TAMKANG JOURNAL OF MATHEMATICS
Volume 52, Number 3, 413-441, September 2021
DOI:10.5556/j.tkjm.52.2021.3563

An Iterative Method for a Common Solution of a Combination of the Split Equilibrium Problem, a Finite Family of Nonexpansive Mapping and a Combination of Variational Inequality Problem

Ihssane Hay, Abdellah Bnouhachem and Themistocles M.Rassias

Abstract. The present paper aims to deal with an iterative algorithm for finding common solution of the combination of the split equilibrium problem and a finite family of nonexpansive mappings and the combination of variational inequality problem in the setting of real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution to these problems. A numerical example is presented to illustrate the proposed method and convergence result. The results and method presented in this paper generalize, extend and unify some known results in the literatures.

1 Introduction.

The theory of equilibrium problems has grown enormously in various branches of the pure and applied sciences, it has been widely studied in the literature. It provides a framework for many problems in finance, economics, networks analysis, optimization and others; see for example [2, 4, 5, 6, 7, 8, 10, 11, 12, 18, 19, 20, 21, 22, 26].

Let $\mathcal{H}$ be a real Hilbert space with the inner product $< . >$ and the norm $\| . \|$. Let $\mathcal{C}$ be nonempty closed convex subset of Hilbert space $\mathcal{H}$. Given a bifunction $F : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}$, the standard equilibrium problem is formulated as follows:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{C} \\ F(x^*, x) \geq 0, \quad \forall x \in \mathcal{C}, \end{cases} \tag{1.1}$$

which was first considered and investigated by Blum and Oettli [1]. The solution set of the equilibrium problem is denoted by $EP(F)$.

Inspired by a wide variety of works in this direction, Kazmi and Rivzi [17] have recently investigated and studied a new form of the equilibrium problem called the split equilibrium problem:

Key words and phrases. Split equilibrium problem; variational inequality problem; fixed point problem; projection method; nonexpansive mapping.
2010 *Mathematics Subject Classification*. 49J30, 47H09, 47J20.
Corresponding author: Abdellah Bnouhachem.

let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Given two bifunctions $F : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}_1$, and $G : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{H}_2$ and a bounded linear operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, the split equilibrium problem is defined by:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{C} \\ F(x^*, x) \geq 0, \quad \forall x \in \mathcal{C}, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \text{Find} \quad y^* = Ax^* \in \mathcal{Q} \\ G(y^*, y) \geq 0, \quad \forall y \in \mathcal{Q}. \end{cases} \tag{1.3}$$

If $G \equiv 0$ and $F(x^*, x) = \langle Bx^*, x - x^* \rangle$, where $B : \mathcal{C} \longrightarrow \mathcal{H}$, is a nonlinear operator, then the split equilibrium problem collapses to the classical variational inequality problem:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{C} \\ \langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C}. \end{cases} \tag{1.4}$$

The set of solutions of (1.4) is denoted $(\mathcal{VIP})_{B,\mathcal{C}}$. It is easy to observe that

$$x^* \in (\mathcal{VIP})_{B,\mathcal{C}} \Longleftrightarrow x^* = P_C[x^* - \rho Bx^*], \quad \text{where} \quad \rho > 0. \tag{1.5}$$

Variational inequalities are being used as a mathematical programming tool in modeling a large class of problems arising in various branches of pure and applied sciences. In recent years, variational inequalities have been generalized and extended novel and new techniques in several directions. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems; see [1-28].
Both classes of problems; variational inequality and split equilibrium problem, have been studied and treated in details in several research works due to its important role on the development of many problems; see for example [3, 9, 13, 14, 15, 16, 17, 28].

In the present paper, motivated by the above works and related literature, we present a new iterative algorithm for finding a common element of the solution set of common fixed points of a finite family of nonexpansive mappings and the solution set of a combination of the split equilibrium problem and the solution set of a combination of variational inequality problem. More precisely, we use the idea of combination of the split equilibrium problem. According to numerical results, the iteration algorithm for the combination of the split equilibrium problem converges faster than the iterative algorithm of the split equilibrium problem. Under appropriate conditions we derive the strong convergence results for this method. Preliminary numerical experiments are included to verify the theoretical assertions of the proposed method. Since the combination of the split equilibrium problem includes the split equilibrium problem and the equilibrium problem as special cases, results presented in this paper continue to hold for these problems.

2 Preliminaries.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces with inner product $< . >$, and norm $\| \, . \, \|$. Let $\mathcal{C}$ and Q be nonempty closed convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For $i \in \{1, ..., N\}$, let $\sum_{i=1}^N a_i F_i : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}_1$, and $\sum_{i=1}^N b_i G_i : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{H}_2$, be two bifunctions, and let $\sum_{i=1}^N c_i B_i : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$, with $a_i \in]0,1[$, $b_i \in]0,1[$, $c_i \in]0,1[$, $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = \sum_{i=1}^N c_i = 1$ and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, be a bounded linear operator. The combination of split equilibrium problem (CSEP) is defined as:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{C} \\ \sum_{i=1}^N a_i F_i(x^*, x) \geq 0, \quad \forall x \in \mathcal{C}, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \text{Find} \quad y^* = Ax^* \in \mathcal{Q} \\ \sum_{i=1}^N b_i G_i(y^*, y) \geq 0, \quad \forall y \in Q. \end{cases} \tag{2.2}$$

The solution set of the (CSEP) is denoted by $\Omega = \{p \in \mathcal{C}; z \in \cap_{i=1}^N EP(F_i)$ such that $Ap \in \cap_{i=1}^N EP(G_i)\}$.

If $F_i = F$ and $G_i = G, \forall i = 1, 2, \ldots, N$, then the combination of split equilibrium problem (2.1)-(2.2) reduces to the split equilibrium problem (1.2)-(1.3). The fixed point problem for a sequence of nonexpansive mappings $S_n : \mathcal{C} \longrightarrow \mathcal{C}$ is to find $x \in C$ such that

$$S_n x = x. \tag{2.3}$$

The set of all fixed points of S_n is denoted by $F(S_n)$. The combination of variational inequality problem (CVIP) is defined as:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{C} \\ \left\langle \sum_{i=1}^N c_i B_i(x^*), x - x^* \right\rangle \geq 0, \quad \forall x \in \mathcal{C}. \end{cases} \tag{2.4}$$

The solution set of the (CVIP) is denoted by $(\mathcal{VIP})_{\sum_{i=1}^N c_i B_i, \mathcal{C}}$.

We introduce the following definitions, which are useful in the following analysis.

Definition 1. Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathbb{R}^n$, and $v \in \mathbb{R}^n$, then the projection of v onto $\mathcal{C}$ is denoted by $P_{\mathcal{C}}(v)$, that is,

$$P_{\mathcal{C}}(v) := \arg\min \{\| v - u \| / u \in \mathcal{C}\}. \tag{2.5}$$

Since $\mathcal{C}$ is convex and closed, the projection onto $\mathcal{C}$ is unique.

Definition 2. The mapping $T : \mathcal{C} \longrightarrow \mathcal{C}$ is said to be :

1) strongly positive if $\langle Tx, x\rangle \geq \rho \parallel x \parallel^2$, for all $x \in \mathcal{H}$, where ρ is a positive constant.

2) β-inverse strongly monotone over $\mathcal{C}$, if there exists a positive real number $\beta > 0$ such that

$$\langle T(u) - T(v), u - v\rangle \quad \geq \quad \beta \parallel T(u) - T(v) \parallel^2; \qquad \forall u, v \in \mathcal{C},$$

3) nonexpansive if

$$\parallel Tx - Ty \parallel \leq \parallel x - y \parallel; \qquad \forall x, y \in \mathcal{C}.$$

4) lower semicontinuous at $x_0 \in \mathcal{C}$ if, for any sequence $\{x_n\}$ in $\mathcal{C}$ with $x_n \longrightarrow x_0$,

$$T(x_0) \leq \lim_{n \longrightarrow \infty} T(x_n).$$

We list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto $\mathcal{C}$.

Lemma 2.1. *(i). If T is nonexpansive, then $I - T$ is 1- inverse strongly monotone;*

(ii). If $T : \mathcal{C} \longrightarrow \mathcal{C}$, is β-inverse strongly monotone, then for all $\lambda \in]0, 2\beta[$, $I - \lambda T$ is nonexpansive.

Lemma 2.2. *Let $\mathcal{C} \subset \mathbb{R}^n$, be a closed convex set, then we have*

1) $\forall\, v \in \mathbb{R}^n, \forall\, w \in \mathcal{C}$

$$\langle v - P_{\mathcal{C}}(v), w - P_{\mathcal{C}}(v)\rangle \;\leq\; 0. \tag{2.6}$$

2) $\forall\, v \in \mathbb{R}^n, \forall\, w \in \mathbb{R}^n$

$$\parallel P_{\mathcal{C}}(v) - P_{\mathcal{C}}(w) \parallel \quad \leq \quad \parallel v - w \parallel . \tag{2.7}$$

3) $\forall\, v \in \mathbb{R}^n, \forall\, u \in \mathcal{C}$

$$\parallel P_{\mathcal{C}}(v) - u \parallel \quad \leq \quad \parallel v - u \parallel . \tag{2.8}$$

Lemma 2.3. *It is well known that in case of a real Hilbert space $\mathcal{H}$, the following assertions:*

(i) $\langle x, y\rangle \;=\; \frac{1}{2}\Big(\parallel x \parallel^2 + \parallel y \parallel^2 - \parallel x - y \parallel^2 \Big),$

(ii) $\parallel \alpha x + (1-\alpha) y \parallel^2 \;=\; \alpha \parallel x \parallel^2 + (1-\alpha) \parallel y \parallel^2 - \alpha(1-\alpha) \parallel x - y \parallel^2,$

hold for all $\alpha \in [0, 1]$ and $x, y \in \mathcal{H}$, such that $x \neq y$.

