Connes Amenability of $l^1$-Munn Algebras

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Abstract. In this paper, we study Connes amenability of $l^1$-Munn algebras. We apply this results to semigroup algebras. We show that for a weakly cancellative semigroup $S$ with finite idempotents, amenability and Connes amenability are equivalent.

1 Introduction

In [4], Eslamzadeh introduced $l^1$-Munn algebras. He used these algebras to characterize amenable semigroup algebras. A special case of these algebras was introduced by Munn [18]. $l^1$-Munn algebras has been studied in some texts. In [1], Blackmore showed the $l^1$-Munn algebra of the group algebra $l^1(G)$ is weakly amenable. Eslamzadeh in [5] and [6] investigated the structure of $l^1$-Munn algebras. Duncan and Paterson used the $l^1$-Munn algebras to study of semigroup algebras of completely simple semigroups [3].

The motivation to study of the theory of amenable von Neumann algebras stems from the fact that they are dual. In [12], it is shown that if $A$ is a von Neumann algebra containing a weak*-dense amenable $C^*$-subalgebra, then for every normal Banach $A$-bimodule $E$, every weak*-continuous derivation $D : A \to E$ is inner. This concept of amenability was called Connes amenability [9]. In [21], Runde extended the notion of Connes-amenability to dual Banach algebras. For a locally compact group $G$, the group algebra $l^1(G)$ and the measure algebra $M(G)$ are two examples of dual Banach algebras. In [23], Runde introduced normal, virtual diagonals for a dual Banach algebra and showed that the existence of a normal virtual diagonal for $M(G)$ is equivalent to it being Connes amenable. Also in [22], it is shown that $G$ is amenable if and only if $M(G)$ is Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable.

The investigation of Connes amenability for dual Banach algebras which are not von Neumann algebra is interesting for many authors, see [24], [2] and [7]. Several authors have generalized the earlier concept of amenability introduced by Lau in [13] (see [14], [15], [16] and [17]).

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Recently, the authors have introduced the $\phi$-version of Connes amenability of dual Banach algebra $A$ that $\phi$ is a homomorphism from $A$ onto $\mathbb{C}$ that lies in predual $A_*$. We study the Runde’s theorem for the case of semigroup algebra of a weakly cancellative semigroup [8]. In this paper, we study Connes amenability of $l^1$-Munn algebras. We use the $l^1$-Munn algebras to study of Connes amenability of semigroup algebras of weakly cancellative semigroups. In order to do this, we follow the argument of [4].

2 Connes amenability of $l^1$-Munn algebras

Let $A$ be a dual Banach algebra with predual $A_*$. A dual Banach $A$-bimodule $E$ is called normal Banach $A$-bimodule if for each $x \in E$, the maps $a \mapsto x.a$, $a \mapsto a.x$ are weak$^*$-continuous ($a \in A$). $A$ is called Connes amenable, if for every normal Banach $A$-bimodule $E$, every weak$^*$-continuous derivation $D : A \to E$ is inner.

Let $E$ be a Banach $A$-bimodule. An element $x \in E$ is called weak$^*$-weakly continuous if the module maps $a \mapsto x.a$, $a \mapsto a.x$ are weak$^*$-weak continuous ($a \in A$). The collection of all weak$^*$-weakly continuous elements of $E$ is denoted by $\sigma wc(E)$. It is shown that, $\sigma wc(E)^*$ is normal [24]. Let $\pi : A \hat{\otimes} A \to A$ be the multiplication map. From Corollary 4.6 in [24], $\pi^*$ maps $A_*$ into $\sigma wc((A \hat{\otimes} A)^*)$. Consequently, $\pi^{**}$ drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((A \hat{\otimes} A)^*) \to A$. An element $M \in \sigma wc((A \hat{\otimes} A)^*)$ is called a $\sigma wc$-virtual diagonal for $A$, if $M.u = u.M$, $u.\pi_{\sigma wc}(M) = u$ for every $u \in A$. In [24], Runde showed that $A$ is Connes amenable if and only if there is a $\sigma wc$-virtual diagonal for $A$.

