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On the r-Stability Index of r-Maximal Closed Hypersurfaces in de Sitter Spaces

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Abstract. It is well-known that some minimal (or maximal) hypersurfaces are stable. However, there is a growing recognition on unstable minimal (or maximal) hypersurfaces by introducing the concept of index of stability. Firstly, the index of stability for minimal hypersurfaces in the Euclidean n-sphere has been introduced by J. Simons [20], which is followed recently by many people (see for instance [3, 9, 18, 21]). Also, Barros and Sousa in [10] have payed attention to the concept of *index of r-stability* (as the r-extension of index of stability) on r-minimal hypersurfaces in the Euclidean n-sphere. They gave low bounds for r-stability index of r-minimal n-dimensional closed hypersurfaces in $\mathbb{S}^{n+1}$. In this paper we give low bounds for the r-stability index of r-maximal closed spacelike hypersurfaces in the de Sitter space S_1^{n+1}.

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1 Introduction

The importance of hypersurfaces with null mean curvature in (semi-) Riemanian manifolds is well-known in physics and mathematics. These hypersurfaces are critical points of the first variational problem of optimizing the area functional. The second variation problem of the area functional leads us to the stability of hypersurfaces. The classification of stable hypersurfaces in Euclidean spheres, as a well-known topic in differential geometry, has been started (firstly) by J. L. Barbosa and M. de Carmo ([5]) and then, it is followed by other researchers (see, for instance, [6, 7, 8]). Similarly, the hypersurfaces with null rth mean curvature in (semi-)Riemannian manifolds, as the critical points of a suit variational problem, play interesting roles in the theory of r-stability. Many people have studied the r-stability of r-minimal hypersurfaces in spheres ([6, 7, 8, 11]). The concept of index of stability, related to the second variational problem, on the unstable hypersurfaces in Euclidean spheres has been introduced by Simons in [20], which is followed by many researchers ([3, 9, 10, 18, 21]). Intuitively, the index of stability of a hypersurface

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gives a rate of distance from being stable. In fact, it gives the number of independent directions in which the hypersurface fails to be of minimum area. In the recent results, some lower bounds have been found for the index of r-minimal hypersurfaces in the standard Euclidean sphere $\mathbb{S}^{n+1}$ of sectional curvature 1. In [9], Barros and Sousa have estimated the index of stability of minimal closed hypersurfaces in the Euclidean spheres. Also, in [10], they gave low bounds for the index of r-stability of some r-minimal hypersurfaces in the Euclidean spheres. They proved that, closed oriented non-totally geodesic minimal hypersurfaces of $\mathbb{S}^{n+1}$ have index of stability greater than or equal to $n+3$, where the equality occurs only when the hypersurface is Clifford tori. Furthermore, they have extended similar results to the closed oriented r-minimal hypersurfaces in $\mathbb{S}^{n+1}$, by estimating the index of r-stability. Up to Clifford tori, for closed oriented hypersurfaces in $\mathbb{S}^{n+1}$ satisfying the conditions $H_{r+1}=0$ and $H_{r+2}<0$, we have $Ind^r(M^n)\geq 2n+5$.

On the other hand, it is well-known that, the complete hypersurfaces with constant mean curvature in Lorentz space forms (especially, in the de Sitter space $\mathbb{S}_1^{n+1}$) have important role in the relativity theory ([15]). Also, we know that, a maximal spacelike entire graph in the Lorentz-Minkowski space-time $\mathbb{R}_1^{n+1}$ is a linear hyperplane. As a generalization of this fact, the totally geodesic hypersurfaces are the only complete spacelike maximal hypersurfaces in $\mathbb{S}_1^{n+1}$. Akutagawa [1] and Ramanathan [19] have showed that, the complete spacelike hypersurfaces with constant mean curvature H in $\mathbb{S}_1^{n+1}$, satisfying the condition $n^2H^2<4(n-1)$ for $n>2$ and $H^2\leq 1$ for $n=2$, are totally umbilical. In this paper, we extend the notion of index of r-stability and give similar results for the r-maximal close spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$. We give some estimator low bounds for r-stability index of some hypersurfaces in $\mathbb{S}_1^{n+1}$.

