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Connes amenability of l^1-Munn algebras

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Abstract. In this paper, we study Connes amenability of l^1-Munn algebras. We apply this results to semigroup algebras. We show that for a weakly cancellative semigroup S with finite idempotents, amenability and Connes amenability are equivalent.

Keywords. Amenability, Banach algebras, derivation, l^1-Munn algebras,semigroup algebras

1 Introduction

In [4], Eslamzadeh introduced l^1-Munn algebras. He used these algebras to characterize amenable semigroup algebras. A special case of these algebras was introduced by Munn [18]. l^1-Munn algebras has been studied in some texts. In [1], Blackmore showed the l^1-Munn algebra of the group algebra $l^1(G)$ is weakly amenable. Eslamzadeh in [5] and [6] investigated the structure of l^1-Munn algebras. Duncan and Paterson used the l^1-Munn algebras to study of semigroup algebras of completely simple semigroups [3].

The motivation to study of the theory of amenable von Neumann algebras stems from the fact that they are dual. In [12], it is shown that if $\mathcal{A}$ is a von Neumann algebra containing a weak*-dense amenable C^*-subalgebra, then for every normal Banach $\mathcal{A}$-bimodule E, every weak*-continuous derivation $D : \mathcal{A} \to E$ is inner. This concept of amenability was called Connes amenability [9]. In [21], Runde extended the notion of Connes-amenability to dual Banach algebras. For a locally compact group G, the group algebra $l^1(G)$ and the measure algebra $M(G)$ are two examples of dual Banach algebras. In [23], Runde introduced normal, virtual diagonals for a dual Banach algebra and showed that the existence of a normal virtual diagonal for $M(G)$ is equivalent to it being Connes amenable. Also in [22], it is shown that G is amenable if and only if $M(G)$ is Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable.

The investigation of Connes amenability for dual Banach algebras which are not von Neumann algebra is interesting for many authors, see [24], [2] and [7]. Several authors have generalized the earlier concept of amenability introduced by Lau in [13] (see [14], [15], [16] and [17]). Recently the authors have introduced the ϕ-version of Connes amenability of dual Banach algebra

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$\mathcal{A}$ that ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ that lies in predual $\mathcal{A}_*$. We study the Runde's theorem for the case of semigroup algebra of a weakly cancellative semigroup [8]. In this paper, we study Connes amenability of l^1-Munn algebras. We use the l^1-Munn algebras to study of Connes amenability of semigroup algebras of weakly cancellative semigroups. In order to do this, we follow the argument of [4].

2 Connes amenability of l^1-Munn algebras

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. A dual Banach $\mathcal{A}$-bimodule E is called normal Banach $\mathcal{A}$-bimodule if for each $x \in E$, the maps $a \mapsto x.a,\ a \mapsto a.x$ are weak*-continuous $(a \in \mathcal{A})$. $\mathcal{A}$ is called Connes amenable, if for every normal Banach $\mathcal{A}$-bimodule E, every weak*-continuous derivation $D : \mathcal{A} \to E$ is inner.

Let E be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is called weak*-weakly continuous if the module maps $a \mapsto x.a,\ a \mapsto a.x$ are weak*-weak continuous $(a \in \mathcal{A})$. The collection of all weak*-weakly continuous elements of E is denoted by $\sigma wc(E)$. It is shown that, $\sigma wc(E)^*$ is normal [24]. Let $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ be the multiplication map. From Corollary 4.6 in [24], π^* maps $\mathcal{A}_*$ into $\sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is called a σwc-virtual diagonal for $\mathcal{A}$, if $M.u = u.M,\ \ u.\pi_{\sigma wc}(M) = u$ for every $u \in \mathcal{A}$. In [24], Runde showed that $\mathcal{A}$ is Connes amenable if and only if there is a σwc-virtual diagonal for $\mathcal{A}$.

