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On subspace-recurrent operators

Mansooreh Moosapoor

Abstract. In this article, subspace-recurrent operators are presented and it is showed that the set of subspace-transitive operators is a strict subset of the set of subspace-recurrent operators. We demonstrate that despite subspace-transitive operators and subspace-hypercyclic operators, subspace-recurrent operators exist on finite dimensional spaces. We establish that operators that have a dense set of periodic points are subspace-recurrent. Especially, if T is an invertible chaotic or an invertible subspace-chaotic operator, then T^n, T^{-n} and λT are subspace-recurrent for any positive integer n and any scalar λ with absolute value 1. Also, we state a subspace-recurrence criterion.

Keywords. Subspace-recurrent operators, subspace-transitive operators, subspace-recurrent criterion, subspace-chaotic operators, periodic points

1 Introduction and Preliminaries

For a given Banach space X and for a given vector $x \in X$, the orbit of x under T is signified by $orb(T,x) = \{x, Tx, T^2x, ...\}$. If there exists an element $x \in X$ such that $\overline{orb(T,x)} = X$ for a bounded and linear operator T, then T is called hypercyclic. The notion of hypercyclicity is related to the closed subspace problem and studied by mathematicians for years.

A related topic to hypercyclicity is topological transitivity. Let U and V be two open sets of X. A bounded and linear operator T is named topologically transitive if $T^{-n}U \cap V \neq \phi$ for some nonnegative integer n. It is well known that hypercyclicity and topological transitivity are equivalent on a complete and separable metric space X. One can see [6] and [11] for more information. Another central notion in the dynamical system is recurrence. A bounded and linear operator $T : X \to X$ is called recurrent if for any $U \subseteq X$ that is nonempty and open, we have $T^{-n}U \cap U \neq \phi$ for some positive integer n. That means a recurrent operator, send back to itself, for any open set U.

It is not hard to see that transitive operators are recurrent. Also, a vector x is named a recurrent vector for T if we can construct an increasing sequence $\{n_k\}$ that make of positive integers such that $T^{n_k}x \to x$ when $k \to \infty$.

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Corresponding author: Mansooreh Moosapoor.

It is established that any bounded and linear operator on a compact metric space, have some recurrent vectors([8]).

The notion of recurrence was introduced many years ago in [10] and [7]. It is also considered recently by authors like Glasner[9], Costakis and Parissis[4] and Chen[3]. One can find interesting theorems about the recurrence of composition operators in [17]. Also, Bonilla et al. introduced the concept of frequent recurrence in [2].

Subspace-hypercyclic operators and subspace-transitive operators were presented in [13] by Madore and Martinez-Avendano and by this, they extended the notion of hypercyclic operators and transitive operators. Let M be a closed and nonempty subspace of X. A bounded and linear operator T on X is named subspace-hypercyclic with respect to M if there is $x \in X$ so that $\overline{orb(T,x) \cap M} = M$. Also, we say T is M-transitive if for arbitrary nonempty open subsets U, V of M, non-negative integer n can be find such that $T^{-n}U \cap V$ contains a nonempty and relatively open subset of M. It is demonstrated in [13] that subspace-transitive operators are subspace-hypercyclic. We can construct subspace-hypercyclic operators that can not be hypercyclic but authors in [1] proved that any hypercyclic operator is subspace-hypercyclic. One can also see [14] and [15].

Now, we interested in knowing that for a closed subspace M of X, if for an operator T on X we have $T^{-n}U \cap U$ be nonempty and relatively open for any relatively open and nonempty set U and some positive integer n, and call it subspace-recurrence, then what other properties does T has? Clearly, subspace-transitive operators are subspace-recurrent. But is the converse true? Are that mean these operators necessarily subspace-transitive?

Also, authors in [13] showed that subspace-hypercyclic operators and consequently subspace-transitive operators do not exist on finite dimensional spaces. Does it true for subspace-recurrent operators?