It is well known that every nonexpansive operator $T : H \rightarrow H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\Big\langle (x - Tx) - (y - Ty), Ty - Tx \Big\rangle \leq \frac{1}{2} \parallel (Tx - x) - (Ty - y) \parallel^2 \tag{2.9}$$

and therefore, we get, for all $(x, y) \in H \times F(T)$,

$$\Big\langle x - Tx, y - Tx \Big\rangle \leq \frac{1}{2} \parallel Tx - x \parallel^2 . \tag{2.10}$$

We assume that both $F : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}_1$ and $G : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{H}_2$ are bifunctions satisfying the following assumptions:

Assumption 2.1

1) $F(x, x) = 0 \quad \forall x \in \mathcal{C}$;

2) F *is monotone i.e.,* $F(x, y) + F(y, x) \leq 0 \quad \forall x, y \in \mathcal{C}$;

3) F *is hemicontinuous i.e.*

$$\forall x, y, z \in \mathcal{C} \quad \lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

4) for each $x \in \mathcal{C}$, $y \longrightarrow F(x, y)$ *is convex and lower semicontinuous mapping.*

Lemma 2.4. *[1] Let* $\mathcal{C}$ *be a nonempty closed convex subset of* $\mathcal{H}$ *and let* F *be a function which satisfies above assumptions, then for each* $x \in \mathcal{H}$*, for* $r > 0$*, there exists* $z \in \mathcal{C}$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \forall y \in \mathcal{C}$$

Moreover, define a mapping $T_r^F : \mathcal{C} \longrightarrow \mathcal{H}$ as follows:

$$T_r^F(x) = \Big\{ z \in \mathcal{C} : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \forall y \in \mathcal{C} \Big\}.$$

Then for all $x \in \mathcal{H}$, we have the following :

(i) T_r^F is single-valued;

(ii) T_r^F is firmly nonexpansive, i.e., for all $x, y \in \mathcal{H}$

$$\parallel T_r^F(x) - T_r^F(y) \parallel^2 \quad \leq \quad \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

(iii) $F(T_r^F) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.5. *[25] Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For every $i \in \{1,...,N\}$, let $F_i : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{H}$ be a bifunction satisfying assumptions 2.1, with $\cap_{i=1}^N EP(F_i) \neq \emptyset$. Then for $a_i \in (0,1)$ and $\sum_{i=1}^N a_i = 1$, we conclude that $\sum_{i=1}^N a_i F_i$, also satisfies the same above assumptions, and*

$$F(T_r) = EP(\sum_{i=1}^N a_i F_i) = \cap_{i=1}^N EP(F_i).$$

Lemma 2.6. *[24] (Demiclosedness principle.) Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space H. If $T : \mathcal{C} \to \mathcal{C}$ is a nonexpansive mapping with Fix(T)$\neq \emptyset$, then the mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in $\mathcal{C}$ weakly converges to x and if $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)x = 0$.*

Lemma 2.7. *[27] Let $\{a_n\}$ be a sequence of positive real numbers such that*

$$a_{n+1} \leq (1-\nu_n)a_n + \delta_n,$$

where $\{\nu_n\}$ is a sequence in $]0,1[$ and $\{\delta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty} \nu_n = \infty$;

2) $\lim_{n\to\infty} \sup \dfrac{\delta_n}{\nu_n} \leq 0$, *or* $\sum_{n=1}^{\infty} |\delta_n| \leq \infty$.

Hence $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. *[25] Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For all $i \in \{1,...,N\}$, let B_i be a strongly positive linear bounded operator on a Hilbert space $\mathcal{H}$ with coefficient $\rho_i > 0$ and $\bar{\rho} = \min_{\{1,...,N\}} \rho_i$. i.e., for all $x \in \mathcal{H}$ $\langle B_i x, x\rangle \geq \rho_i \parallel x \parallel^2$. Moreover for each $i \in \{1,...,N\}$, $c_i \subset]0,1[$, with $\sum_{i=1}^N c_i = 1$. Then the following properties :*

(i) For all $0 < \lambda <\parallel B_i \parallel^{-1}$, we have $\parallel I - \lambda \sum_{i=1}^N c_i B_i \parallel \leq 1 - \lambda\bar{\rho}$, and $\parallel I - \lambda \sum_{i=1}^N c_i B_i \parallel$ is nonexpansive mapping.

(ii) $(\mathcal{VIP})_{(\sum_{i=1}^N c_i B_i, \mathcal{C})} = \cap_{i=1}^N (\mathcal{VIP})_{(B_i,\mathcal{C})}$.

3 The proposed method and some properties

In this section, we suggest and analyze our method for finding common solutions of the combination of the split equilibrium problem (2.1)-(2.2) and a finite family of nonexpansive mappings (2.3) and the combination of variational inequality problem (2.4).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $\mathcal{C}$ (respectively $\mathcal{Q}$) be a nonempty closed convex subset of $\mathcal{H}_1$ (respectively $\mathcal{H}_2$). Let $F_i : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}_1$ and $G_i : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{H}_2$ be two finite family bifunctions satisfying Assumptions 2.1 such that G_i is upper semicontinuous mapping. Let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator, and let B_i be strongly positive linear bounded operator on a Hilbert space $\mathcal{H}_1$ with coefficient $\rho_i > 0$ and $\bar{\rho} = \min_{i\in\{1,\ldots,N\}} \rho_i$. Let $S_n : \mathcal{C} \longrightarrow \mathcal{C}$ be a sequence of nonexpansive mappings such that $\Gamma = \Big(\cap_{n=1}^{\infty} F(S_n)\Big) \cap \Omega \cap \Big(\cap_{i=1}^{N} (\mathcal{VIP})_{B_i,\mathcal{C}}\Big)$ and let $S : \mathcal{C} \longrightarrow \mathcal{C}$ be a mapping such that $\lim_{n\to\infty} S_n x = Sx$.

Algorithm 3.1 For a given $x_1 \in \mathcal{C}_1 = \mathcal{C}$, arbitrarily, let the iterative sequences $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{\sum_{i=1}^{N} a_i F_i}\big(I - \gamma A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)A\big)x_n \\ z_n = P_{\mathcal{C}}[u_n - \lambda_n \sum_{i=1}^{N} c_i B_i u_n] \\ y_n = \alpha_n S_n x_n + (1-\alpha_n) z_n \\ \mathcal{C}_{n+1} = \{p \in \mathcal{C}_n : \| y_n - p \| \leq \| x_n - p \|\} \\ x_{n+1} = \gamma_n P_{\mathcal{C}_{n+1}} x_1 + (1-\gamma_n) y_n, \quad n \geq 1. \end{cases} \tag{3.1}$$

Let $\gamma \in \left]0, \frac{1}{L}\right[$ such that L is the spectral radius of the operator A^*A, where A^* is the adjoint of A. Let $\{s_n\}$ and $\{r_n\}$ be two positive real sequences, and let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ be three sequences in $]0, 1[$, satisfying the following conditions:

C1) $0 < a \leq \alpha_n, \gamma_n \leq b < 1$.

C2) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = \sum_{i=1}^{N} c_i = 1$.

C3) $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

C4) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

C5) $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

C6) $\lim_{n\to\infty} \inf r_n > 0$ and $\lim_{n\to\infty} \sup s_n > 0$.

If for $i = 1, 2, \ldots, N$, $F_i = F$, $G_i = G$, $B_i = B$, $H_1 = H_2 = H$, $S_n = S$ and $s_n = r_n$, then Algorithm 3.1 reduces to Algorithm 3.2 for finding the common solutions of split equilibrium problem (1.2)-(1.3), variational inequality problem (1.3) and a finite family of nonexpansive mappings (2.3).

Algorithm 3.2 For a given $x_1 \in \mathcal{C} = \mathcal{C}$, arbitrarily, let the iterative sequences $\{u_n\}$, $\{z_n\}$, $\{y_n\}$

and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^F(I - \gamma A^*(I - T_{r_n}^G)A)x_n \\ z_n = P_{\mathcal{C}}[u_n - \lambda_n B u_n] \\ y_n = \alpha_n S x_n + (1-\alpha_n) z_n \\ \mathcal{C}_{n+1} = \{p \in \mathcal{C}_n : \| y_n - p \| \leq \| x_n - p \|\} \\ x_{n+1} = \gamma_n P_{\mathcal{C}_{n+1}} x_1 + (1-\gamma_n) y_n, \quad n \geq 1. \end{cases} \tag{3.2}$$

Let $\gamma \in \left]0, \frac{1}{L}\right[$ where L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Let $\{r_n\}$ be a positive real sequence, and let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ be three sequences in $]0,1[$, satisfying the following conditions:

C1) $0 < a \leq \alpha_n, \gamma_n \leq b < 1$.

C2) $\lim\limits_{n\to\infty} \lambda_n = 0$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

C3) $\lim\limits_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

C4) $\lim\limits_{n\to\infty} \gamma_n = 0$; and $\sum_{n=1}^{\infty} \gamma_n = \infty$; and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

C5) $\lim\limits_{n\to\infty} \inf r_n > 0$.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by the Algorithm 3.1. Then we have*

(i) *$\{x_n\}$ is well defined for every $n \in \mathbb{N}^*$ and bounded.*

(ii) $\Gamma \subset \mathcal{C}_{n+1}$.

Proof. We show that the sequence $\{x_n\}$ is well defined for every $n \in \mathbb{N}^*$. To prove that, we will show that $\mathcal{C}_n$ is a closed convex subset for all $n \geq 1$. Indeed, $\mathcal{C}_1 = \mathcal{C}$ is closed convex. Assume that $\mathcal{C}_k$ is closed convex; we have to prove that so is $\mathcal{C}_{k+1}$.

Let $p_m \in \mathcal{C}_{k+1} \subset \mathcal{C}_k$ such that $p_m \to p$ then $p \in \mathcal{C}_k$; (because $\mathcal{C}_k$ is closed) thus

$$\| y_k - p_m \| \leq \| x_k - p_m \| .$$

This implies that

$$\begin{aligned} \| y_k - p \| &\leq \| y_k - p_m \| + \| p_m - p \| \\ &\leq \| x_k - p_m \| + \| p_m - p \|, \end{aligned}$$

by taking $\lim\limits_{m\to\infty}$ on both sides of the above estimate, hence we get

$$\lim_{m\to\infty} \| y_k - p \| = \| y_k - p \| \leq \lim_{m\to\infty} (\| x_k - p_m \| + \| p_m - p \|)$$

$$\leq \quad \| x_k - p \| \ .$$

Then $p \in \mathcal{C}_{k+1}$, it follows that $\mathcal{C}_{k+1}$ is closed.