Let $\mathcal{A}$ be a unital Banach algebra, let I and J be nonempty sets and $P = (p_{ij}) \in M_{J\times I}(\mathcal{A})$ be such that $\|P\|_\infty = sup\{\|p_{ji}\| : j \in J, i \in I\} \le 1$. The set $M_{I\times J}(\mathcal{A})$ of all $I \times J$ matrices $a = (a_{ij})$ on $\mathcal{A}$ with l^1-norm and the product $A \odot B = APB, (A, B \in M_{I\times J}(\mathcal{A}))$ is a Banach algebra that is called l^1-Munn algebra on $\mathcal{A}$ with sandwich matrix P. It is denoted by $\mathcal{LM}(\mathcal{A}, P, I, J)$ [4]. Also ξ_{ij} is denoted the element of $M_{I\times J}(\mathbb{C})$ with 1 in (i, j)th place and 0 elsewhere. Throughout we use the notations of [4]. We define $\Gamma : M_{I\times J}(\mathcal{A}_*) \to M_{I\times J}(\mathcal{A})$ by $\langle(\Gamma(f))_{ij}, a\xi_{ij}\rangle \to \langle(f_{ij}), a\xi_{ij}\rangle$, then $M_{I\times J}(\mathcal{A})$ is a dual space with predual $M_{I\times J}(\mathcal{A}_*)$. It is clear that above multiplication in $\mathcal{LM}(\mathcal{A}, P, I, J)$ is separately weak*-continuous and from Proposition 1.2 in [21], $\mathcal{LM}(\mathcal{A}, P, I, J)$ is a dual Banach algebra.

Theorem 2.1. *Let $\mathcal{A}$ be a unital dual Banach algebra. The following are equivalent:*

(i) $\mathcal{LM}(\mathcal{A}, P, I, J)$ is Connes amenable;

(ii) $\mathcal{A}$ is Connes amenable, I and J are finite and P is invertible.

Proof. (i)⇒ (ii) Since $\mathcal{LM}(\mathcal{A}, P, I, J)$ is Connes amenable dual Banach algebra, then from Proposition 4.1 in [21], it has a bounded approximate identity. By Lemma 3.7 and Lemma 3.5 in [4], I and J are finite, P is invertible and $\mathcal{LM}(\mathcal{A}, P, I, J)$ is topologically algebra isomorphic to $\mathcal{LM}(\mathcal{A}, I_m, I, J)$ where I_m is the identity matrix with dimension m and $|I| = |J| = m$. It is known that $\mathcal{LM}(\mathcal{A}, I_m, I, J)$ is isometrically algebra isomorphic to $\mathcal{M}_m\widehat{\otimes}\mathcal{A}$ where $\mathcal{M}_m$ is the algebra of $m \times m$ complex matrices [19]. Using the idea of Theorem 4.1 in [4] and Theorem 4.8 in [24], we obtain the desired proof.

By Theorem 4.8 in [24], there exists

$$M \in \sigma wc(((\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A}))^*)^*$$

such that

$$M.u = u.M, \quad u.\pi_{\sigma wc}(M) = u, \ u \in (\mathcal{M}_m\widehat{\otimes}\mathcal{A}).$$

Now as [24], we consider those elements of $\sigma wc(((\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A}))^*)^*$ that lies in the canonical image of $(\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A})$ and we write $M=\Sigma_{i,j=1}^m(\xi_{ij}\otimes a_{ij})\otimes\Sigma_{r,l=1}^m(\xi_{rl}\otimes b_{rl})$.

Let E be a normal Banach $\mathcal{A}$-bimodule with predual E_* and $D:\mathcal{A}\longrightarrow E$ be a derivation that is weak*-continuous. By a similar argument in Lemma 3.3 in [8], we may assume that E is a normal dual Banach $\mathcal{A}$-bimodule such that its predual is essential. Let

$$\begin{aligned}\psi:(\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A}) &\longrightarrow (\mathcal{M}_m\widehat{\otimes}\mathcal{M}_m)\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{A})\\ \psi((c\otimes x)\otimes(d\otimes y)) &= (c\otimes d)\otimes(x\otimes y)\ (x,y\in\mathcal{A},c,d\in\mathcal{M}_m)\end{aligned}$$

be the onto linear isometry. Let $c\in\mathcal{A}$ and $c=\Sigma_{s,t=1}^m\xi_{st}\otimes c_{st}$ that $c_{11}=c, c_{st}=0$ if $s\neq 1$ or $t\neq 1$. We have

$$\begin{aligned}c=c.\pi_{\sigma wc}(M) &= \Sigma_{s,t=1}^m\xi_{st}\otimes c_{st}.\Sigma_{i,j,l=1}^m(\xi_{il}\otimes a_{ij}b_{jl})\\ &= \Sigma_{i,j,s,l=1}^m\xi_{sl}\otimes c_{si}a_{ij}b_{jl}).\end{aligned}$$