Chaotic operators are an important subset of transitive operators. Remind that we know an operator T on a Banach space X as a chaotic operator, if it is transitive and its periodic points be a dense set in X. Also, subspace-chaotic operators are an important subset of subspace-transitive operators. Recall that we say a bounded and linear operator T on X is M-chaotic if it is M-transitive and the set of its periodic points in M is dense in M ([16]). In this paper, we also want to know the relations between subspace-chaotic and subspace-recurrent operators.

In this paper, X denotes an F-space, a complex and complete metrizable topological vector space. Also, $B(X)$ indicates the set of all bounded linear operators on X and we call its elements, by operators. In the article, M shows a nonzero and closed subspace of X.

In Section 2, we present some examples of subspace-recurrent operators. We show that there are subspace-recurrent operators that are not subspace-transitive and by this, we conclude that the set of subspace-transitive operators is a strict subset of the set of subspace-recurrent operators. We also define subspace-recurrent vectors and prove that an operator is M-recurrent if and only if it has a dense set of M-recurrent vectors.

In Section 3, we prove that if an operator has a dense set of periodic points, then it is subspace-recurrent. Especially, if T is an invertible and chaotic or an invertible subspace-chaotic operator, then T^n, T^{-n} and λT are subspace-recurrent for any positive integer n and any scalar λ with $|\lambda| = 1$. We show that surprisingly, subspace-recurrent operators exist on finite dimensional spaces.

In Section 4, we give some conditions that under them, the operator becomes subspace-recurrent. Especially, we state a subspace-recurrent criterion.

2 Insight into Subspace-recurrent Operators

At the beginning of this section, we say the main definition.

Definition 1. An operator T is said M-recurrent or subspace-recurrent with respect to M if for every open and nonempty subset U of M, a positive integer n can be find such that $T^{-n}(U) \cap U$ is nonempty and open in M.

It is mentioned in Section 1 that subspace-transitive operators are subspace-recurrent. So, we can make an example as follows.

Example 1. Assume that B is the backward shift on l^2. Let $T = 5B$, and consider

$$M := \{\{a_n\}_{n=0}^{\infty} : a_{2n} = 0 \text{ for all } n\}.$$

By [[13], Example 3.7], T is M-transitive and accordingly, it is M-recurrent.

The next lemma shows that if T is an M-recurrent operator, then for open and nonempty set $U \subseteq M$, $T^{-n}(U)$ hit U for infinitely many n.

Lemma 2.1. *Suppose that $T \in B(X)$ is M-recurrent. Then,*

$$\{n \in \mathbb{N} : T^{-n}(U) \cap U \quad \textit{is nonempty and open in } M\}$$

is infinite for any open and nonempty subset U of M.

Proof. Suppose, on the contrary, that the set

$$\{n \in \mathbb{N} : T^{-n}(U) \cap U \quad \text{is nonempty and open in } M\}$$

is finite for an open set $U \subseteq M$. Without loose of generality, we can take that

$$\{n \in \mathbb{N} : T^{-n}(U) \cap U \quad \text{is nonempty and open in } M\} = \{1, 2, 3, ..., k\}.$$

So, $T^{-k}(U) \cap U$ is a nonempty and open subset of M. By definition of an M-recurrent operator, a natural number m can be find so that

$$T^{-m}(T^{-k}(U) \cap U) \cap (T^{-k}(U) \cap U)$$

is nonempty and open in M. Hence, $T^{-(m+k)}(U) \cap U \neq \phi$ and open in M. But $m + k$ is greater than k and this is a contradiction. □

Now, we define a subspace-recurrent vector as follows.

Definition 2. If an increasing sequence $\{n_k\}$ of positive integers exists such that $T^{n_k}(x) \in M$ and $T^{n_k}(x) \to x$, where $x \in M$, we call x is an M-recurrent vector. Equivalently, x is an M-recurrent vector if for any $\varepsilon > 0$ the set $\{n \in \mathbb{Z}^+ : T^n(x) \in M \text{ and } d(T^n(x), x) < \varepsilon\}$ is infinite.

We offer the symbol $Rec_M(T)$ to show the set of all M-recurrent vectors for the operator T.

The next theorem presents an equivalent condition for subspace-recurrence.