2 Preliminaries

Here, we recall some basic preliminaries from [13, 16, 17]. By $\mathbb{R}_p^m$ we mean the vector space $\mathbb{R}^m$ with metric $<x,y>:=-\Sigma_{i=1}^p x_iy_i+\Sigma_{j>p}x_jy_j$. Especially, $\mathbb{R}_0^m=\mathbb{R}^m$, and $\mathbb{R}_1^m$ is the Minkowski space. For $c>0$, the pseudo-sphere $\mathbb{S}_q^{n+1}(c)=\{y\in\mathbb{R}_q^{n+2}|<y,y>=c^2\}$ denotes the Euclidean sphere (when $q=0$) and the de Sitter space (when $q=1$) of radius c and curvature $1/c^2$. Similarly, $\mathbb{H}_q^{n+1}(-c)=\{y\in\mathbb{R}_{q+1}^{n+2}|<y,y>=-c^2\}$ denotes the hyperbolic space (when $q=0$) and the anti-de Sitter space (when $q=1$) of radius c and curvature $-1/c^2$. The simply connected space form $\tilde{M}_q^{n+1}(c)$ of curvature c and index q denotes $\mathbb{R}_q^{n+1}$ for $c=0$, $\mathbb{S}_q^{n+1}=\mathbb{S}_q^{n+1}(1)$ for $c=1$ and $\mathbb{H}_q^{n+1}=\mathbb{H}_q^{n+1}(-1)$ for $c=-1$. When $q=0$, we take a component of H_0^{n+1}. The Weingarten formula for a spacelike hypersurface $x:M^n\to\mathbb{S}_1^{n+1}$ is $\bar{\nabla}_VW=\nabla_VW-<SV,W>\mathbf{N}$, for $V,W\in\chi(M)$, where S is the shape operator of M associated to a unit normal vector field $\mathbf{N}$ on M with $<\mathbf{N},\mathbf{N}>=-1$. Since $\mathbb{S}_1^{n+1}$ is time-oriented, on each orientable spacelike hypersurface in $\mathbb{S}_1^{n+1}$ there is a global unit normal timelike vector field $\mathbf{N}$ such that the shape operator associated to $\mathbf{N}$ is diagonalizable. We denote the eigenvalues of its shape operator (i.e. the principal curvatures of M) by the functions $\kappa_1, ..., \kappa_n$ on M. The jth elementary symmetric function $s_j := \sum_{1\le i_1<...<i_j\le n} \kappa_{i_1}...\kappa_{i_j}$ can be used to define the jth mean curvature function H_j on M as $\binom{n}{j}H_j = (-1)^j s_j$. By definition, a spacelike hypersurface $x : M^n \to \tilde{M}_q^{n+1}(c)$ with null $(j+1)$th mean curvature is said to be *j-maximal*.

In particular, $H_1 = -(1/n)tr(S)$ is the ordinary mean curvature function. The normal vector field $\mathbf{H} = H_1\mathbf{N}$ is called the mean curvature vector field on M. There is a relation between the scalar curvature of M and the 2nd mean curvature as $tr(Ric) = n(n-1)(c - H_2)$. In general, since the sign of H_j depends on the chosen orientation only in the odd case, H_j is extrinsic (respectively, intrinsic) when j is an odd (respectively, an even) number.

For a spacelike hypersurface $x : M^n \to \tilde{M}_q^{n+1}(c)$, as in [17], the jth Newton transformation $P_j : \chi(M) \to \chi(M)$, associated with the shape operator S, is defined, inductively, by

$$P_0 = I, P_j = (-1)^j s_j I + S \circ P_{j-1} (j = 1, ..., n),$$

where I is the identity on $\chi(M)$. It can be seen that P_j has an explicit formula,

$$P_j = (-1)^j \Sigma_{l=0}^{j} (-1)^l s_{j-l} S^l = \sum_{l=0}^{j} \tbinom{n}{j-l} H_{j-l} S^l,$$

where, $H_0 = 1$ and $S^0 = I$. According to the characteristic polynomial of S, $Q_S(t) = det(tI - S) = \sum_{l=0}^{n} (-1)^{n-l} s_{n-l} t^l$, the Cayley-Hamilton theorem gives $P_n = 0$.