Then

$$\Sigma_{i,j,s,t=1}^m\xi_{st}\otimes(c_{st}-c_{si}a_{ij}b_{jt})=0. \tag{2.1}$$

Also

$$\begin{aligned}c.M &= \Sigma_{s,t=1}^m(\xi_{st}\otimes c_{st}).(\Sigma_{i,j=1}^m(\xi_{ij}\otimes a_{ij})\otimes\Sigma_{r,l=1}^m(\xi_{rl}\otimes b_{rl}))\\ &= (\Sigma_{i,j=1}^m(\xi_{ij}\otimes a_{ij})\otimes\Sigma_{r,l=1}^m(\xi_{rl}\otimes b_{rl})).\Sigma_{s,t=1}^m(\xi_{st}\otimes c_{st})=M.c.\end{aligned}$$

Therefore

$$\Sigma_{s,t,r,l,j=1}^m(\xi_{sj}\otimes c_{st}a_{tj})\otimes(\xi_{rl}\otimes b_{rl})=\Sigma_{i,j,r,l,t=1}^m(\xi_{ij}\otimes a_{ij})\otimes(\xi_{rt}\otimes b_{rl}c_{lt}).$$

Apply ψ, we have

$$\Sigma_{i,t,r,l,j=1}^m(\xi_{ij}\otimes\xi_{rl})\otimes(c_{it}a_{tj}\otimes b_{rl})=\Sigma_{i,j,r,l,t=1}^m(\xi_{ij}\otimes\xi_{rt})\otimes(a_{ij}\otimes b_{rl}c_{lt}).$$

Suppose that $c=\Sigma_{s,t=1}^m\xi_{st}\otimes c_{st}$ that $c_{11}=c, c_{st}=0$ if $s\neq 1$ or $t\neq 1$. Then

$$\Sigma_{r,j=1}^m(\xi_{1j}\otimes\xi_{r1})\otimes(c_{11}a_{1j}\otimes b_{r1})=\Sigma_{r,j=1}^m(\xi_{1j}\otimes\xi_{r1})\otimes(a_{1j}\otimes b_{r1}c_{11}). \tag{2.2}$$

Define

$$\begin{aligned}\theta:((\mathcal{M}_m\widehat{\otimes}\mathcal{M}_m)\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{A})) &\longrightarrow E\\ \theta(\Sigma_{i,j,r,l=1}^m(\xi_{ij}\otimes\xi_{rl})\otimes(a_{ij}\otimes b_{rl})) &= \Sigma_{i,j,r,l=1}^m a_{ij}D(b_{rl}).\end{aligned}$$

It is easy to see that ψ and θ are weak*-continuous. Now consider

$$\lambda=\theta O\psi:(\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A})\to E.$$

From Lemma 4.9 in [24] λ^* maps E_* into $\sigma wc(((\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A}))^*)$ and so $(\lambda^*|_{E_*})^*$ maps $\sigma wc(((\mathcal{M}_m\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\mathcal{A}))^*)^*$ into E. We apply θ on (2.2) and we get

$$\Sigma_{r,j=1}^m c_{11}a_{1j}D(b_{r1})=\Sigma_{r,j=1}^m a_{1j}D(b_{r1}c_{11}). \tag{2.3}$$

Put $M_1=\Sigma_{r,j=1}^m(\xi_{1j}\otimes a_{1j})\otimes(\xi_{r1}\otimes b_{r1})$ and $M'=\lambda(M_1)$. We obtain from (2.1) and (2.3),

$$\begin{aligned}\langle x,c.M'\rangle &= \langle x,\Sigma_{r,j=1}^m c_{11}a_{1j}D(b_{r1})\rangle\\ &= \langle x,\Sigma_{r,j=1}^m a_{1j}D(b_{r1}c_{11})\rangle\end{aligned}$$

$$\begin{aligned} &= \langle x, \Sigma_{r,j=1}^{m} a_{1j} D(b_{r1}).c_{11} + \Sigma_{r,j=1}^{m} a_{1j} b_{r1} D(c_{11}) \rangle \\ &= \langle x, M'.c \rangle + \langle x, D(c) \rangle \end{aligned}$$

for all $x \in E_*$. Consequently $D(c) = M'.c - c.M'$.