Now we set $p = \lambda x + (1-\lambda)y$, for every $x, y \in \mathcal{C}_{k+1}$ then $p \in \mathcal{C}_k$ (because $\mathcal{C}_k$ is convex). From lemma 2.3, we get

$$\begin{aligned}
\| y_k - p \|^2 &= \| y_k - \lambda x - (1-\lambda)y \|^2 \\
&= \| \lambda(y_k - x) + (1-\lambda)(y_k - y) \|^2 \\
&= \lambda \| y_k - x \|^2 + (1-\lambda) \| y_k - y \|^2 - \lambda(1-\lambda) \| y - x \|^2 \\
&\leq \lambda \| x_k - x \|^2 + (1-\lambda) \| x_k - y \|^2 - \lambda(1-\lambda) \| y - x_k + x_k - x \|^2 \\
&= \| \lambda(x_k - x) + (1-\lambda)(x_k - y) \|^2 \\
&= \| x_k - p \|^2,
\end{aligned}$$

thus $p \in \mathcal{C}_{k+1}$ then $\mathcal{C}_{k+1}$ is convex. Therefore $\mathcal{C}_n$ is closed convex for all $n \geq 1$. Since for every $x_1 \in \mathcal{C}$, $P_{\mathcal{C}_{n+1}} x_1$ is well-defined, and $y_n \in \mathcal{C}_{n+1}$, then x_n is well defined. Obviously, $\Gamma \subset \mathcal{C}_1$. Assume that $\Gamma \subset \mathcal{C}_n$, and we show that $\Gamma \subset \mathcal{C}_{n+1}$. Without loss of generality, we assume that for all $n \geq 0$, and for every $i \in \{1, ..., N\}$, $0 < \lambda_n <\| B_i \|^{-1}$. Applying Lemma 2.8, we get that $I - \lambda_n \sum_{i=1}^N c_i B_i$ is a nonexpansive mapping. Then from (1.5) for all $p \in \Gamma$, $p = P_{\mathcal{C}}[p - \lambda_n \sum_{i=1}^N c_i B_i p]$ then according to (2.7), we obtain

$$\begin{aligned}
\| z_n - p \|^2 &= \| P_{\mathcal{C}}[u_n - \lambda_n \sum_{i=1}^N c_i B_i u_n] - P_{\mathcal{C}}[p - \lambda_n \sum_{i=1}^N c_i B_i p] \|^2 \\
&\leq \| u_n - \lambda_n \sum_{i=1}^N c_i B_i u_n - p + \lambda_n \sum_{i=1}^N c_i B_i p \|^2 \\
&= \| (I - \lambda_n \sum_{i=1}^N c_i B_i) u_n - (I - \lambda_n \sum_{i=1}^N c_i B_i) p \|^2 \\
&\leq \| u_n - p \|^2 \ . \qquad (3.3)
\end{aligned}$$

For all $p \in \Gamma$, we have $p = T_{r_n}^{\sum_{i=1}^N a_i F_i} p$ and $(I - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})A)p = p$, then

$$\begin{aligned}
&\| A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax - A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ay \|^2 \\
&\qquad = \| A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay) \|^2 \\
&\qquad = \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay), A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay) \rangle \\
&\qquad = \langle (I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay), AA^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay) \rangle \\
&\qquad \leq L \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i})(Ax - Ay) \|^2 \ .
\end{aligned}$$

Then

$$\| (I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay) \|^2 \geq \frac{1}{L} \| A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay) \|^2 . \tag{3.4}$$

Since $T_{s_n}^{\sum_{i=1}^{N} b_i G_i}$ is nonexpansive, it follows from Lemma 2.1, that $I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}$ is 1- inverse strongly monotone, then

$$\begin{aligned} \| (I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay) \|^2 &\leq \left\langle (I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay), (Ax - Ay) \right\rangle \\ &= \left\langle A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay), x - y) \right\rangle \end{aligned} \tag{3.5}$$

and hence using (3.4) and (3.5), we obtain

$$\left\langle A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay), x - y) \right\rangle \geq \frac{1}{L} \| A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})(Ax - Ay) \|^2;$$

this implies that $A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})A$ is $\frac{1}{L}$-inverse strongly monotone, by using Lemma 2.1 we get $I - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})A$ is nonexpansive for each $\gamma \in]0, \frac{1}{L}[$. Therefore, we obtain

$$\begin{aligned} \| u_n - p \| = \| & T_{r_n}^{\sum_{i=1}^{N} a_i F_i} \big(I - \gamma A^* \big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big) A\big) x_n \\ & - T_{r_n}^{\sum_{i=1}^{N} a_i F_i}(I - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i})A)p \| \\ \leq & \| x_n - p \| . \end{aligned} \tag{3.6}$$

Let $p \in \Gamma$, then the following results can be immediately obtained from Lemma 2.3 (ii), nonexpansiveness of S_i, (3.3) and (3.6).

$$\begin{aligned} \| y_n - p \|^2 &= \| \alpha_n S_n x_n + (1 - \alpha_n) z_n - p \|^2 \\ &= \| \alpha_n S_n x_n + (1 - \alpha_n) z_n - \alpha_n S_n p - (1 - \alpha_n) S_n p \|^2 \\ &= \alpha_n \| S_n x_n - S_n p \|^2 + (1 - \alpha_n) \| z_n - p \|^2 - \underbrace{\alpha_n (1 - \alpha_n)}_{\geq 0} \| S_n x_n - z_n \|^2 \quad (3.7) \\ &\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| u_n - p \|^2 \quad (3.8) \\ &\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| x_n - p \|^2 \\ &= \| x_n - p \|^2, \quad (3.9) \end{aligned}$$

which yields that $p \in \mathcal{C}_{n+1}$, therefore $\Gamma \subset \mathcal{C}_{n+1}$.

Next, we show that the sequence $\{x_n\}$ is bounded. Note that $\| P_{\mathcal{C}_{n+1}} x_1 - x_1 \|^2 \leq \| x^* - x_1 \|^2$ for all $x^* \in \mathcal{C}_{n+1}$. In particular, we have

$$\| P_{\mathcal{C}_{n+1}} x_1 - x_1 \|^2 \leq \| P_\Gamma x_1 - x_1 \|^2 .$$

Then, we get

$$\begin{aligned}
\| x_{n+1} - x_1 \|^2 &= \| \gamma_n (P_{\mathcal{C}_{n+1}} x_1 - x_1) + (1-\gamma_n)(y_n - x_1) \|^2 \\
&\leq \gamma_n \| P_{\mathcal{C}_{n+1}} x_1 - x_1 \|^2 + (1-\gamma_n) \| y_n - x_1 \|^2 \\
&\leq \gamma_n \| P_\Gamma x_1 - x_1 \|^2 + (1-\gamma_n) \| x_n - x_1 \|^2 \qquad \text{– see (3.9) –} \\
&\leq \max\{\| P_\Gamma x_1 - x_1 \|^2, \| x_n - x_1 \|^2\} \\
&\ \ \vdots \\
&\leq \max\{\| P_\Gamma x_1 - x_1 \|^2, 0\} \\
&= \| P_\Gamma x_1 - x_1 \|^2 .
\end{aligned}$$

Therefore it follows from the above inequalities that $\| x_{n+1} - x_1 \| < \infty$, hence $\{x_n\}$ is bounded, and from inequalities (3.6), (3.3), and (3.9) we conclude the boundless of $\{u_n\}$, $\{z_n\}$, and $\{y_n\}$.
□ □

Lemma 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, we have*

(a) $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$;

(b) $\lim_{n\to\infty} \| y_n - x_n \| = 0$;

(c) $\lim_{n\to\infty} \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \| = 0$;

(d) $\lim_{n\to\infty} \| S_n x_n - z_n \| = 0$;

(e) $\lim_{n\to\infty} \| x_n - z_n \| = 0$.

Proof. From (2.7), we have

$$\begin{aligned}
&\| z_n - z_{n-1} \| \\
&= \| P_{\mathcal{C}}[u_n - \lambda_n \sum_{i=1}^N c_i B_i u_n] - P_{\mathcal{C}}[u_{n-1} - \lambda_{n-1} \sum_{i=1}^N c_i B_i u_{n-1}] \| \\
&\leq \| (I - \lambda_n \sum_{i=1}^N c_i B_i) u_n - (I - \lambda_{n-1} \sum_{i=1}^N c_i B_i) u_{n-1} \| \\
&= \| (I - \lambda_n \sum_{i=1}^N c_i B_i) u_n - (I - \lambda_n \sum_{i=1}^N c_i B_i) u_{n-1} + (I - \lambda_n \sum_{i=1}^N c_i B_i) u_{n-1} \\
&\qquad - (I - \lambda_{n-1} \sum_{i=1}^N c_i B_i) u_{n-1} \|
\end{aligned}$$

$$\leq \| (I-\lambda_n\sum_{i=1}^{N}c_iB_i)u_n-(I-\lambda_n\sum_{i=1}^{N}c_iB_i)u_{n-1}\| + \| (\lambda_{n-1}-\lambda_n)\sum_{i=1}^{N}c_iB_iu_{n-1}\| .$$

According to Lemma 2.8, we obtain

$$\| z_n - z_{n-1} \| \quad \leq \quad \| u_n - u_{n-1} \| + |\lambda_n - \lambda_{n-1}| \, \| \sum_{i=1}^{N} c_iB_iu_{n-1} \| . \tag{3.10}$$