Let $v_1, ..., v_n$ be a local orthonormal tangent frame of principal directions on M such that $Sv_i = \kappa_i v_i$ for $i = 1, 2, ..., n$. Clearly, we have $P_j v_i = \mu_{i,j} v_i$, for $i = 1, 2, ..., n$, where $\mu_{i,j} = (-1)^j \Sigma_{i_1<...<i_j, i_l\ne i} \kappa_{i_1}...\kappa_{i_j}$, (for $j = 0, 1, ..., n-1$). Using the identity

$$\kappa_i \mu_{i,j} = \mu_{i,j+1} - (-1)^{j+1} s_{j+1} = \mu_{i,j+1} - \tbinom{n}{j+1} H_{j+1},$$

the following formulae can be obtained easily:

$$\begin{aligned} & tr(P_j) = (-1)^j (n-j) s_j = c_j H_j, \\ & tr(S \circ P_j) = (-1)^j (j+1) s_{j+1} = -c_j H_{j+1}, \\ & tr(S^2 \circ P_j) = \tbinom{n}{j+1} \left(n H_1 H_{j+1} - (n-j-1) H_{j+2} \right), \\ & tr(P_j \circ \nabla_X S) = -\tbinom{n}{j+1} < \nabla H_{j+1}, X >, \qquad (X \in \chi(M)), \end{aligned} \tag{2.1}$$

where $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$ and ∇ stands for the gradient operator. For any vector $a \in \mathbb{R}_1^{n+2}$, we define two height functions $\lambda_a :=< x, a >$ and $\gamma_a :=< \mathbf{N}, a >$. From [4, 17], we have $\nabla \lambda_a = a^T$ and $\nabla \gamma_a = -S a^T$.

Notation: We will use the following notations:
(1) $\Lambda := \{\lambda_a | a \in \mathbb{R}_1^{n+2}\}$, $\bar{\Lambda} := span(\Lambda \cup \{1\})$;

(2) $\Gamma := \{\gamma_a | a \in \mathbb{R}^{n+2}_1\}$, $\bar{\Gamma} := span(\Gamma \cup \{1\})$;
(3) $\Omega := span(\Lambda \cup \Gamma)$, $\bar{\Omega} := span(\Omega \cup \{1\})$.
We note that Λ and Γ are linear subspaces of $\mathcal{C}^\infty(M)$, respectively generated by $\mathcal{B} := \{\lambda_{e_i}\}_{i=1}^{n+2}$ and $\hat{\mathcal{B}} := \{\gamma_{e_i}\}_{i=1}^{n+2}$, where $\{e_i\}_{i=1}^{n+2}$ is the canonical basis on the Lorentz-Minkowski space $\mathbb{R}^{n+2}_1$.

Definition 1. The *jth linearized operator* $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a second order differential operator defined by $L_j(f) := tr(P_j \circ \nabla^2 f)$, where $\nabla^2 f$ is given by $< \nabla^2 f(X), Y >= Hess(f)(X,Y)$ for every $X, Y \in \chi(M)$.

From [3, 17], for $j = 1, ..., n-1$, we have the following equalities:

$$L_j\lambda_a = c_j H_{j+1}\gamma_a - cc_j H_j \lambda_a,$$
$$L_j\gamma_a = \binom{n}{j+1} grad(H_{j+1}) + \binom{n}{j+1}[nH_1H_{j+1} - (n-j-1)H_{j+2}]\gamma_a - cc_j H_{j+1}\lambda_a.$$

Definition 2. Let $x : M^n \to \mathbb{S}^{n+1}_1$ be a closed connected orientable spacelike hypersurface isometrically immersed into $\mathbb{S}^{n+1}_1$. A smooth map $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}^{n+1}_1$ is called a *variation* of x if it satisfies the following conditions:

(1) For each $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \to \mathbb{S}^{n+1}_1$ defined by $X_t(p) := X(p,t)$, is a spacelike immersion.