Let $A$ be a unital Banach algebra, let $I$ and $J$ be nonempty sets and $P = (p_{ij}) \in M_{I \times J}(A)$ be such that $\|P\|_{\infty} = \sup\{\|p_{ij}\| : j \in J, i \in I\} \leq 1$. The set $M_{I \times J}(A)$ of all $I \times J$ matrices $a = (a_{ij})$ on $A$ with $l^1$-norm and the product $A \circ B = APB, (A, B \in M_{I \times J}(A))$ is a Banach algebra that is called $l^1$-Munn algebra on $A$ with sandwich matrix $P$. It is denoted by $LM(A, P, I, J)$ [4]. Also $\xi_{ij}$ is denoted the element of $M_{I \times J}(\mathbb{C})$ with 1 in $(i, j)$th place and 0 elsewhere. Throughout we use the notations of [4]. We define $\Gamma : M_{I \times J}(A_*) \to M_{I \times J}(A)$ by $\langle (\Gamma(f))_{ij}, a\xi_{ij} \rangle \to \langle (f_{ij}), a\xi_{ij} \rangle$, then $M_{I \times J}(A)$ is a dual space with predual $M_{I \times J}(A_*)$. It is clear that above multiplication in $LM(A, P, I, J)$ is separately weak$^*$-continuous and from Proposition 1.2 in [21], $LM(A, P, I, J)$ is a dual Banach algebra.

**Theorem 2.1.** Let $A$ be a unital dual Banach algebra. The following are equivalent:

(i) $LM(A, P, I, J)$ is Connes amenable;

(ii) $A$ is Connes amenable, $I$ and $J$ are finite and $P$ is invertible.
Proof. (i)$\Rightarrow$ (ii) Since $\mathcal{L}M(A, P, I, J)$ is Connes amenable dual Banach algebra, then from Proposition 4.1 in [21], it has a bounded approximate identity. By Lemma 3.7 and Lemma 3.5 in [4], $I$ and $J$ are finite, $P$ is invertible and $\mathcal{L}M(A, P, I, J)$ is topologically algebra isomorphic to $\mathcal{L}M(A, I_m, I, J)$ where $I_m$ is the identity matrix with dimension $m$ and $|I| = |J| = m$. It is known that $\mathcal{L}M(A, I_m, I, J)$ is isometrically algebra isomorphic to $\mathcal{M}_m \hat{\otimes} A$ where $\mathcal{M}_m$ is the algebra of $m \times m$ complex matrices [19]. Using the idea of Theorem 4.1 in [4] and Theorem 4.8 in [24], we obtain the desired proof.

By Theorem 4.8 in [24], there exists $M \in \sigma_{wc}(((\mathcal{M}_m \hat{\otimes} A) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} A))^*)^*$ such that $M.u = u.M$, $u.\pi_{\sigma_{wc}}(M) = u$, $u \in (\mathcal{M}_m \hat{\otimes} A)$. Now as [24], we consider those elements of $\sigma_{wc}(((\mathcal{M}_m \hat{\otimes} A) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} A))^*)^*$ that lies in the canonical image of $(\mathcal{M}_m \hat{\otimes} A) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} A)$ and we write $M = \sum_{i,j=1}^m (\xi_{ij} \otimes a_{ij}) \otimes \sum_{r,l=1}^m (\xi_{rl} \otimes b_{rl})$.

Let $E$ be a normal Banach $A$-bimodule with predual $E_*$ and $D : A \rightarrow E$ be a derivation that is weak$^*$-continuous. By a similar argument in Lemma 3.3 in [8], we may assume that $E$ is a normal dual Banach $A$-bimodule such that its predual is essential. Let

$$\psi : (\mathcal{M}_m \hat{\otimes} A) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} A) \rightarrow (\mathcal{M}_m \hat{\otimes} \mathcal{M}_m) \hat{\otimes} (A \hat{\otimes} A)$$

be the onto linear isometry. Let $c \in A$ and $c = \sum_{s,t=1}^m \xi_{st} \otimes c_{st}$ that $c_{11} = c$, $c_{st} = 0$ if $s \neq 1$ or $t \neq 1$. We have

$$c = c.\pi_{\sigma_{wc}}(M) = \sum_{s,t=1}^m \xi_{st} \otimes c_{st}.\sum_{i,j,l=1}^m (\xi_{it} \otimes a_{ij}b_{jl}) = \sum_{i,j,s,l=1}^m \xi_{st} \otimes c_{si}a_{ij}b_{jl}. \sum_{m}$$

Then

$$\sum_{i,j,s,t=1}^m \xi_{st} \otimes (c_{st} - c_{si}a_{ij}b_{jl}) = 0. \quad (2.1)$$