Next, we apply the definition of y_n, and the inequality (3.10), we get

$$\begin{aligned}
&\| y_n - y_{n-1} \| \\
&\quad = \| \alpha_n S_n x_n + (1-\alpha_n) z_n - \alpha_{n-1} S_{n-1} x_{n-1} - (1-\alpha_{n-1}) z_{n-1} \| \\
&\quad = \| \alpha_n (S_n x_n - S_n x_{n-1}) + (\alpha_n - \alpha_{n-1}) S_n x_{n-1} + \alpha_{n-1}(S_n x_{n-1} - S_{n-1}x_{n-1}) \\
&\qquad + (1-\alpha_n)(z_n - z_{n-1}) + (\alpha_{n-1} - \alpha_n) z_{n-1} \| \\
&\quad \leq \alpha_n \| x_n - x_{n-1} \| + |\alpha_n - \alpha_{n-1}|(\| S_n x_{n-1} \| + \| z_{n-1} \|) \\
&\qquad + \alpha_{n-1} \| S_n x_{n-1} - S_{n-1} x_{n-1} \| + (1-\alpha_n) \| z_n - z_{n-1} \| \\
&\quad \leq \alpha_n \| x_n - x_{n-1} \| + |\alpha_n - \alpha_{n-1}|(\| S_n x_{n-1} \| + \| z_{n-1} \|) + \alpha_{n-1} \| S_n x_{n-1} - S_{n-1} x_{n-1} \| \\
&\quad + (1-\alpha_n)\Big(\| u_n - u_{n-1} \| + |\lambda_n - \lambda_{n-1}| \, \| \sum_{i=1}^{N} c_iB_iu_{n-1} \| \Big).
\end{aligned}$$

Using the above inequalities, we obtain the following result, for all $n \geq 1$,

$$\begin{aligned}
&\| x_{n+1} - x_n \| \\
&\quad = \| \gamma_n (P_{\mathcal{C}_{n+1}} x_1 - P_{\mathcal{C}_n} x_1) + (\gamma_n - \gamma_{n-1}) P_{\mathcal{C}_n} x_1 + \gamma_n (y_{n-1} - y_n) + (\gamma_{n-1} - \gamma_n) y_{n-1} \\
&\qquad + y_n - y_{n-1} \| \\
&\quad \leq \| P_{\mathcal{C}_{n+1}} x_1 - P_{\mathcal{C}_n} x_1 \| + |\gamma_n - \gamma_{n-1}|(\| P_{\mathcal{C}_n} x_1 \| + \| y_{n-1} \|) + (1-\gamma_n) \| y_n - y_{n-1} \| \\
&\quad \leq \| P_{\mathcal{C}_{n+1}} x_1 - P_{\mathcal{C}_n} x_1 \| + |\gamma_n - \gamma_{n-1}|(\| P_{\mathcal{C}_n} x_1 \| + \| y_{n-1} \|) \\
&\quad + (1-\gamma_n)\Bigg(\alpha_n \| x_n - x_{n-1} \| + |\alpha_n - \alpha_{n-1}|(\| S_n x_{n-1} \| + \| z_{n-1} \|) + \\
&\quad \alpha_{n-1} \| S_n x_{n-1} - S_{n-1} x_{n-1} \| + (1-\alpha_n)\Big(\| u_n - u_{n-1} \| + |\lambda_n - \lambda_{n-1}| \, \| \sum_{i=1}^{N} c_iB_iu_{n-1} \| \Big)\Bigg) \\
&\quad \leq (1-\gamma_n) \| x_n - x_{n-1} \| + \| P_{\mathcal{C}_{n+1}} x_1 - P_{\mathcal{C}_n} x_1 \| + |\gamma_n - \gamma_{n-1}|(\| P_{\mathcal{C}_n} x_1 \| + \| y_{n-1} \|) \\
&\qquad + |\alpha_n - \alpha_{n-1}|(\| S_n x_{n-1} \| + \| z_{n-1} \|) + \alpha_{n-1}(\| S_n x_{n-1} \| + \| S_{n-1} x_{n-1} \|) \\
&\qquad + (1-\alpha_n)\Big(\| u_n \| + \| u_{n-1} \| + |\lambda_n - \lambda_{n-1}| \, \| \sum_{i=1}^{N} c_iB_iu_{n-1} \| \Big)
\end{aligned}$$

$$\leq (1-\gamma_n) \parallel x_n - x_{n-1} \parallel + M\Big(1 + |\gamma_n - \gamma_{n-1}| + |\alpha_n - \alpha_{n-1}| + \alpha_{n-1} - \alpha_n + 1 + |\lambda_n - \lambda_{n-1}|\Big).$$

where

$$M = \max\Big\{ \sup_{n\geq 1}(\parallel P_{\mathcal{C}_{n+1}}x_1 \parallel + \parallel P_{\mathcal{C}_n}x_1 \parallel), \sup_{n\geq 1}(\parallel P_{\mathcal{C}_n}x_1 \parallel + \parallel y_{n-1} \parallel), \sup_{n\geq 1}(\parallel S_n x_{n-1} \parallel - \parallel z_{n-1} \parallel), \sup_{n\geq 1}(\parallel S_n x_{n-1} \parallel + \parallel S_{n-1}x_{n-1} \parallel), \sup_{n\geq 1}(\parallel u_n \parallel + \parallel u_{n-1} \parallel), \sup_{n\geq 1}(\parallel \sum_{i=1}^{N} c_i B_i u_{n-1} \parallel)\Big\}.$$

Setting $\delta_n = M\Big(2 + |\gamma_n - \gamma_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}|\Big)$ and by using conditions **C3)**, **C4)** and **C5)**, it follows that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Hence from Lemma 2.7 we conclude directly that $\lim_{n\to\infty} \parallel x_{n+1} - x_n \parallel = 0$ which proves the result **(a).**

Now, we show the assertion **(b)**. We have

$$\begin{aligned}
\parallel y_n - x_n \parallel &\leq \parallel y_n - x_{n+1} \parallel + \parallel x_{n+1} - x_n \parallel \\
&= \parallel y_n - \gamma_n P_{\mathcal{C}_{n+1}}x_1 - (1-\gamma_n)y_n \parallel + \parallel x_{n+1} - x_n \parallel \\
&= \parallel \gamma_n(y_n - P_{\mathcal{C}_{n+1}}x_1) \parallel + \parallel x_{n+1} - x_n \parallel \\
&\leq \gamma_n \parallel y_n - P_{\mathcal{C}_{n+1}}x_1 \parallel + \parallel x_{n+1} - x_n \parallel \\
&\leq \gamma_n \parallel x_n - P_{\mathcal{C}_{n+1}}x_1 \parallel + \parallel x_{n+1} - x_n \parallel \\
&= \gamma_n \parallel x_n - P_{\Gamma}x_1 \parallel + \parallel x_{n+1} - x_n \parallel \\
&\leq \gamma_n(\parallel x_n \parallel + \parallel P_{\Gamma}x_1 \parallel) + \parallel x_{n+1} - x_n \parallel .
\end{aligned}$$

This implies with **C5)** condition and **(a)**, $\lim_{n\to\infty} \parallel y_n - x_n \parallel = 0$, thus **(b)** is proved.

Now we prove the assertion **(c)**. For $p \in \Gamma$, we have $p = T_{r_n}^{\sum_{i=1}^{N} a_i F_i} p$. Using Lemma 2.4 we get

$$\begin{aligned}
&\|u_n - p \|^2 \\
&= \parallel T_{r_n}^{\sum_{i=1}^{N} a_i F_i}\big(I - \gamma A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)A\big)x_n - T_{r_n}^{\sum_{i=1}^{N} a_i F_i} p \parallel^2 \\
&\leq \parallel x_n - p - \gamma A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n \parallel^2 \\
&= \parallel x_n - p \parallel^2 + \gamma^2 \parallel A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n \parallel^2 - 2\gamma\big\langle x_n - p, A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n\big\rangle \\
&= \parallel x_n - p \parallel^2 + \gamma^2\big\langle A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n, A^*\big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n\big\rangle \\
&\quad - 2\gamma\big\langle A(x_n - p), \big(I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}\big)Ax_n\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \| x_n - p \|^2 + \gamma^2 L \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 \\
&- 2\gamma \big\langle A(x_n - p) + (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n - (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n, \big(I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \big\rangle \\
&= \| x_n - p \|^2 + \gamma^2 L \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 - 2\gamma \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 \\
&- 2\gamma \big\langle T_{s_n}^{\sum_{i=1}^N b_i G_i} A x_n - Ap, \big(I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \big\rangle.
\end{aligned}
$$

It follows from (2.10) that

$$
\begin{aligned}
\| u_n - p \|^2 &\leq \| x_n - p \|^2 + \gamma^2 L \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 - 2\gamma \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 \\
&+ 2\gamma \frac{1}{2} \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 \\
&= \| x_n - p \|^2 + \gamma(\gamma L - 1) \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 . \qquad (3.11)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\| y_n - p \|^2 \\
&\quad = \| \alpha_n S_n x_n + (1 - \alpha_n) z_n - p \|^2 \\
&\quad = \| \alpha_n (S_n x_n - S_n p) + (1 - \alpha_n)(z_n - p) \|^2 \\
&\quad = \alpha_n \| S_n x_n - S_n p \|^2 + (1 - \alpha_n) \| z_n - p \|^2 - \alpha_n (1 - \alpha_n) \| S_n x_n - z_n \|^2 \\
&\quad \leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| u_n - p \|^2 \\
&\quad \leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n)(\| x_n - p \|^2 + \gamma(\gamma L - 1) \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2) \\
&\quad = \| x_n - p \|^2 + (1 - \alpha_n)\gamma(\gamma L - 1) \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 .
\end{aligned}
$$

Hence

$$
\begin{aligned}
(1 - \alpha_n)\gamma(1 - \gamma L) \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i}) A x_n \|^2 &\leq \| x_n - p \|^2 - \| y_n - p \|^2 \\
&= \| x_n - y_n \| (\| x_n - p \| + \| y_n - p \|).
\end{aligned}
$$

Using **(b),** condition **C1)** and letting $n \longrightarrow \infty$, we obtain the desired result.