(2) $X_0 = x$ and for every $t \in (-\epsilon, \epsilon)$, $X_t|_{\partial M} = x|_{\partial M}$.

Now, we introduce some notations that will be used in the rest. dM_t denotes the volume element of M endowed with the metric induced by X_t and $\mathbf{N}_t$ denotes the unit normal vector field along X_t. The variational vector field associated to the variation X is the vector field $\frac{\partial X}{\partial t}|_{t=0}$. Putting $f := -\langle \frac{\partial X}{\partial t}, \mathbf{N}_t \rangle$, we have the equality $\frac{\partial X}{\partial t} = fN_t + (\frac{\partial X}{\partial t})^\top$, where $\top$ stands for tangent component. Throughout this paper, we will assume that M is compact. If $f : M^n \to \mathbb{R}$ is a smooth function and $\int f dM = 0$, then there exists a volume-preserving normal variation of M^n whose variational field is $f\mathbf{N}$ (see [22]).

Lemma 2.1. *Let $x : M^n \to \mathbb{S}^{n+1}_1$ be a closed spacelike hypersurface of the de Sitter space, $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}^{n+1}_1$ be a variation of x and $f := -\langle \frac{\partial X}{\partial t}, \mathbf{N}_t \rangle$. Then, for $r = 0, 1, \cdots, n-1$ we have :*

$$\frac{\partial s_{r+1}}{\partial t} = (-1)^{r+1}(L_r f + tr(P_r)f - tr(S^2 \circ P_r)f) + \langle \left(\frac{\partial X}{\partial t}\right)^\perp, \nabla s_{r+1} \rangle.$$

Proof. See the proof of Lemma 2.2 in [12]. □

The rth area functional $A_r : (-\epsilon, \epsilon) \to \mathbb{R}$, associated to a variation X of $x : M^n \to \mathbb{S}^{n+1}_1$, is defined by $A_r(t) := \int_M F_r(t) dM_t$, where $F_r(t)$ is recursively given by $F_0(t) \equiv 1$, $F_1(t) := -s_1(t)$ and $F_r(t) := (-1)^r s_r(t) - \frac{n-r+1}{r-1} F_{r-2}(t)$ for $2 \le r \le n-1$. In the case $r = 0$, the functional A_0 is the classical area functional. If $s_{r+1} = 0$, there is a function $f : M \to \mathbb{R}$ supported in a compact domain $K \subset M$, that satisfies the following lemma.

Lemma 2.2. *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a closed oriented spacelike hypersurface of the de Sitter space with constant $(r+1)$th mean curvature, $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ be a variation of x and $f := -\langle \frac{\partial X}{\partial t}, \mathbf{N}_t \rangle$. Then, we have*

$$\mathrm{A}_r''(t) = (r+1) \int_M [L_r f + tr(P_r) f - tr(S^2 \circ P_r) f] f dM_t.$$

Proof. It is derived from Proposition 2.3 in [12]. □

Associated to each variation $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ of x, we consider a Jacobi functional J_r which is a second order self-adjoint differential operator defined by $J_r := L_r + [tr(P_r) - tr(S^2 \circ P_r)]I$. So, we define a function $B_r : \mathcal{C}_c^\infty(M) \to \mathbb{R}$ by rule $B_r(f) := \int_M f J_r f dM$, where $\mathcal{C}_c^\infty(M)$ stands for the set of compactly supported smooth functions on M. The spacelike hypersurfaces isometrically immersed into $\mathbb{S}_1^{n+1}$ maximizing the function B_r can be interested as stable hypersurfaces. The above discussion shows that M must have zero $(r+1)$th mean curvature.