Also

$$c.M = \sum_{s,t=1}^m (\xi_{st} \otimes c_{st}).(\sum_{i,j,l=1}^m (\xi_{ij} \otimes a_{ij}) \otimes \sum_{r,l=1}^m (\xi_{rl} \otimes b_{rl})) = (\sum_{i,j,l=1}^m (\xi_{ij} \otimes a_{ij}) \otimes \sum_{r,l=1}^m (\xi_{rl} \otimes b_{rl})).\sum_{s,t=1}^m (\xi_{st} \otimes c_{st}) = M.c.$$ 

Therefore

$$\sum_{s,t,r,l,j=1}^m (\xi_{sj} \otimes c_{st}a_{ij}) \otimes (\xi_{rl} \otimes b_{rl}) = \sum_{i,j,r,l,t=1}^m (\xi_{ij} \otimes a_{ij}) \otimes (\xi_{rl} \otimes b_{rl}c_{lt}).$$
Apply $\psi$, we have
\[
\sum_{i,t,r,l,j=1}^m (\xi_{ij} \otimes \xi_{rt}) \otimes (c_{it} a_{tj} \otimes b_{rl}) = \sum_{i,j,r,t,l=1}^m (\xi_{ij} \otimes \xi_{rt}) \otimes (a_{ij} \otimes b_{rl} c_{lt}).
\]
Suppose that $c = \sum_{s,t=1}^m c_{st} \otimes c_{st}$ that $c_{11} = c, c_{st} = 0$ if $s \neq 1$ or $t \neq 1$. Then
\[
\sum_{r,j=1}^m (\xi_{1j} \otimes \xi_{r1}) \otimes (c_{11} a_{1j} \otimes b_{r1}) = \sum_{r,j=1}^m (\xi_{1j} \otimes \xi_{r1}) \otimes (a_{1j} \otimes b_{r1} c_{11}). \tag{2.2}
\]
Define
\[
\theta : ((\mathcal{M}_m \hat{\otimes} \mathcal{M}_m) \hat{\otimes} (\mathcal{A} \hat{\otimes} \mathcal{A})) \longrightarrow E
\]
\[
\theta (\sum_{i,j,r,t=1}^m (\xi_{ij} \otimes \xi_{rt}) \otimes (a_{ij} \otimes b_{rt})) = \sum_{r,j=1}^m a_{1j} D(b_{r1}).
\]
It is easy to see that $\psi$ and $\theta$ are weak$^*$-continuous. Now consider
\[
\lambda = \theta O \psi : (\mathcal{M}_m \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} \mathcal{A}) \rightarrow E.
\]
From Lemma 4.9 in [24] $\lambda^*$ maps $E_*$ into $\sigma wc(((\mathcal{M}_m \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} \mathcal{A}))^*)$ and so $(\lambda^*|_{E_*})^*$ maps $\sigma wc(((\mathcal{M}_m \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{M}_m \hat{\otimes} \mathcal{A}))^*)^*$ into $E$. We apply $\theta$ on (2.2) and we get
\[
\sum_{r,j=1}^m c_{11} a_{1j} D(b_{r1}) = \sum_{r,j=1}^m a_{1j} D(b_{r1} c_{11}). \tag{2.3}
\]
Put $M_1 = \sum_{r,j=1}^m (\xi_{1j} \otimes a_{1j}) \otimes (\xi_{r1} \otimes b_{r1})$ and $M' = \lambda(M_1)$. We obtain from (2.1) and (2.3),
\[
\langle x, c. M' \rangle = \langle x, \sum_{r,j=1}^m c_{11} a_{1j} D(b_{r1}) \rangle = \langle x, \sum_{r,j=1}^m a_{1j} D(b_{r1} c_{11}) \rangle = \langle x, \sum_{r,j=1}^m a_{1j} D(b_{r1}). c_{11} + \sum_{r,j=1}^m a_{1j} b_{r1} D(c_{11}) \rangle = \langle x, M'.c \rangle + \langle x, D(c) \rangle
\]
for all $x \in E_*$. Consequently $D(c) = M'.c - c. M'$.