In the following, we show the assertion **(d);** let $p \in \Gamma$, from (3.7), (3.3) and (3.6), we get

$$
\begin{aligned}
\| y_n - p \|^2 &\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| z_n - p \|^2 - \alpha_n (1 - \alpha_n) \| S_n x_n - z_n \|^2 \\
&\leq \| x_n - p \|^2 - \alpha_n (1 - \alpha_n) \| S_n x_n - z_n \|^2 .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\alpha_n (1 - \alpha_n) \| S_n x_n - z_n \|^2 &\leq \| x_n - p \|^2 - \| y_n - p \|^2 \\
&= (\| x_n - p \| + \| y_n - p \|) \| x_n - y_n \| .
\end{aligned}
$$

From the condition **C1)**, it holds that

$$a(1-b)\parallel S_n x_n - z_n \parallel^2 \;\leq\; (\parallel x_n - p \parallel + \parallel y_n - p \parallel) \parallel x_n - y_n \parallel . \tag{3.12}$$

By using **(b)**, it can be easily seen that $\lim_{n\to\infty} \parallel S_n x_n - z_n \parallel = 0$.

Next, we show the assertion **(e),** we have

$$\parallel y_n - z_n \parallel \;=\; \alpha_n \parallel S_n x_n - z_n \parallel . \tag{3.13}$$

Furthermore

$$\begin{aligned}
\parallel z_n - x_n \parallel &\leq \parallel z_n - x_{n+1} \parallel + \parallel x_{n+1} - x_n \parallel \\
&= \parallel z_n - \gamma_n P_{\mathcal{C}_{n+1}} x_1 - (1-\gamma_n) y_n \parallel + \parallel x_{n+1} - x_n \parallel \\
&\leq \gamma_n \parallel z_n - P_{\mathcal{C}_{n+1}} x_1 \parallel + (1-\gamma_n) \parallel z_n - y_n \parallel + \parallel x_{n+1} - x_n \parallel \\
&\leq \gamma_n (\parallel P_{\mathcal{C}_{n+1}} x_1 \parallel + \parallel z_n \parallel) + (1-\gamma_n) \parallel y_n - z_n \parallel + \parallel x_{n+1} - x_n \parallel .
\end{aligned}$$

Combining the above inequality with (3.13), we obtain

$$\parallel z_n - x_n \parallel \;\leq\; \gamma_n (\parallel P_{\mathcal{C}_{n+1}} x_1 \parallel + \parallel z_n \parallel) + (1-\gamma_n)\alpha_n \parallel S_n x_n - z_n \parallel + \parallel x_{n+1} - x_n \parallel .$$

We apply **(d)**, **(a)** and the conditions **C4)** and **C5)** of Algorithm 3.1, so we get immediately that $\lim_{n\to\infty} \parallel z_n - x_n \parallel = 0$. □ □

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $q \in \Gamma$, where $\Gamma = \Big(\cap_{n=1}^{\infty} F(S_n) \Big) \cap \Omega \cap \Big(\cap_{i=1}^{N} (\mathcal{VIP})_{B_i,\mathcal{C}} \Big)$, and*
$\Omega = \Big\{ p \in \mathcal{C} : p \in \cap_{i=1}^{N} EP(F_i)$ and $Ap \in \cap_{i=1}^{N} EP(G_i)$, for all $i \in \{1, ..., N\} \Big\}$.

Proof. Firstly, we show that $q \in \cap_{n=1}^{\infty} F(S_n)$. We know that x_n is bounded, then there exist a subsequence x_{n_j} such that $x_{n_j} \rightharpoonup q$.
Since $\parallel S_{n_j} x_{n_j} - x_{n_j} \parallel \leq \parallel S_n x_{n_j} - z_{n_j} \parallel + \parallel z_{n_j} - x_{n_j} \parallel$. From **(d)** and **(e)** of Lemma 3.2, we get $\lim_{j\to\infty} \parallel S_{n_j} x_{n_j} - x_{n_j} \parallel = 0$. Using Lemma 2.6, we get $S_n q - q = 0$, then $q \in F(S_n)$ for every $n \in \mathbb{N}^*$. Hence $q \in \cap_{n=1}^{\infty} F(S_n)$.

Next, we show that $q \in \Omega = \Big\{ p \in \mathcal{C} : p \in \cap_{i=1}^{N} EP(F_i) \text{ and } Ap \in \cap_{i=1}^{N} EP(G_i) \Big\}$.
Let us first prove that $\lim_{n\to\infty} \parallel u_n - x_n \parallel = 0$. For a given $p \in \Gamma$, we easily obtain

$$\begin{aligned}
&\parallel u_n - p \parallel^2 \\
&\quad = \parallel T_{r_n}^{\sum_{i=1}^{N} a_i F_i} (x_n - \gamma A^* (I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}) A x_n) - T_{r_n}^{\sum_{i=1}^{N} a_i F_i} p \parallel^2 \\
&\quad \leq \langle T_{r_n}^{\sum_{i=1}^{N} a_i F_i} (x_n - \gamma A^* (I - T_{s_n}^{\sum_{i=1}^{N} b_i G_i}) A x_n)
\end{aligned}$$

$$- T_{r_n}^{\sum_{i=1}^N a_i F_i} p, x_n - p - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \rangle$$
$$= \langle u_n - p, x_n - p - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \rangle.$$

From Lemma 2.3, it follows that

$$\begin{aligned}
&\| u_n - p \|^2 \\
&\leq \frac{1}{2}\Big(\| u_n - p \|^2 + \| x_n - p - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \|^2 \\
&\quad - \| u_n - x_n + \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \|^2 \Big) \\
&\leq \frac{1}{2}\Big(\| u_n - p \|^2 + \| x_n - p \|^2 - 2\gamma \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, x_n - p \rangle \\
&\quad + \gamma^2 \| A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \|^2 - \| u_n - x_n \|^2 \\
&\quad - 2\gamma \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, u_n - x_n \rangle - \gamma^2 \| A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \|^2 \Big).
\end{aligned}$$

Then

$$\| u_n - p \|^2 \leq \| x_n - p \|^2 - \| u_n - x_n \|^2 - 2\gamma \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, u_n - p \rangle. \quad (3.14)$$

Substituting (3.8) into (3.14), we obtain

$$\begin{aligned}
\| y_n - p \|^2 &\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n)\Big(\| x_n - p \|^2 - \| u_n - x_n \|^2 \\
&\quad - 2\gamma \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, u_n - p \rangle \Big) \\
&= \| x_n - p \|^2 - (1 - \alpha_n) \| u_n - x_n \|^2 - 2\gamma(1 - \alpha_n) \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, u_n - p \rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(1 - \alpha_n) \| u_n - x_n \|^2 \\
&\leq \| x_n - p \|^2 - \| y_n - p \|^2 - 2\gamma(1 - \alpha_n) \langle A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n, u_n - p \rangle \\
&\leq (\| x_n - p \| + \| y_n - p \|) \| y_n - x_n \| \\
&+ 2\gamma(1 - \alpha_n) \| (I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n \| \| A(u_n - p) \| .
\end{aligned}$$

Letting $n \longrightarrow \infty$ and using **(b)**, **(c)** and both conditions **C1)** and **C2)** of Algorithm 3.1, we get

$$\lim_{n \to \infty} \| u_n - x_n \| = 0. \quad (3.15)$$

Now, we are ready to prove that for every $i \in \{1, ..., N\}$, we have $p \in \cap_{i=1}^N EP(F_i)$. Indeed; for all $i \in \{1, ..., N\}$, we have from the definition of u_n

$$u_n = T_{r_n}^{\sum_{i=1}^N a_i F_i}(I - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})A)x_n, \quad n \geq 1.$$

Then

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\Big\langle y - u_n, u_n - x_n + \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n\Big\rangle \geq 0 \qquad \forall y \in \mathcal{C},$$

which implies

$$-\sum_{i=1}^{N} a_i F_i(u_n, y) \leq \frac{1}{r_n}\Big(\Big\langle y - u_n, u_n - x_n\Big\rangle + \gamma\Big\langle y - u_n, A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n\Big\rangle\Big).$$

By monotonicity of F_i and from Lemma 2.5, we can write the above inequality as follows

$$\sum_{i=1}^{N} a_i F_i(y, u_n) \leq \frac{1}{r_n}\Big(\Big\langle y - u_n, u_n - x_n\Big\rangle + \gamma\Big\langle y - u_n, A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n\Big\rangle\Big).$$

Therefore

$$\sum_{i=1}^{N} a_i F_i(y, u_{n_j}) \leq \frac{1}{r_{n_j}}\Big(\| y - u_{n_j} \| \| u_{n_j} - x_{n_j} \| + \gamma \| A(y - u_{n_j}) \| \| (I - T_{s_{n_j}}^{\sum_{i=1}^N b_i G_i})Ax_{n_j} \| \Big).$$

From Lemma 3.2, (3.15) and the condition **C6)** of Algorithm 3.1, we can conclude that for every $i \in \{1, ..., N\}$, $\lim_{j\to\infty} \sum_{i=1}^N a_i F_i(y, u_{n_j}) \leq 0$, and since $\sum_{i=1}^N a_i F_i$ is lower semicontinuous in the last argument (see Lemma 2.5), hence we get for each $i \in \{1, ..., N\}$, and for every $y \in \mathcal{C}$

$$\sum_{i=1}^{N} a_i F_i(y, q) \leq 0.$$

Setting $y_t = ty + (1-t)q$ for some $0 < t \leq 1$, then $y_t \in \mathcal{C}$. Using the assertions 1 and 4 of Assumption 2.1, we get

$$\begin{aligned} 0 &= \sum_{i=1}^{N} a_i F_i(y_t, y_t) \\ &= \sum_{i=1}^{N} a_i F_i(y_t, ty + (1-t)q) \\ &\leq t\sum_{i=1}^{N} a_i F_i(y_t, y) + \underbrace{(1-t)\sum_{i=1}^{N} a_i F_i(y_t, q)}_{\leq 0} \\ &\leq t\sum_{i=1}^{N} a_i F_i(y_t, y). \end{aligned}$$

Then for all $i \in \{1, ..., N\}$, $\sum_{i=1}^N a_i F_i(ty + (1-t)q, y) \geq 0$, letting $t \longrightarrow 0$, by the hemicontinuity of $\sum_{i=1}^N a_i F_i$ (see Lemma 2.5), we obtain that $\sum_{i=1}^N a_i F_i(q, y) \geq 0$, for each $i \in \{1, ..., N\}$.