Definition 3. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a closed spacelike hypersurface of zero $(r+1)$th mean curvature isometrically immersed into $\mathbb{S}_1^{n+1}$. We say that x is *r-stable*, if $B_r(f) \leq 0$ for every function $f \in \mathcal{C}_c^\infty(M)$.

Definition 4. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a closed spacelike hypersurface of zero $(r+1)$th mean curvature isometrically immersed into $\mathbb{S}_1^{n+1}$. the *index of r-stability* of M^n, denoted by $Ind^r(M^n)$ is the maximal dimension of the set $\{f \in \mathcal{C}_c^\infty(M) | B_r(f) > 0\}$.

3 Some examples of spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$

In this section, we show some examples of complete spacelike hypersurfaces with constant $(r+1)$th mean curvature in the de Sitter space.

Example 1. Let $\mathbf{a} \in \mathbb{R}_1^{n+2}$ be a fixed vector and $\sigma =< \mathbf{a}, \mathbf{a} >$. For each positive real number c where $c^2 \geq -\sigma$, The spacelike hypersurface

$$M_c := \{y \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} | < y, \mathbf{a} >= \sqrt{c^2 + \sigma}\}$$

is a totally umbilical hypersurface in $\mathbb{S}_1^{n+1}$. The Gauss map on M_c is $\mathbf{N}(x) = \frac{1}{c}(\mathbf{a} - \sqrt{c^2 + \sigma} x)$, and the principal curvatures of M_c are $\kappa_i = \frac{1}{c}\sqrt{c^2 + \sigma}$ for $i = 1, 2, \cdots, n$. So, for each $1 \leq k \leq n$, we have $H_k = (-1)^k [\frac{1}{c}\sqrt{c^2 + \sigma}]^k$. When $\sigma = -1$ and $c \geq 1$, $M_c = \mathbb{S}^n(c)$. When $\sigma = 1$ and $c > 0$, $M_c = \mathbb{H}^n(-c)$.

Example 2. Let $g : \mathbb{S}_1^{n+1} \to \mathbb{R}$ be defined by $g(x_1, \cdots, x_{n+2}) := -x_1 + x_2$. We consider the hypersurface $M_t := g^{-1}(e^{-t})$, for each $t \in \mathbb{R}$. In fact, it can be restated as

$$M_t = \{(f(y) + sinh\, t, f(y) + cosh\, t, y) \in \mathbb{S}_1^{n+1} | y \in \mathbb{R}^n\},$$

where $f(y) = \frac{-e^t}{2}\Sigma_{i=1}^n y_i^2$. The Gauss map on M_t is $\mathbf{N}(x) = e^t\mathbf{a} - x$, where $\mathbf{a} := (-1, 1, 0, ..., 0) \in \mathbb{R}_1^{n+2}$. Hence. the principal curvatures of M_t are $\kappa_1 = ... = \kappa_n = 1$, and then we have $H_k = (-1)^k$ for $k = 1, 2, \cdots, n$.

Example 3. We define a smooth function $f : \mathbb{S}_1^{n+1} \to \mathbb{R}$ by $f(y) := y_{m+2}^2 + ... + y_{n+2}^2$ (where, $0 < m < n$). For each real number $d > 1$, the hypersurface $M_d = f^{-1}(d^2)$ is a spacelike hypersurface of $\mathbb{S}_1^{n+1}$. The Gauss map on M_d will be

$$N(y) = \frac{-d}{\sqrt{d^2 - 1}}(y_1, ..., y_{m+1}, (1 - \frac{1}{d^2})y_{m+2}, ..., (1 - \frac{1}{d^2})y_{n+2}).$$

In fact, M_d denotes the standard product $H^m(-\sqrt{d^2 - 1}) \times S^{n-m}(d) \subset \mathbb{S}_1^{n+1}$. Similar to Example 3.4 in [?], the principal curvatures of M_d are $\kappa_1 = ... = \kappa_m = \frac{d}{\sqrt{d^2-1}}$, $\kappa_{m+1} = ... = \kappa_n = \frac{\sqrt{d^2-1}}{d}$. Then, the mean curvatures of M_d (of all orders) are constant.