(2) $\Rightarrow$ (1) Let $E$ be a normal Banach $\mathcal{M}_n \hat{\otimes} \mathcal{A}$-bimodule and $D : \mathcal{M}_n \hat{\otimes} \mathcal{A} \longrightarrow E$ be a derivation that is weak$^*$-continuous. Let $e_\mathcal{A}$ denote the identity of $\mathcal{A}$. We define
\[
\xi_{ij} \cdot x = (\xi_{ij} \otimes e_\mathcal{A}).x, \quad x \cdot \xi_{ij} = x \cdot (\xi_{ij} \otimes e_\mathcal{A}) \quad (i, j \in 1, \ldots, m).
\]
So $E$ is a normal Banach $\mathcal{M}_n$-bimodule.

Put $D_{\mathcal{M}_n} : \mathcal{M}_n \longrightarrow E, D_{\mathcal{M}_n} (\xi_{ij}) = D(\xi_{ij} \otimes e_\mathcal{A})$, then
\[
D_{\mathcal{M}_n} (\xi_{ij} \xi_{kl}) = D(\xi_{ij} \xi_{kl} \otimes e_\mathcal{A}) = D(\xi_{ij} \otimes e_\mathcal{A}) \cdot (\xi_{kl} \otimes e_\mathcal{A}) + (\xi_{ij} \otimes e_\mathcal{A}) \cdot D(\xi_{kl} \otimes e_\mathcal{A}) = D(\xi_{ij} \otimes e_\mathcal{A}) \cdot \xi_{kl} + \xi_{ij} \bullet D(\xi_{kl} \otimes e_\mathcal{A})
\]
Hence, there exists $u \in E$ such that $D_{\mathcal{M}_n} = ad_u$. Therefore, $\hat{D} = D(\xi_{ij} \otimes e_A) - ad_u$ vanishes on $\mathcal{M}_n \otimes e_A$.

Let $I$ be the identity matrix. Then $E$ is an $\mathcal{A}$-bimodule for the maps defined by

$$a \circ x = (I \otimes a).x, \quad x \circ a = x.(I \otimes a), \quad (a \in \mathcal{A}, \ x \in E).$$

Let us now $D_A(a) = \hat{D}(I \otimes a) (a \in \mathcal{A})$. Define $K = \{ e \in E_* : \langle \hat{D}(I \otimes a), e \rangle = 0 \}$. Note that $(\frac{E_*}{K})^* = \hat{D}(I \otimes a)^{w_k^*}$. Further $\hat{D}(I \otimes a)^{w_k^*}$ is a commutative normal Banach $\mathcal{A}$-bimodule. Then, there is $\nu \in \hat{D}(I \otimes a)^{w_k^*}$ such that $\hat{D}(I \otimes a) = (I \otimes a).\nu - \nu. (I \otimes a)$. This complete the proof.

\section{Semigroup algebra}

In this section, we apply these results to semigroup algebra $l^1(S)$. For a semigroup $S$ and $s \in S$, we define maps $L_s, R_s : S \to S$ by $L_s(t) = st, \ R_s(t) = ts, \ t \in S$. If for each $s \in S, R_s$ and $L_s$ are finite-to-one maps, then we say that $S$ is weakly cancellative. Before turning our results, we note that if $S$ is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [2]. It is known that $(l^1(S) \bar{\otimes} l^1(S))' = B(l^1(S), l^\infty(S)) = l^1(S \times S)' = l^\infty(S \times S)$, where $T \in B(l^1(S), l^\infty(S))$ is identified with $T \in l^\infty(S \times S)$, where $T(s, t) = \langle T(\delta_s), \delta_t \rangle$. By the Krein-Smulian Theorem, $T$ is weakly compact if and only if the set $\{ T(\delta_s) : s \in S \}$ is relatively weakly compact.

A semigroup $S$ is simple if the only ideal in $S$ is $S$. A semigroup $S$ with zero is called 0-simple if $\{0\}$ and $S$ are the only ideals and $S.S \neq 0$. An element $p \in S$ is an idempotent if $p^2 = p$, the set of idempotents of $S$ is denoted by $E(S)$. For $p, q \in E(S)$, set $p \leq q$ if $pq = qp = p$. An element $e \in E(S)$ is called primitive if it is nonzero and is minimal in the set of nonzero idempotents. $S$ is called completely simple if it is simple and contains a primitive idempotent.