Then $q \in EP(\sum_{i=1}^{N} a_i F_i)$ for every $i \in \{1, ..., N\}$. According to Lemma 2.5, we easily deduce that $q \in \cap_{i=1}^{N} EP(F_i)$.
Next, we show that for all $i \in \{1, ..., N\}$, $Aq \in \cap_{i=1}^{N} EP(G_i)$.
For all $i \in \{1, ..., N\}$, we have A is bounded and since $x_{n_j} \rightharpoonup q$ then $Ax_{n_j} \rightharpoonup Aq$, from Lemma 3.2 **(c)**, we get $T_{s_{n_j},i}^{\sum_{i=1}^{N} b_i G_i} Ax_{n_j} \rightharpoonup Aq$, thus

$$G_i(T_{s_{n_j}}^{\sum_{i=1}^{N} b_i G_i} Ax_{n_j}, z) + \frac{1}{s_{n_j}} \Big\langle z - T_{s_{n_j}}^{\sum_{i=1}^{N} b_i G_i} Ax_{n_j}, T_{s_{n_j},i}^{\sum_{i=1}^{N} b_i G_i} Ax_{n_j} - Ax_{n_j} \Big\rangle \geq 0 \qquad \forall z \in \mathcal{C}.$$

Taking the limit sup of each side of the above inequality, and using the fact that G_i is upper semicontinuous in the first argument and using the condition $C5)$, we conclude that for every $i \in \{1, ...N\}$,

$$G_i(Aq, z) \geq 0, \qquad \forall z \in \mathcal{C}$$

which implies that $Aq \in EP(G_i)$, then $Aq \in \cap_{i=1}^{N} EP(G_i)$ and hence $q \in \Omega$.

Furthermore, we show that $q \in \cap_{i=1}^{N} (\mathcal{VIP})_{B_i, \mathcal{C}}$. Let

$$Tv = \begin{cases} \sum_{i=1}^{N} c_i B_i v + N_{\mathcal{C}} v, & \forall v \in \mathcal{C}, \\ \emptyset & \text{otherwise} \end{cases} \tag{3.16}$$

where $N_C v := \{w \in H : \langle w, v - u \rangle \geq 0, \quad \forall u \in C\}$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $0 \in Tv$ if and only if $v \in (\mathcal{VIP})_{\sum_{i=1}^{N} c_i B_i, \mathcal{C}}$ (see [23]).
Let $G(T)$ denote the graph of T, and let $(v, u) \in G(T)$; since $u - \sum_{i=1}^{N} c_i B_i \in N_{\mathcal{C}} v$, and $z_n \in \mathcal{C}$, we have

$$\big\langle v - z_n, u - \sum_{i=1}^{N} c_i B_i v \big\rangle \geq 0. \tag{3.17}$$

It follows from $z_n = P_C[u_n - \lambda_n \sum_{i=1}^{N} c_i B_i u_n]$ and $v \in C$ that

$$\big\langle v - z_n, z_n - (u_n - \lambda_n \sum_{i=1}^{N} c_i B_i u_n) \big\rangle \geq 0.$$

Then in particular, it follows that

$$\big\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + \sum_{i=1}^{N} c_i B_i u_{n_k} \big\rangle \geq 0. \tag{3.18}$$

Using the fact that $\sum_{i=1}^{N} c_i B_i$ is strongly positive, (3.17), and (3.18), we obtain

$$\big\langle v - z_{n_k}, u \big\rangle \;\; \geq \;\; \big\langle v - z_{n_k}, \sum_{i=1}^{N} c_i B_i v \big\rangle$$

$$
\begin{aligned}
&\geq \Big\langle v - z_{n_k}, \sum_{i=1}^{N} c_i B_i v\Big\rangle - \Big\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + \sum_{i=1}^{N} c_i B_i u_{n_k}\Big\rangle \\
&= \underbrace{\Big\langle v - z_{n_k}, \sum_{i=1}^{N} c_i B_i (v - z_{n_k})\Big\rangle}_{\geq 0} - \Big\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\Big\rangle \\
&+ \Big\langle v - z_{n_k}, \sum_{i=1}^{N} c_i B_i z_{n_k} - \sum_{i=1}^{N} c_i B_i u_{n_k}\Big\rangle \\
&\geq -\Big\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\Big\rangle + \Big\langle v - z_{n_k}, \sum_{i=1}^{N} c_i B_i z_{n_k} - \sum_{i=1}^{N} c_i B_i u_{n_k}\Big\rangle.
\end{aligned}
$$

Note that

$$\| z_n - u_n \| \leq \| z_n - x_n \| + \| x_n - u_n \|,$$

it follows from Lemma 3.2 **(e)** and (3.15) that $\lim_{n\to\infty} \| z_n - u_n \| = 0$, since $\lim_{k\to\infty} x_{n_k} = q$, thus $\lim_{k\to\infty} z_{n_k} = q$ and $\lim_{k\to\infty} u_{n_k} = q$. Hence letting $k \to \infty$, we have $\langle v - q, u\rangle \geq 0$.
Since T is maximal monotone, we have $q \in T^{-1}0$, and hence $q \in (\mathcal{VIP})_{(\sum_{i=1}^{N} c_i B_i, \mathcal{C})} = \cap_{i=1}^{N} (\mathcal{VIP})_{(B_i, \mathcal{C})}$. We finally deduce that

$$q \in \Gamma = \Big(\cap_{n=1}^{\infty} F(S_n)\Big) \cap \Omega \cap \Big(\cap_{i=1}^{N} (\mathcal{VIP})_{B_i, \mathcal{C}}\Big).$$

Next, we show that $\{x_n\}$ converges strongly to $q \in \Gamma$.
Indeed, as $q \in \Gamma$, we have

$$
\begin{aligned}
\| x_{n+1} - q \|^2 &= \| \gamma_n (P_{\mathcal{C}_{n+1}} x_1 - q) + (1 - \gamma_n)(y_n - q) \|^2 \\
&\leq \gamma_n \| P_{\mathcal{C}_{n+1}} x_1 - q \|^2 + (1 - \gamma_n) \| y_n - q \|^2 \\
&\leq \| P_{\mathcal{C}_{n+1}} x_1 - q \|^2 + (1 - \gamma_n) \| x_n - q \|^2 \\
&= (1 - \gamma_n) \| x_n - q \|^2 + \delta_n;
\end{aligned}
$$

where $\delta_n = \| P_{\mathcal{C}_{n+1}} x_1 - q \|^2$. Since

$$
\begin{cases}
\sum_{n=1}^{\infty} \gamma_n = \infty; \\
\sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} \| P_{\mathcal{C}_{n+1}} x_1 - q \|^2 < \infty.
\end{cases}
$$

Thus all the conditions of Lemma 2.7 are satisfied. Hence we deduce that $\lim_{n\to\infty} x_n = q \in \Gamma$. This completes the proof. □ □

4 Numerical example

To verify the theoretical assertions, we consider the following example. All codes were written in Matlab. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, and let $\mathcal{C} = [0, 30]$ and $\mathcal{Q} =]-\infty, 0]$ two closed convex subsets of $\mathbb{R}$. For all $i \in \{1, ..., N\}$, we define A and S_i as follows:

$$\begin{aligned} A \ &: \ \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \\ &\quad x \ \longrightarrow 3x \end{aligned}$$

and

$$\begin{aligned} S_i \ &: \ \mathcal{C} \longrightarrow \mathcal{C} \\ &\quad x \longrightarrow \frac{x}{10i}. \end{aligned}$$

and

$$\begin{aligned} B_i \ &: \ \mathcal{C} \longrightarrow \mathcal{H}_1 \\ &\quad x \longrightarrow \frac{ix}{2}. \end{aligned}$$

It is obvious that A is linear bounded and S_i is nonexpansive. Since $S_i(0) = 0$ then $F(S_i) = \{0\}$. We define the bifunctions F_i and G_i by:

$$\begin{aligned} F_i \ &: \ \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{H}_1 \\ &\quad (u, v) \longrightarrow F_i(u, v) = i(u+1)(v-u) \end{aligned}$$

and

$$\begin{aligned} G_i \ &: \ \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{H}_2 \\ &\quad (x, y) \longrightarrow G_i(x, y) = i(x-10)(y-x) \end{aligned}$$

We define a_i, b_i and c_i, as follows :

$$a_i = \frac{3}{4^i} + \frac{1}{4^N N},$$
$$b_i = c_i = \frac{8}{9^i} + \frac{1}{9^N N}.$$

Let $r_n = \dfrac{n}{n+1}$, $s_n = \dfrac{n}{2n+3}$, $\lambda_n = \dfrac{1}{n+1}$, $\alpha_n = \dfrac{1}{n}$, $\gamma_n = \dfrac{1}{n+1}$, and $\gamma = \dfrac{1}{11}$. It is easy to see that F_i, G_i, r_n, s_n, λ_n, α_n, γ_n, and γ are satisfying all conditions of Algorithm 3.1. In order to simplify the notation we use $\sigma = \sum_{i=1}^{N} ib_i = \sum_{i=1}^{N} ic_i$ and $\rho = \sum_{i=1}^{N} ia_i$.