(2) $\Rightarrow$ (1) Let E be a normal Banach $\mathcal{M}_n \widehat{\otimes} \mathcal{A}$-bimodule and $D : \mathcal{M}_n \widehat{\otimes} \mathcal{A} \longrightarrow E$ be a derivation that is weak*-continuous. Let $e_{\mathcal{A}}$ denote the identity of $\mathcal{A}$. We define

$$\xi_{ij} \bullet x = (\xi_{ij} \otimes e_{\mathcal{A}}).\ x, \quad x \bullet \xi_{ij} = x.\ (\xi_{ij} \otimes e_{\mathcal{A}}) \ \ (i, j \in 1, .., m).$$

So E is a normal Banach $\mathcal{M}_n$-bimodule.

Put $D_{\mathcal{M}_n} : \mathcal{M}_n \longrightarrow E$, $D_{\mathcal{M}_n}(\xi_{ij}) = D(\xi_{ij} \otimes e_{\mathcal{A}})$, then

$$\begin{aligned} D_{\mathcal{M}_n}(\xi_{ij}\xi_{kl}) &= D(\xi_{ij}\xi_{kl} \otimes e_{\mathcal{A}}) \\ &= D(\xi_{ij} \otimes e_{\mathcal{A}}).\ (\xi_{kl} \otimes e_{\mathcal{A}}) + (\xi_{ij} \otimes e_{\mathcal{A}}).\ D(\xi_{kl} \otimes e_{\mathcal{A}}) \\ &= D(\xi_{ij} \otimes e_{\mathcal{A}}) \bullet\ \xi_{kl} + \xi_{ij} \bullet\ D(\xi_{kl} \otimes e_{\mathcal{A}}) \\ &= D_{\mathcal{M}_n}(\xi_{ij}) \bullet \xi_{kl} + \xi_{ij} \bullet D_{\mathcal{M}_n}(\xi_{kl}). \end{aligned}$$

Hence, there exists $u \in E$ such that $D_{\mathcal{M}_n} = ad_u$. Therefore, $\tilde{D} = D(\xi_{ij} \otimes e_{\mathcal{A}}) - ad_u$ vanishes on $\mathcal{M}_n \otimes e_{\mathcal{A}}$.

Let I be the identity matrix. Then E is an $\mathcal{A}$-bimodule for the maps defined by

$$a \circ x = (I \otimes a).\ x, \quad x \circ a = x.\ (I \otimes a), \quad (a \in \mathcal{A}, \quad x \in E).$$

Let us now $D_{\mathcal{A}}(a) = \tilde{D}(I \otimes a)$ $(a \in \mathcal{A})$. Define $K = \{e \in E_* : \langle \tilde{D}(I \otimes a), e \rangle = 0\}$. Note that $(\frac{E_*}{K})^* = \overline{\tilde{D}(I \otimes a)}^{w_k^*}$. Further $\overline{\tilde{D}(I \otimes a)}^{w_k^*}$ is a commutative normal Banach $\mathcal{A}$-bimodule. Then, there is $\nu \in \overline{\tilde{D}(I \otimes a)}^{w_k^*}$ such that $\tilde{D}(I \otimes a) = (I \otimes a).\ \nu - \nu.\ (I \otimes a)$. This complete the proof. $\square$

3 Semigroup algebra

In this section, we apply these results to semigroup algebra $l^1(S)$. For a semigroup S and $s \in S$, we define maps $L_s, R_s : S \to S$ by $L_s(t) = st,\ R_s(t) = ts,\ t \in S$. If for each $s \in S$, R_s and L_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our results, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [2]. It is known that $(l^1(S) \widehat{\otimes} l^1(S))' = B(l^1(S), l^\infty(S)) = l^1(S \times S)' = l^\infty(S \times S)$, where $T \in B(l^1(S), l^\infty(S))$ is identified with $T \in l^\infty(S \times S)$, where $T(s,t) = \langle T(\delta_s), \delta_t \rangle$. By the Krein-Smulian Theorem, T is weakly compact if and only if the set $\{T(\delta_s) : s \in S\}$ is relatively weakly compact.