Theorem 2.1. *For an operator $T \in B(X)$, T is M-recurrent if and only if $\overline{Rec_M(T)} = M$.*

Proof. Consider that $\overline{Rec_M(T)} = M$. Allow U_M be an open and nonempty set in M. By hypothesis, $\overline{Rec_M(T)} = M$. So, U_M includes a recurrent vector like y. But U_M is open. So, there exists $\varepsilon > 0$ such that $B_M = B(y, \varepsilon) \cap M \subseteq U_M$. By Definition 2, $n \in \mathbb{N}$ can be find such that $T^n(y) \in M$ and $||T^n(y) - y|| < \varepsilon$. Therefore, $y \in T^{-n}(U_M)$ and hence, $y \in T^{-n}(U_M) \cap U_M$.

Now, assume that T is M-recurrent. Let $U = B(x_0, \varepsilon_0) \cap M$ for some $x_0 \in M$ and $\varepsilon_0 < 1$. T is M-recurrent. So, there exists $n_1 \in \mathbb{N}$ such that $U_1 = T^{-n_1}(U) \cap U$ is nonempty and open in M. Therefore, we can find $x_1 \in M$ and $\varepsilon_1 < \frac{1}{2}$ such that

$$U_2 = B(x_1, \varepsilon_1) \cap M \subseteq U_1 = T^{-n_1}(U) \cap U.$$

Again, $U_2 = B(x_1, \varepsilon_1) \cap M$ is open in M. So, we can find n_2 with $n_2 > n_1$ so that $T^{-n_2}(U_2) \cap U_2$ is nonempty and open in M. Hence, we can find $x_2 \in M$ and $\varepsilon_2 < \frac{1}{2^2}$ such that

$$U_3 = B(x_2, \varepsilon_2) \cap M \subseteq U_2 = B(x_1, \varepsilon_1) \cap M.$$

By induction, we can make a sequence $\{n_k\}$ that is increasing and their elements are positive integers and we can create a sequence $\{\varepsilon_k\}$ of real numbers such that $\varepsilon_k < \frac{1}{2^k}$ and

$$B(x_k, \varepsilon_k) \cap M \subseteq B(x_{k-1}, \varepsilon_{k-1}) \cap M$$

and

$$\begin{aligned} &T^{n_k}(B(x_k, \varepsilon_k) \cap M) \subseteq T^{n_k}(T^{-n_k}(B(x_{k-1}, \varepsilon_{k-1}) \cap M) \cap (B(x_{k-1}, \varepsilon_{k-1}) \cap M)) \\ &\subseteq B(x_{k-1}, \varepsilon_{k-1}) \cap M. \end{aligned}$$

Now, by Cantor theorem, $\cap_n (B(x_n, \varepsilon_n) \cap M) = \{y\}$ for some $y \in M$, since M is complete. So, $T^{n_k}(y) \to y$ and hence, y is an M-recurrent vector.

□

In the following lemma, we prove that subspace-recurrence of T^p for a positive integer p implies subspace-recurrence of T.

Lemma 2.2. *Suppose that $T \in B(X)$ and consider $p > 1$ is an integer. Then,*

(i) If T^p is an M-recurrent operator, then T is an M-recurrent operator.

(ii) $Rec_M(T^p) \subseteq Rec_M(T)$.

Proof. Proof of (i) is clear by definition. For proving (ii), let $x \in Rec_M(T^p)$. Hence, one can find an increasing sequence $\{n_k\}$ of positive integers with $(T^p)^{n_k}(x) \in M$ and $(T^p)^{n_k}(x) \to x$. So, $T^{pn_k}(x) \to x$ and pn_k is increasing. Therefore, $x \in Rec_M(T)$.

□

3 Periodic Points and Subspace-recurrent Operators

In this section, we discuss operators that have a dense set of periodic points. We consider relations between them and subspace-recurrent operators. We begin by saying a lemma. The proof of the lemma is not hard by using the definition of the subspace-recurrent vector.

Lemma 3.1. *Consider that $x \in M$ is a periodic point for T. Then x is an M-recurrent vector for T.*

By Lemma 3.1, we can present the subsequent example.