First, we compute $v_n = T_{s_n}^{\sum_{i=1}^N b_i G_i} A x_n$, for every $x_n \in \mathcal{C}$, $v_n \in \mathcal{Q}$ such that

$$\sum_{i=1}^{N} b_i G_i(v_n, y) + \frac{1}{s_n}\Big\langle y - v_n, v_n - Ax_n \Big\rangle \geq 0. \tag{4.1}$$

Clearly

$$\begin{aligned}
(4.1) \iff & \sigma(v_n - 10)(y - v_n) + \frac{1}{s_n}\Big\langle y - v_n, v_n - Ax_n \Big\rangle \geq 0, \\
\iff & s_n\sigma(v_n - 10)(y - v_n) + (y - v_n)(v_n - 3x_n) \geq 0, \\
\iff & (y - v_n)(\sigma s_n(v_n - 10) + (v_n - 3x_n)) \geq 0, \\
\iff & (y - v_n)(v_n(\sigma s_n + 1) - (10\sigma s_n + 3x_n)) \geq 0,
\end{aligned}$$

since $T_{s_n}^{\sum_{i=1}^N b_i G_i}$ is a single-valued, then $v_n = \dfrac{10\sigma s_n + 3x_n}{\sigma s_n + 1}$, which holds

$$T_{s_n}^{\sum_{i=1}^N b_i G_i} A x_n = \frac{10\sigma s_n + 3x_n}{\sigma s_n + 1}.$$

Now, we determine $w \in \mathcal{C}$, such that $w = x_n - \gamma A^*(I - T_{s_n}^{\sum_{i=1}^N b_i G_i})Ax_n$, then

$$w = x_n - \gamma(9x_n - 3\frac{10\sigma s_n + 3x_n}{\sigma s_n + 1}).$$

In order to compute $u_n = T_{r_n}^{\sum_{i=1}^N a_i F_i} w$, we will find $u_n \in \mathcal{C}$ which satisfies

$$\sum_{i=1}^{N} a_i F_i(u_n, z) + \frac{1}{r_n}\Big\langle z - u_n, u_n - w \Big\rangle \geq 0 \quad \forall z \in \mathcal{C},$$

which is equivalent to

$$\begin{aligned}
\iff & \rho r_n(u_n + 1)(z - u_n) + (z - u_n)(u_n - w) \geq 0, \\
\iff & (z - u_n)(\rho r_n(u_n + 1) + (u_n - w)) \geq 0, \\
\iff & (z - u_n)(u_n(\rho r_n + 1) - (w - \rho r_n)) \geq 0.
\end{aligned}$$

Thus, we get $u_n = \dfrac{w - \rho r_n}{\rho r_n + 1}$, which holds

$$u_n = \frac{1 - 9\gamma}{\rho r_n + 1} x_n + \frac{3\gamma(10\sigma s_n + 3x_n)}{(\sigma s_n + 1)(\rho r_n + 1)} - \frac{\rho r_n}{\rho r_n + 1}.$$

Then

$$z_n = P_{\mathcal{C}}[u_n - \lambda_n \sigma \frac{u_n}{2}]$$

$$y_n = \frac{\alpha_n}{10n} x_n + (1-\alpha_n) z_n.$$

and thus

$$x_{n+1} = \gamma_n P_{\mathcal{C}_{n+1}} x_1 + (1-\gamma_n) y_n,$$

since for $x_1 \in \mathcal{C} = \mathcal{C}_1$, we get $0 \le y_1 \le x_1 \le 30$, then

$$\mathcal{C}_2 = \{p \in \mathcal{C} : |y_1 - p| \le |x_1 - p|\} = \left[0, \frac{y_1 + x_1}{2}\right],$$

it can be clearly seen that $\frac{y_1 + x_1}{2} \le x_1$, which implies $x_2 = P_{\mathcal{C}_2} x_1 = \frac{y_{n,1} + x_1}{2}$. Therefore by the same process we easily get $\mathcal{C}_{n+1} = \left[0, \frac{y_n + x_n}{2}\right]$, consequently $P_{\mathcal{C}_{n+1}} x_1 = \frac{y_n + x_n}{2}$. Thus x_{n+1} can be rewrited as follows

$$x_{n+1} = \gamma_n \frac{y_n + x_n}{2} + (1-\gamma_n) y_n.$$

Hence, the new form of the algorithm is given by:

$$\begin{cases} u_n = \dfrac{1-9\gamma}{\rho r_n + 1} x_n + \dfrac{3\gamma(10\sigma s_n + 3x_n)}{(\sigma s_n + 1)(\rho r_n + 1)} - \dfrac{\rho r_n}{\rho r_n + 1}, \\ z_n = P_{\mathcal{C}}[u_n - \lambda_n \sigma \dfrac{u_n}{2}], \\ y_n = \dfrac{\alpha_n}{10} x_n + (1-\alpha_n) z_n, \\ x_{n+1} = \gamma_n \dfrac{y_n + x_n}{2} + (1-\gamma_n) y_n, \quad n \ge 1. \end{cases} \tag{4.2}$$

	Algorithm 3.1				**Algorithm 3.2**			
n	u_n	z_n	y_n	x_n	u_n	z_n	y_n	x_n
1	4.998900	3.593000	1.000000	10.000000	5.727300	4.295500	1.000000	10.000000
2	1.258800	1.022800	0.592630	3.250000	1.559100	1.299200	0.730870	3.250000
3	0.274130	0.235580	0.168560	1.035500	0.484100	0.423580	0.295180	1.150700
4	-0.030834	0.000000	0.001731	0.276930	0.134250	0.120830	0.093135	0.402120
5	-0.125720	0.000000	0.000117	0.029251	0.010956	0.0100430	0.008531	0.124030
6	-0.137460	0.000000	0.000007	0.002545	-0.034467	0.000000	0.000050	0.018156
7	-0.139900	0.000000	0.000000	0.000188	-0.041879	0.000000	0.000003	0.001344
8	-0.141200	0.000000	0.000000	0.000012	-0.042746	0.000000	0.000000	0.000086
9	-0.142190	0.000000	0.000000	0.000000	-0.043060	0.000000	0.000000	0.000005
10	-0.142980	0.000000	0.000000	0.000000	-0.043290	0.000000	0.000000	0.000000

Table 1: the values of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ with initial value $x_1 = 10$.

Figures 1 and 2 clearly show the behavior of the sequence $\{x_n\}$ generated by the Algorithm 3.1, which converges to the same solution i.e., $0 \in \Gamma = \left(\cap_{i=1}^{\infty} F(S_i)\right) \cap \Omega \cap \left(\cap_{i=1}^{N} (\mathcal{VIP})_{B_i,\mathcal{C}}\right)$.

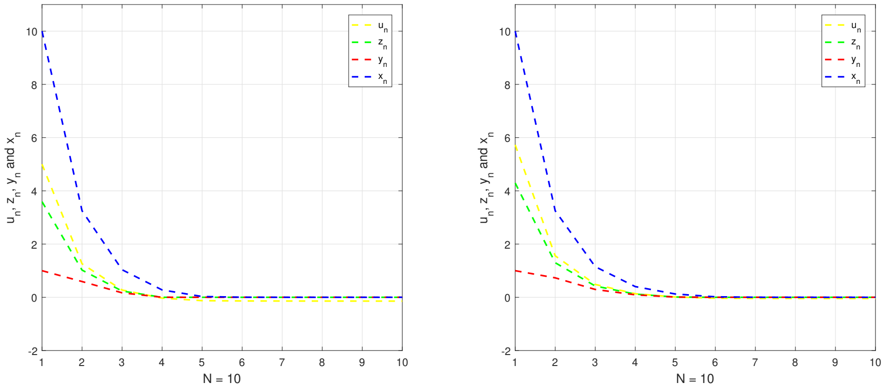

Figure 1: **Convergence of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ with initial value $x_1 = 10$, for Algorithm 3.1 and Algorithm 3.2.**

And the algorithm 3.2, which converges to 0. i.e., $0 \in \Gamma = \left(\cap_{i=1}^{\infty} F(S_i) \right) \cap \Omega \cap (\mathcal{VIP})_{B,\mathcal{C}}$.

Figures 3 and 4 show again that the sequence x_n generated by the Algorithm 3.1. converges to 0, where $0 \in \Gamma = \left(\cap_{i=1}^{\infty} F(S_i) \right) \cap \Omega \cap \left(\cap_{i=1}^{N} (\mathcal{VIP})_{B_i,\mathcal{C}} \right)$. Similary Algorithm 3.2 converges to 0, where $0 \in \Gamma = \left(\cap_{i=1}^{\infty} F(S_i) \right) \cap \Omega \cap (\mathcal{VIP})_{B,\mathcal{C}}$.

In the following we compare the proposed method with those in [20] and [13].

Remark 1. Table 3 and Figure 5, show that the sequence $\{x_n\}$ converges faster than those in [20] and [13].

5 Conclusions

In this paper, we suggested and analyzed an iterative method for finding the approximate element of the common set of solutions of the combination of the split equilibrium problem (2.1)-(2.2), a finite family of nonexpansive mapping (2.3) and the combination of variational inequality problem (2.4) in real Hilbert spaces. We proved that the sequences generated by the proposed iterative method converge strongly to a common solution to these problems. We also discussed a numerical example to demonstrate the applicability of the iterative algorithm. The method and results presented in this paper can be viewed as a refinement and improvement of some existing methods for solving a variational inequality problem and a split equilibrium problem.

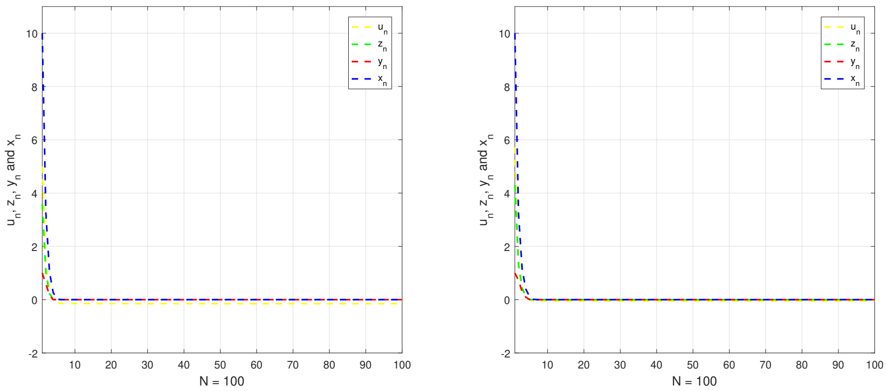

Figure 2: **Convergence of** $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ **and** $\{x_n\}$ **with initial value** $x_1 = 10$, **for Algorithm 3.1 and Algorithm 3.2.**