A semigroup S is simple if the only ideal in S is S. A semigroup S with zero is called 0-simple if $\{0\}$ and S are the only ideals and $S.S \neq 0$. An element $p \in S$ is an idempotent if $p^2 = p$, the set of idempotents of S is denoted by $E(S)$. For $p, q \in E(S)$, set $p \leq q$ if $pq = qp = p$. An element $e \in E(S)$ is called primitive if it is nonzero and is minimal in the set of nonzero idempotents. S is called completely simple if it is simple and contains a primitive idempotent.

Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$. Let $P_G = (a_{ij}) \in M_{J \times I}(G)$. For $x \in G$, let $(x)_{ij}$ be the element of $M_{I \times J}(G^0)$ with x in $(i,j)^{th}$ place and 0 elsewhere. The set of all $(x)_{ij}$ matrices is denoted by S. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{kl} = (x a_{jk} y)_{il} \ \ (x, y \in G, i, k \in I, j, l \in J).$$

We write $S = \mathcal{M}(G, P, I, J)$. S is called Rees matrix semigroup with sandwich matrix P. It is known that S is a completely simple semigroup and each completely simple semigroup is isomorphic to one constructed in this manner [10]. Similarly, we have the semigroup $\mathcal{M}^0(G, P, I, J)$ where the elements of this semigroup are those of $\mathcal{M}(G, P, I, J)$, together with the element 0 so that 0 is a matrix with 0 everywhere and $P_G = (a_{ij}) \in M_{J\times I}(G^0)$.

Proposition 3.1. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. If $l^1(S)$ is Connes amenable, then $E(S)$ is finite.*

Proof. For each $s \in S$, we put $[ss^{-1}] = \{x \in S : \ xs = s\}$ and $[s^{-1}s] = \{x \in S : \ sx = s\}$ and $\chi(s) = sS \cap [ss^{-1}]$. We follow [3] and consider the equivalence relation R on $E(S)$ by sRt if $s \in \chi(t)$. By this relation, $E(S)$ is partitioned into the sets $\chi(s)$. Suppose via contradiction, there exists an infinite sequence of sets $\chi(k_n)$, $\{k_n\}_{n\in\mathbb{N}} \in S$.

Let $M \in (l^1(S)\widehat{\otimes}l^1(S))^{**} = l^\infty(S \times S)'$ be a σwc-virtual diagonal for $l^1(S)$ which satisfies Theorem 5.9 in [2]. Therefore $\langle M, f(hk, g) - f(h, kg)\rangle = 0$ for each $k \in S$ and $f \in l^\infty(S\times S)$. Then $\langle M, f(h', g)\rangle = 0, (h' = hk, \ g \notin kS)$. In particular $\langle M, f(h', g)\rangle = 0(h' = hk, g \notin kS, ghk = k)$.

Now consider the multiplication map $\pi : l^1(S)\widehat{\otimes}l^1(S) \to l^1(S)$ and $f' \in l^\infty(S)$ such that $\pi' f' = f$. We have $\langle \pi'(f'), \delta_g \otimes \delta_h\rangle = \langle f', \delta_{gh}\rangle$. Also

$$\langle \delta_k.M, \pi'(f')\rangle = \langle \delta_k.(e_{l^1(S)} \otimes e_{l^1(S)}), \pi'(f')\rangle = \langle (\delta_k.e_{l^1(S)} \otimes e_{l^1(S)}), \pi'(f')\rangle.$$

Consequently

$$\begin{aligned}\langle M, f(hk, g)\rangle &= \langle M, \pi'(f'.\delta_k)\rangle = \langle M, \langle f', \delta_{ghk}\rangle(g, h)\rangle \\ &= \langle f'\delta_k, e_{\mathcal{A}}\rangle(g, h) \ (ghk = k).\end{aligned}$$

This implies that $\langle M, f(h', g)\rangle = \langle f'\delta_k, e_{\mathcal{A}}\rangle(g \in kS, gh'k = k)$. Write $f(h, g) = \langle T(\delta_h), \delta_g\rangle$ where $T \in B(l^1(S), l^\infty(S))$. Since T is weakly compact. By Theorem 5.9 in [2], $\{T(\delta_i) : i \in I\}$ is relatively weakly compact and so totally bounded [20]. Let

$$Z(k) = \{(h, g) \in S \times S : g \in kS, \ ghk = k\}.$$

Since S is weakly cancellative, then each $\chi(k_n)$ is finite and also $Z(k_n)$ is contained in pairwise disjoint sets $\chi(k_n)$. Choose distinct elements $k_1, k_2, .., k_n \in S$ with

$$L = min|\langle f', \delta_{k_i}\rangle|, \quad nL \geq \|M\|\Sigma^n_{i=1(h,g)\in Z(k_i)} sup\{|\langle T(\delta_h), \delta_g\rangle|\}.$$