Example 2. Assume that $T = 2B$, where B is the backward shift on l^2. It is shown in [16] that $T \oplus I$ is subspace-chaotic with respect to $M := l^2 \oplus \{0\}$. So, the set of periodic points for T in M is dense in M. But any periodic point is an M-recurrent vector by Lemma 3.1. So, T has a dense set of M-recurrent vectors. Hence, T is an M-recurrent operator by Theorem 2.1.

Moreover, we can extend our statements as it is shown in the next theorem.

Theorem 3.1. *Let $T \in B(X)$. Suppose that the set of periodic points of T is dense in X. Then T is subspace-recurrent with respect to a closed and nontrivial subspace M of X.*

Proof. Assume that $Per(T)$ is the set of periodic points for T. According to the hypothesis, $\overline{Per(T)} = X$. So, by [[1], Theorem 2.1], one can find a closed and nontrivial subspace M of X such that $\overline{Per(T) \cap M} = M$. Hence, the set of periodic points of T in M is dense in M and so, T has a dense set of M-recurrent vectors in M. Consequently, T is an M-recurrent operator by Theorem 2.1.

□

It is established in [13] that subspace-hypercyclic operators and consequently subspace-transitive operators do not exist on finite dimensional spaces. But in the next example, we make examples of subspace-recurrent operators on finite dimensional spaces.

Example 3. (a) The tent map $T : [0,1] \to [0,1]$ is determined by,

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}; \\ 2 - 2x, & \frac{1}{2} < x \le 1. \end{cases}$$

The tent map has a periodic point in any interval $[\frac{m}{2^n}, \frac{m+1}{2^n}]$ [[11], Example 1.24]. Hence, it's periodic points make a dense subset of $[0,1]$.

(b) Recall that a rotation $T : \mathbb{T} \to \mathbb{T}$ is defined by $z \to e^{i\alpha} z$ where $\alpha \in [0, 2\pi)$. If T is a rational rotation, we can detect $N \ge 1$ such that $T^N = I$ [[11], Example 1.24]. So, every point is a periodic point for a rational rotation.

By Theorem 3.1, these operators are subspace-recurrent.

By Example 3, and this fact that subspace-hypercyclic operators and subspace-transitive operators don't exist on finite dimensional spaces, we can say the following corollaries.

Corollary 3.2. *There are finite dimensional spaces that support subspace-recurrent operators.*

Corollary 3.3. *There exist subspace-recurrent operators, where they are not subspace-hypercyclic nor subspace-transitive.*

In fact, we can deduce that the set of subspace-transitive operators is a proper subset of subspace-recurrent operators.

The next theorem shows that if the set of periodic points of T is dense in M, then T^n, λT with $|\lambda| = 1$ and T^{-n}, when T is invertible, are all M-recurrent.

Theorem 3.4. *Consider that $T \in B(X)$. If the set of periodic points of T is dense in M, then:*

(i) T^n *is an* M*-recurrent operator, for every* $n \in \mathbb{N}$*.*

(ii) λT *is* M*-recurrent, where* $\lambda \in \mathbb{C}$ *and* $|\lambda| = 1$*.*

(iii) T^{-n} *is* M*-recurrent for every* $n \in \mathbb{N}$*, when* T *is invertible.*

Proof. For proving (i), let n be a positive integer greater than 1. It is sufficient to show that T^n has a dense set of periodic points in M. Let $x \in M$ be a periodic point for T. So, we can detect $p \in \mathbb{N}$ so that $T^p(x) = x$. In fact,

$$(T^n)^p(x) = T^{np}(x) = \underbrace{T^n \cdots T^n}_{p}(x) = x.$$

This completes the proof.

For proving (ii), note this point that if x is any periodic point for T, then for λ with $|\lambda| = 1$, $\frac{x}{\lambda}$ is a periodic point for λT. For this, suppose that $T^p(x) = x$. Note that,

$$(\lambda T)^p(\frac{x}{\lambda}) = \lambda^p T^p(\frac{x}{\lambda}) = \lambda^p \frac{1}{\lambda^p} T^p(x) = x.$$

Also, we know that:

$$\overline{\{\frac{x}{\lambda} : x \in Per(T)\}} = \overline{\frac{1}{\lambda}\{x : x \in Per(T)\}} = \overline{\{x : x \in Per(T)\}} = M.$$

So, λT has a dense set of periodic points in M and hence, it is an M-recurrent operator.