Let $G$ be a group, $I$ and $J$ be arbitrary nonempty sets and $G^0 = G \cup \{0\}$. Let $P_G = (a_{ij}) \in M_{J \times I}(G)$. For $x \in G$, let $(x)_{ij}$ be the element of $M_{I \times J}(G^0)$ with $x$ in $(i, j)^{th}$ place and $0$ elsewhere. The set of all $(x)_{ij}$ matrices is denoted by $S$. Multiplication in $S$ is given by the formula

$$(x)_{ij}(y)_{kl} = (xa_{jk}y)_{il} \ (x, y \in G; i, k \in I, j, l \in J).$$

We write $S = \mathcal{M}(G, P, I, J)$. $S$ is called Rees matrix semigroup with sandwich matrix $P$. It is known that $S$ is a completely simple semigroup and each completely simple semigroup is isomorphic to one constructed in this manner [10]. Similarly, we have the semigroup $\mathcal{M}^0(G, P, I, J)$ where the elements of this semigroup are those of $\mathcal{M}(G, P, I, J)$, together with the element $0$ so that $0$ is a matrix with $0$ everywhere and $P_G = (a_{ij}) \in M_{J \times I}(G^0)$. 

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Proposition 3.1. Let $S$ be a weakly cancellative semigroup and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. If $l^1(S)$ is Connes amenable, then $E(S)$ is finite.

Proof. For each $s \in S$, we put $[ss^{-1}] = \{x \in S : xs = s\}$ and $[s^{-1}s] = \{x \in S : sx = s\}$ and $\chi(s) = sS \cap [ss^{-1}]$. We follow [3] and consider the equivalence relation $R$ on $E(S)$ by $sRt$ if $s \in \chi(t)$. By this relation, $E(S)$ is partitioned into the sets $\chi(s)$. Suppose via contradiction, there exists an infinite sequence of sets $\chi(k_n)$, $\{k_n\}_{n \in \mathbb{N}} \in S$.

Let $M \in (l^1(S) \hat{\otimes} l^1(S))^{\ast\ast} = l^\infty(S \times S)'$ be a $\sigma wc$-virtual diagonal for $l^1(S)$ which satisfies Theorem 5.9 in [2]. Therefore $\langle M, f(hk, g) - f(h, kg) \rangle = 0$ for each $k \in S$ and $f \in l^\infty(S \times S)$. Then $\langle M, f(h', g) \rangle = 0, (h' = hk, g \notin kS)$. In particular $\langle M, f(h', g) \rangle = 0(h' = hk, g \notin kS, ghk = k)$.

Now consider the multiplication map $\pi : l^1(S) \hat{\otimes} l^1(S) \to l^1(S)$ and $f' \in l^\infty(S)$ such that $\pi' f' = f$. We have $\langle \pi'(f'), \delta_g \otimes \delta_h \rangle = \langle f', \delta_{gh} \rangle$. Also

$$\langle \delta_k, M, \pi'(f') \rangle = \langle \delta_k, (e_{l^1(S)} \otimes e_{l^1(S)}), \pi'(f') \rangle = \langle (\delta_k, e_{l^1(S)} \otimes e_{l^1(S)}), \pi'(f') \rangle.$$

Consequently

$$\langle M, f(hk, g) \rangle = \langle M, \pi'(f', \delta_k) \rangle = \langle M, f', \delta_{ghk} \rangle(g, h)$$

$$\quad = \langle f', \delta_k, e_A \rangle(g, h) (ghk = k).$$

This implies that $\langle M, f(h', g) \rangle = \langle f', \delta_k, e_A \rangle(g \in kS, gh'k = k)$. Write $f(h, g) = \langle T(\delta_h), \delta_g \rangle$ where $T \in B(l^1(S), l^\infty(S))$. Since $T$ is weakly compact. By Theorem 5.9 in [2], $\{T(\delta_i) : i \in I\}$ is relatively weakly compact and so totally bounded [20]. Let

$$Z(k) = \{(h, g) \in S \times S : g \in kS, ghk = k\}.$$

Since $S$ is weakly cancellative, then each $\chi(k_n)$ is finite and also $Z(k_n)$ is contained in pairwise disjoint sets $\chi(k_n)$. Choose distinct elements $k_1, k_2, ..., k_n \in S$ with

$$L = \min |\langle f', \delta_{k_i} \rangle|, \quad nL \geq \| M \| \Sigma_{i=1}^n \sup \{|\langle T(\delta_h), \delta_g \rangle|\}.$$ 