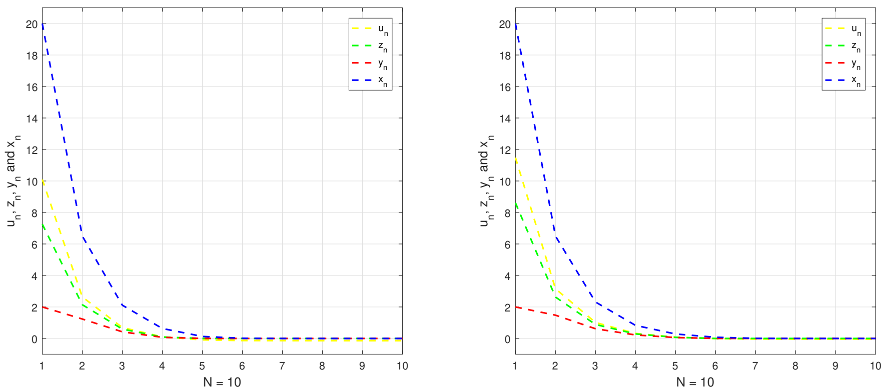

Figure 3: **Convergence of** $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ **and** $\{x_n\}$ **with initial value** $x_1 = 20$, **for Algorithm 3.1 and Algorithm 3.2.**

	Algorithm 3.1				Algorithm 3.2			
n	u_n	z_n	y_n	x_n	u_n	z_n	y_n	x_n
1	10.097000	7.257400	2.000000	20.000000	11.485000	8.613600	2.000000	20.000000
2	2.636900	2.142500	1.233800	6.500000	3.154500	2.628800	1.476900	6.500000
3	0.692060	0.594740	0.419950	2.111500	1.012900	0.886280	0.616570	2.314100
4	0.099939	0.088696	0.070468	0.631390	0.319560	0.287600	0.220880	0.828760
5	-0.091017	0.000000	0.000506	0.126560	0.077398	0.070948	0.057885	0.281670
6	-0.134510	0.000000	0.000003	0.011011	-0.010381	0.000000	0.000213	0.076534
7	-0.139680	0.000000	0.000002	0.000815	-0.040124	0.000000	0.000011	0.005664
8	-0.141190	0.000000	0.000000	0.000052	-0.042634	0.000000	0.000000	0.000365
9	-0.142190	0.000000	0.000000	0.000003	-0.043054	0.000000	0.000000	0.000020
10	-0.142980	0.000000	0.000000	0.000000	-0.043290	0.000000	0.000000	0.000001

Table 2: The values of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ with initial value $x_1 = 20$.

	Algorithm 3.1	Algorithm [20]	Algorithm [13]
n	x_n	x_n	x_n
1	5.000000	5.000000	5.000000
2	1.625000	2.500000	5.000000
3	0.497560	1.250000	3.792000
4	0.099701	0.625000	2.571000
5	0.010531	0.312500	1.639600
6	0.000916	0.156250	1.005600
7	0.000068	0.078125	0.599980
8	0.000004	0.039063	0.350430
9	0.000000	0.019531	0.200930
10	0.000000	0.009766	0.113100

Table 3: the values $\{x_n\}$ with initial value $x_1 = 5$. with three different methods.

References

[1] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student, **63** (1994), 123-145.

[2] A. Bnouhachem, An iterative algorithm for system of generalized equilibrium problems and fixed point problem, Fix. Point Th. Appl. **2014**(235) (2014), 1-21.

[3] A. Bnouhachem, A modified projection method for a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem, Fix. Point Th. Appl. **2014**(22) (2014), 1-25.

[4] A. Bnouhachem, S. Al-Homidan and Q. H. Ansari, An iterative method for common solutions

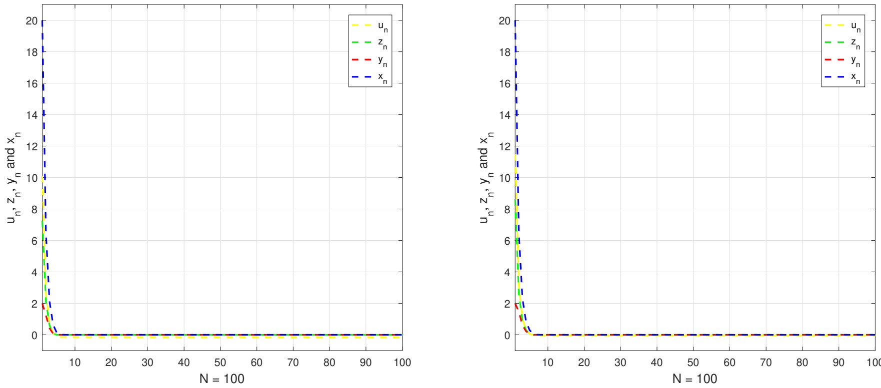

Figure 4: **Convergence of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ with initial value $x_1 = 20$, for Algorithm 3.1 and Algorithm 3.2.**

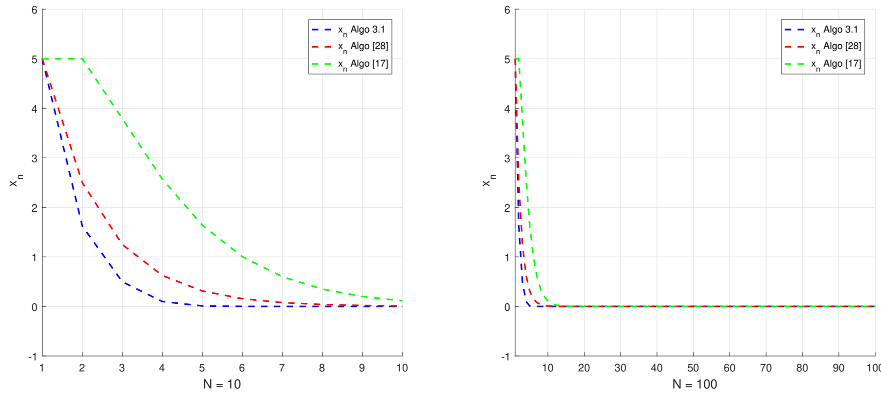

Figure 5: **Convergence of $\{x_n\}$ with initial value $x_1 = 5$, with different number of iterations.**

of equilibrium problems and hierarchical fixed point problems, Fix. Point Th. Appl. **194** (2014), 1-21.

[5] A. Bnouhachem, A hybrid iterative method for a combination of equilibria problem, a combination of variational inequality problems and a hierarchical fixed point problem, Fix. Point Th. Appl. **163**(2014), 1-29.

[6] A. Bnouhachem, Strong convergence algorithm for approximating the common solutions of a variational inequality, a mixed equilibrium problem and a hierarchical fixed-point problem, J. Inequal. Appl. **154** (2014), 1-24.

[7] L. C. Ceng and J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. **214**(1) (2008), 186-201.

[8] S. S. Chang, H. W. Joseph Lee and C. K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal. **70** (2009), 3307-3319.

[9] C. E. Chidume and M. O. Nnakwe, Iterative algorithms for split variational inequalities and generalized split feasibility problems with applications, J. Nonlinear Var. Anal. **3** (2019), 127-140.

[10] F. Cianciaruso, G. Marino, L. Muglia and Y. Yao, A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, Fix. Point Th. Appl. **2010** (2010), Article ID 383740, 19 pages.

[11] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming using proximal like algorithms, Math. Prog. **78** (1997), 29-41.

[12] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. **6** (2005), 117-136.

[13] J. Deepho, W. Kumam and P. Kumam, A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems, J. Math. Model. Algor. **13**(4) (2014), 405-423.

[14] J. Deepho, J. M. Moreno and P. Kumam, A viscosity of Cesaro mean approximation method for split generalized equilibrium, variational inequality and fixed point problems, J. Nonlinear Sci. Appl. **9**, 1475-1496 (2016).

[15] M. Farid, The subgradient extragradient method for solving mixed equilibrium problems and fixed point problems in Hilbert spaces, J. Appl. Numer. Optim. **1** (2019), 335-345.

[16] K. R. Kazmi and S. H. Rizvi, Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup, Math. Sci. **7**(1) (2013), 1-10.

[17] K. R. Kazmi and S. H. Rizvi, Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem, J. Egypt. Math. Society **21**(1) (2013), 44-51.

[18] A. Latif, L. C. Ceng and Q. H. Ansari, Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of equilibrium problem and fixed point problems, Fix. Point Th. Appl. **2012**(186) (2012), 1-26.

[19] C. Marino, L. Muglia and J. C. Yao, Viscosity methods for common solutions of equilibrium and variational inequality problems via multi-step iterative algorithms and common fixed points. Nonlinear Anal. **75**(4) (2012), 1787-1798.

[20] W. Phuengrattana and K. Lerkchaiyaphum, On solving the split generalized equilibrium problem and the fixed point problem for a countable family of nonexpansive multivalued mappings, Fix. Point Th. Appl. **2018**(1) (2018), 1-17.

[21] X. Qin, M. Shang and Y. Su, A general iterative method for equilibrium problem and fixed point problem in Hilbert spaces, Nonlinear Anal. **69**(11) (2008), 3897-3909.

[22] S. Reich and S. Sabach, Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces, optimization theory and related topics, Contemp. Math. **568** (2012), 225-240.

[23] R. T. Rockafellar, On the maximality of sums nonlinear monotone operators, Trans. Amer. Math. Soc. **149** (1970), 75-88 (1970).

[24] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexs, C. R. Acad. Sci. Paris. **258** (1964), 4413-4416.

[25] S. Suwannaut and A. Kangtunyakarn, The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem, Fix. Point Th. Appl. **291** (2013), 1-26.

[26] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. **69**(3) (2008), 1025-1033.

[27] H. K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. **66**(1) (2002), 240-256.

[28] J. Zhao, D. Hou and H. Zong, Several iterative algorithms for solving the multiple-set split common fixed-point problem of averaged operators, J. Nonlinear Funct. Anal. **2019** (2019), Article ID 39.

Ihssane Hay Equipe MAISI, Ibn Zohr University, ENSA, BP 1136, Agadir, Morocco.

E-mail: ihssane.hay@edu.uiz.ac.ma

Abdellah Bnouhachem Equipe MAISI, Ibn Zohr University, ENSA, BP 1136, Agadir, Morocco.

E-mail: babedallah@yahoo.com

Themistocles M.Rassias Department of Mathematics, National Technical University of Athens Zografou Campus 157 80, Athens, Greece.

E-mail: trassias@math.ntua.gr