For proving (iii), let $x \in M$ be a periodic point for T. Consequently, we can discover $p \in \mathbb{N}$ such that $T^p(x) = x$. So, $T^{-p}(T^p(x)) = T^{-p}(x)$. But T is invertible. Therefore, $T^{-p}(T^p(x)) = x$ and hence, $T^{-p}(x) = x$. So, x is a periodic point for T^{-1}. Then $Per(T^{-1}) \cap M$ is dense in M. Similar to (i), we have T^{-n} is M-recurrent.

□

Costakis, Manoussos and Parissis proved in [5] that if T be an invertible operator, then the recurrence of T and T^{-1} are equivalent. By Theorem 3.4, we can conclude that if T is invertible and its periodic points in M are dense in M, then T is M-recurrent if and only if T^{-1} is M-recurrent. Now, the following question arises:

Question: Assume that T is an invertible operator. Can we infer that M-recurrence of T and T^{-1} are equivalent?

Also, we have the following corollaries from Theorem 3.4.

Corollary 3.5. *Consider that* $T \in B(X)$ *is an invertible operator. If* T *is an* M*-chaotic operator, then:*

(i) T^n *is an* M*-recurrent operator, for any* $n \in \mathbb{N}$*.*

(ii) λT *is an* M*-recurrent operator, for any* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$*.*

(iii) If T *is invertible, then* T^{-n} *is an* M*-recurrent operator for any* $n \in \mathbb{N}$*.*

Proof. If T is an M-chaotic operator, by definition of a subspace-chaotic operator, T has a dense set of periodic points in M. So, Theorem 3.4 completes the proof.

□

Corollary 3.6. *Consider that $T \in B(X)$ is an invertible operator. If T is a chaotic operator, then :*

(i) T^n *is subspace-recurrent, for any* $n \in \mathbb{N}$.

(ii) λT *is subspace-recurrent operator, for any* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

(iii) If T *is invertible, then* T^{-n} *is subspace-recurrent for any* $n \in \mathbb{N}$.

Proof. According to the hypothesis, T is chaotic. So, by definition of a chaotic operator, T has a dense set of periodic points in X. Like to proof of Theorem 3.4, T^n, has a dense set of periodic points in X . By Theorem 3.1, there is a closed and nontrivial subspace M_n of X such that T^n is M_n-recurrent. Similarly, λT, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and T^{-n} are subspace-recurrent.

□

By Corollary 3.6, we can make examples as follows.

Example 4. Recall that the Birkhoff's operator on $H(\mathbb{C})$, space of entire functions, is defined by

$$T_a f(z) = f(z + a), \quad a \neq 0.$$

Birkhoff's operators are chaotic[[11], Example 2.35]. Therefore, for any $n \in \mathbb{N}$ and any scalar λ with $|\lambda| = 1$, ${T_a}^n$, ${T_{-a}}^n$ and λT_a are subspace-recurrent.

4 Some Sufficient Conditions for Subspace-recurrence

In this section, we give conditions for an operator to be subspace-recurrent. We state a subspace-recurrence criterion and we construct an example by this criterion.

Lemma 4.1. *Assume that $T \in B(X)$ and assume that Z is a dense subset of M. If we can build an increasing sequence $\{n_k\}$ of positive integers so that:*

(i) $T^{n_k}x \to x$ *for any* $x \in Z$,

(ii) $T^{n_k}(M) \subseteq M$,

then T is M-recurrent.

Proof. Consider $U \subseteq M$ is a relatively open and nonempty set. By hypothesis, Z is dense in M. So, there is $x \in U \cap Z$. By condition (i), we can detect a positive integer n_k such that $T^{n_k}x \in U$. Consequently, $x \in T^{-n_k}(U) \cap U$. But $T^{n_k}(M) \subseteq M$ and hence, $T^{n_k}|_M : M \to M$ is continuous. Therefore $T^{-n_k}(U)$ is open in M. Hence, $T^{-n_k}(U) \cap U$ is nonempty and open in M.