Therefore

$$nL \leq \Sigma_{i=1}^n |\langle f', \delta_{k_i} \rangle| = \Sigma_{i=1}^n \sup \{|\langle M, f(h, g) \rangle|\} \leq \| M \| \Sigma_{i=1}^n \sup \{|\langle T(\delta_h), \delta_g \rangle|\}.$$

This is a contradiction. \qed
Example 1. Let $S$ be the natural numbers $\mathbb{N}$, with the product

$$(m, n) \rightarrow m \vee n = \max\{m, n\}.$$ 

$S$ is a semigroup with identity 1 and weakly cancellative. Clearly $E(S) = S$. Then $L^1(S)$ is a dual Banach algebra with predual $c_0(S)$ and $L^1(S)$ is not Connes amenable as $E(S)$ is infinite.

It is easy to see that each weakly cancellative semigroup is simple. In fact, suppose that $I$ be a left ideal of $S$ containing a nonzero element $i$, then

$$S = Si \subseteq SI \subseteq I$$

and so $I = S$. Consequently, if $S$ is a weakly cancellative semigroup with $E(S)$ finite, then $S$ is completely simple semigroup and Rees matrix semigroup of the form $S = M(G, P, I, J)$.

Theorem 3.1. Let $S$ be a weakly cancellative semigroup and let $L^1(S)$ be unital with unit $e_{L^1(S)}$. If $L^1(S)$ is Connes amenable, then $S$ is a Rees matrix semigroup of the form $S = M(G, P, I, J)$, $LM(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable.

Proof. By Proposition 3.1, $S$ is a simple semigroup with $E(S)$ finite, then $S$ is a completely simple semigroup. Therefore, $S$ is a Rees matrix semigroup of the form $S = M(G, P, I, J)$. By Proposition 5.6 in [4], $L^1(S)$ is isometrically algebra isomorphic to $LM(l^1(G), P, I, J)$. Since $L^1(S)$ is Connes amenable, then $LM(l^1(G), P, I, J)$ is Connes amenable and it has an identity. Also by Theorem 2.1, $L^1(G)$ is Connes amenable.

Theorem 3.2. Let $S$ be a weakly cancellative semigroup with $E(S)$ finite and let $L^1(S)$ be unital with unit $e_{L^1(S)}$. Then $S$ is a Rees matrix semigroup of the form $S = M(G, P, I, J)$. With above notation, the following are equivalent:

(i) $L^1(S)$ is Connes amenable;

(ii) $LM(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable;

(iii) $L^1(S)$ is amenable.

Proof. (i) $\Rightarrow$ (ii) is Theorem 3.1. (ii) $\Rightarrow$ (i) $LM(l^1(G), P, I, J)$ has an identity, then $I$ and $J$ are finite and $P$ is invertible [4]. Since $l^1(G)$ is Connes amenable, then from Theorem 2.1, $LM(l^1(G), P, I, J)$ is Connes amenable. By Proposition 5.6 in [4], $l^1(S)$ is isometrically algebra isomorphic to $LM(l^1(G), P, I, J)$ and $l^1(S)$ is Connes amenable. (ii) $\Leftrightarrow$ (iii) By Theorem 5.3 in [22], $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable. Then, this is Theorem 5.9 in [4].
Theorem 3.3. Let $S = \mathcal{M}(G, P, I, J)$ be a weakly cancellative semigroup. Let $l^1(S)$ be unital with unit $e_{1_{l^1(S)}}$. With above notation, the following are equivalent:

(i) $l^1(S)$ is Connes amenable;

(ii) $l^1(S)$ is amenable.

Proof. This follows in the same manner as the proof of Theorem 3.2. \qed

Example 2. Let $G$ be an amenable group. Let $J$ be finite of order $n$. Let $S = \mathcal{M}(G, P, 1, J)$ where $P$ is invertible. Let $(a)_{1j}$ and $(b)_{1l}$ be two non-zero elements of $S = \mathcal{M}(G, P, 1, J)$. It is easy to see that

$$(a)_{1j}(p_{j1}^{-1}a^{-1}b)_{1l} = (b)_{1l}.$$ 

Then $S$ is a weakly cancellative semigroup. By [11], $l^1(G)$ is amenable. By Proposition 5.6 and Theorem 4.1 in [4], $l^1(S)$ is amenable. Also $l^1(G)$ is Connes amenable and by Theorem 2.1, $l^1(S)$ is Connes amenable.

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