Therefore

$$\begin{aligned}nL \leq \Sigma^n_{i=1}|\langle f', \delta_{k_i}\rangle| &= \Sigma^n_{i=1(h,g)\in Z(k_i)}|\langle M, f(h, g)\rangle| \\ &\leq \|M\|\Sigma^n_{i=1(h,g)\in Z(k_i)} sup\{|\langle T(\delta_h), \delta_g\rangle|\}.\end{aligned}$$

This is a contradiction. □

Example 1. Let S be the natural numbers $\mathbb{N}$, with the product

$$(m, n) \to m \vee n = max\{m, n\}.$$

S is a semigroup with identity 1 and weakly cancellative. Clearly $E(S) = S$. Then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ and $l^1(S)$ is not Connes amenable as $E(S)$ is infinite.

It is easy to see that each weakly cancellative semigroup is simple. In fact, suppose that $\mathcal{I}$ be a left ideal of S containing a nonzero element i, then

$$S = Si \subseteq S\mathcal{I} \subseteq \mathcal{I}$$

and so $\mathcal{I} = S$. Consequently, if S is a weakly cancellative semigroup with $E(S)$ finite, then S is completely simple semigroup and Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$.

Theorem 3.1. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. If $l^1(S)$ is Connes amenable, then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$, $\mathcal{L}M(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable.*

Proof. By Proposition 3.1, S is a simple semigroup with $E(S)$ finite, then S is a completely simple semigroup. Therefore, S is a Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$. By Proposition 5.6 in [4], $l^1(S)$ is isometrically algebra isomorphic to $\mathcal{L}M(l^1(G), P, I, J)$. Since $l^1(S)$ is Connes amenable, then $\mathcal{L}M(l^1(G), P, I, J)$ is Connes amenable and it has an identity. Also by Theorem 2.1, $l^1(G)$ is Connes amenable. □

Theorem 3.2. *Let S be a weakly cancellative semigroup with $E(S)$ finite and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. Then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$. With above notation, the following are equivalent:*

(i) *$l^1(S)$ is Connes amenable;*

(ii) *$\mathcal{L}M(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable;*

(iii) *$l^1(S)$ is amenable.*

Proof. $(i) \Rightarrow (ii)$ is Theorem 3.1.
$(ii) \Rightarrow (i)$ $\mathcal{L}M(l^1(G), P, I, J)$ has an identity, then I and J are finite and P is invertible [4]. Since $l^1(G)$ is Connes amenable, then from Theorem 2.1, $\mathcal{L}M(l^1(G), P, I, J)$ is Connes amenable. By Proposition 5.6 in [4], $l^1(S)$ is isometrically algebra isomorphic to $\mathcal{L}M(l^1(G), P, I, J)$ and $l^1(S)$ is Connes amenable.
$(ii) \Leftrightarrow (iii)$ By Theorem 5.3 in [22], $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable. Then, this is Theorem 5.9 in [4]. □

Theorem 3.3. *Let $S = \mathcal{M}(G, P, I, J)$ be a weakly cancellative semigroup. Let $l^1(S)$ be unital with unit $e_{l^1(S)}$. With above notation, the following are equivalent:*

(i) *$l^1(S)$ is Connes amenable;*

(i) *$l^1(S)$ is amenable.*

Proof. This follows in the same manner as the proof of Theorem 3.2. □

Example 2. Let G be an amenable group. Let J be finite of order n. Let $S = \mathcal{M}(G, P, 1, J)$ where P is invertible. Let $(a)_{1j}$ and $(b)_{1l}$ be two non-zero elements of $S = \mathcal{M}(G, P, 1, J)$. It is easy to see that

$$(a)_{1j}({p_{j1}}^{-1}a^{-1}b)_{1l} = (b)_{1l}.$$

Then S is a weakly cancellative semigroup. By [11], $l^1(G)$ is amenable. By Proposition 5.6 and Theorem 4.1 in [4], $l^1(S)$ is amenable. Also $l^1(G)$ is Connes amenable and by Theorem 2.1, $l^1(S)$ is Connes amenable.

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