□

Theorem 4.1. *(Subspace-recurrence Criterion) Assume that $T \in B(X)$ and assume that M is a closed and nonzero subspace of X. Suppose that a dense set Z of M and an increasing sequence $\{n_k\}$ of positive integers are existed so that:*

(i) $T^{n_k}x \to 0$ *for any* $x \in Z$,

(ii) For every $x \in Z$, *a sequence* $\{x_k\}$ *can be determined such that* $x_k \in M$ *and* $x_k \to 0$ *and* $T^{n_k}x_k \to x$,

(iii) $T^{n_k}(M) \subseteq M$.

Then T is M-recurrent.

Proof. Let $U \subseteq M$ be a relatively open set. By hypothesis, Z is dense in M. So, $x \in U \cap Z$ and $\varepsilon > 0$ can be find such that $B(x, \varepsilon) \cap M \subseteq U$.

By (i), $T^{n_k}x \to 0$ and by (ii), there exists a sequence $\{x_k\}$ in M such that $x_k \to 0$ and $T^{n_k}x_k \to x$. Hence, we can find a positive integer k such that:

$$||T^{n_k}(x)|| < \frac{\varepsilon}{2}, ||x_k|| < \varepsilon \text{ and } ||T^{n_k}(x_k) - x|| < \frac{\varepsilon}{2}.$$

Now,

$$||(x + x_k) - x|| = ||x_k|| < \varepsilon.$$

Therefore, $x + x_k \in U$. Also, we have:

$$\begin{aligned}
&||T^{n_k}(x + x_k) - x|| = ||T^{n_k}(x) + T^{n_k}(x_k) - x|| \\
&\leq ||T^{n_k}(x)|| + ||T^{n_k}(x_k) - x|| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $T^{n_k}(x + x_k) \in U$ and then $x + x_k \in T^{-n_k}(U) \cap U$. But $T^{-n_k}(U)$ is open in M, by the continuity of since $T^{n_k}|_M$. Therefore, $T^{-n_k}(U) \cap U$ is nonempty and open in M.

□

Example 5. Let $T = \lambda B$, where λ is a scalar with $|\lambda| > 1$. Let

$$M := \{\{a_n\} \in l^2 : a_{3k} = 0 \text{ for all } k\}.$$

Consider Z be the set consists of every finite sequence in l^2. So, Z is dense in l^2. Let $x \in Z$. Then, we can detect a natural number m such that for any $k > m$, we have $x_k = 0$. Hence, $T^{n_k}x \to 0$. But x is an arbitrary element of Z. Hence, condition (i) of Theorem 4.1 holds.

Now, let S be the forward shift on l^2 and let $x_k = \frac{1}{\lambda^{3k}} S^{3k}x$, where x is an arbitrary and fix element of Z. It is not hard to see that $x_k \in M$. Also, since $|\lambda| > 1$, we have

$$||x_k|| = \frac{1}{|\lambda^{3k}|}||x|| \to 0.$$

On the other hand:

$$T^{3k}(x_k) = T^{3k}(\frac{1}{\lambda^{3k}} S^{3k}x) = (\lambda B)^{3k}(\frac{1}{\lambda^{3k}} S^{3k}x) = x.$$

So, condition (ii) of Theorem 4.1 holds.

Also, for any $x \in M$,

$$T^{3k}(x_0, x_1, x_2, x_3, ..., x_6, ..., x_{3n}, ...) = (x_{3k}, x_{3k+1}, x_{3k+2}, x_{3k+3}, ...).$$

and $x_{3n} = 0$ for any n. Consequently $T^{3k}(M) \subseteq M$. Hence, condition (iii) of Theorem 4.1 holds and therefore T is an M-recurrent operator.

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Mansooreh Moosapoor Department of Mathematics, Farhangian University, Tehran, Iran

E-mail: mosapor110@gmail.com; m.mosapour@cfu.ac.ir