find that $\Lambda = 29,0059$ and $\Lambda_1 = 26.6923$.
We are going now to illustrate Theorem 3.1, for this end, we take

$$\begin{aligned} f(t,x(t)) &= \left(\frac{\cos t}{94} + \frac{e^{-t}}{t^2 + 94e^{-t}}\right)x, \\ g(t,x(t)) &= \frac{\sin t}{35 + e^t}\left(\frac{|x|}{1 + \|x\|} + \cos x\right), \\ K(t,s,x(t)) &= \frac{e^{s-t-1}}{64}\left(x + 2e^{-|x|}\right). \end{aligned} \tag{5.3}$$

Clearly

$$|f(t,x(t))| \leq \left(\frac{|\cos t|}{94} + \frac{e^{-t}}{t^2 + 94e^{-t}}\right)\|x\| \leq \left(\frac{1}{94} + \frac{e^{-t}}{t^2 + 94e^{-t}}\right)\|x\|,$$

$$|g(t,x(t))| \leq \frac{|\sin t|}{35 + e^t}\left(\frac{\|x\|}{1 + \|x\|} + |\cos x|\right) \leq \frac{1}{35 + e^t}\left(1 + \|x\|\right),$$

$$|K(t,s,x(t))| \leq \frac{e^{s-t-1}}{64}\left(\|x\| + 2e^{-\|x\|}\right) \leq \frac{e^{s-t-1}}{64}\left(\|x\| + 2\right),$$

with

$$P_1(t) = \frac{1}{94} + \frac{e^{-t}}{t^2 + 94e^{-t}}; \|P_1\| = \frac{1}{47};$$

$$P_2(t) = \frac{1}{35+e^t}; \|P_2\| = \frac{1}{36};$$
$$P_3(t,s) = \frac{e^{s-t-1}}{64}; \|P_3\| = \frac{1}{64},$$
$$\psi_1(\|x\|) = \|x\|;\ \psi_2(\|x\|) = \|x\|+2;\ \psi_3(\|x\|) = 2+\|x\|.$$
$$P = \max\left\{\frac{1}{47}, \frac{1}{36}, \frac{1}{64}\right\} = \frac{1}{36};$$
$$\psi = \max\left\{\|x\|,\quad 1+\|x\|,\quad 2+\|x\|\right\} = 2+\|x\|.$$

By (H2), we find $M > \|P\|\psi(M)\Lambda = 8.2935$.
Since all the conditions of Theorem 3.1 are satisfied, there exists at least one solution on $[0,1]$ for the problem (5.1)-(5.2) with the functions are given by (5.3).
We illustrate Theorem 3.2, for this end, we take

$$f(t,x(t)) = \frac{\sin x}{42}, +e^{-t}\cos t,$$
$$K(t,s,x(t)) = \frac{e^{s-t}}{48}\cos x,$$
$$g(t,x(t)) = \frac{|x|}{64(1+|x|)} + 6t.$$

Note that $L_1 = \frac{1}{42},\ L_2 = \frac{1}{48},\ L_3 = \frac{1}{64}$.
Moreover,

$$|f(t,x(t))| = \frac{|\sin x|}{42} + e^{-t}|\cos t| \le \frac{1}{42} + e^{-t}|\cos t| = \mu_1(t),$$
$$|K(t,s,x(t))| \le \frac{e^s}{48} = \mu_3(s),$$
$$|g(t,x(t))| \le \frac{1}{64} + 6t = \mu_2(t).$$

Obviously, $\|\mu_1\| = \frac{43}{42};\quad \|\mu_2\| = \frac{385}{64};\quad \|\mu_3\| = \frac{e}{48};$
and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{42},$$
$$\|\mu\| = \max\{\|\mu_1\|, \|\mu_2\|, \|\mu_3\|\} = \frac{385}{64};$$

we get,

$$L\Lambda = 0.7069, \qquad L\Lambda_1 = 0.6355, \qquad \|\mu\|\Lambda = 174.4886$$

Assumptions of Theorem 3.2 are satisfied. Hence, there exists at least one solution for the problem (5.1)-(5.2) on $[0,1]$.

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