In particular, $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ is called a hyperbolic cylinder and $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$ is called a spherical cylinder.

4 Result

The main ingredient to determine the r-maximal closed spacelike hypersurfaces of the de-Sitter space is the following result derived recently by Caminha [13].

Theorem 4.1. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a r-maximal closed spacelike hypersurface of the de-Sitter space $\mathbb{S}_1^{n+1}$. Then, we have $H_k = 0$ on M, for all k's where $r + 1 \le k \le n$.

First, we recall some properties of the spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$.

Proposition 4.1. ([2]) Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a spacelike hypersurface in $\mathbb{S}_1^{n+1}$, ($n \ge 2$).
(i) If M^n is compact, then it is diffeomorphic to S^n;
(ii) If M^n is compact and totally umbilical in $\mathbb{S}_1^{n+1}$ (for $n \ge 2$), then it is a round n-sphere.

Proposition 4.2. ([14]) Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a complete oriented r-maximal compact spacelike hypersurface in $\mathbb{S}_1^{n+1}$ (where $n \ge 2$) with positive rth mean curvature such that the rank of its shape operator is greater than r. Then, M^n is not r-stable.

Lemma 4.1. ([13]) Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a closed spacelike hypersurface isometrically immersed into the de-Sitter space $\mathbb{S}_1^{n+1}$, $\kappa_1, ..., \kappa_n$ be the principal curvatures and H_r be the rth mean curvature of M^n. Then we have:

(i) For $0 < r < n$, $H_r^2 \geq H_{r-1}H_{r+1}$. If $r = 1$ or if $r > 1$ and $H_{r+1} \neq 0$, then the equality happens if and only if $\kappa_1 = ... = \kappa_n$;

(ii) $H_1 \geq (H_2)^{1/2} \geq ... \geq (H_k)^{1/k}$ if $H_i > 0$ for $i = 1, ..., k$;

Now, we state the auxiliary results on the r-stability and index of r-stability of spacelike hypersurfaces in the de Sitter space.

Lemma 4.2. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a connected orientable spacelike hypersurface isometrically immersed into the de Sitter space. Then we have
(i) $L_k(\lambda_a) = -c_k H_k \lambda_a + c_k H_{k+1}\gamma_a$;
(ii) $L_k(\gamma_a) = \binom{n}{k+1}[nH_1H_{k+1} - (n-k-1)H_{k+2}]\gamma_a - c_k H_{k+1}\lambda_a + \binom{n}{k+1} < \nabla H_{k+1}, a >$.

Theorem 4.2. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be an isometric immersion of a complete connected orientable spacelike hypersurface of the de Sitter space.
(i) If M^n is not totally geodesic in $\mathbb{S}_1^{n+1}$, then we have $\dim(\Lambda) = dim(\Gamma) = n+2$;
(ii) If M^n is not totally umbilical in $\mathbb{S}_1^{n+1}$, then we have $\dim(\bar{\Lambda})$ $=\dim(\bar{\Gamma}) = n+3$.

Proof. (i) Remember that Λ is the linear subspace of $\mathcal{C}^\infty(M)$, generated by $\mathcal{B} := \{\lambda_{e_i}\}_{i=1}^{n+2}$, where $\{e_i\}_{i=1}^{n+2}$ is the canonical basis on the Lorentz-Minkowski space $\mathbb{R}_1^{n+2}$. We show that $\mathcal{B}$ is linearly independent. Assuming $\mathcal{B}$ to be linearly dependent, we have a non-zero finite sequence of real numbers $\{r_i\}_{i=1}^{n+2}$ such that $\sum_{i=1}^{n+2} r_i\lambda_{e_i} \equiv 0$. Putting $v := \sum_{i=1}^{n+2} r_ie_i$ we have $\lambda_v \equiv 0$. One can assume that v is a non zero vector with $< v, v >\in \{-1, 0, 1\}$. On the other hand, the image of M by the isometric immersion x lies in the spacelike hyperplane with a timelike normal vector. Remember that, the totally umbilic spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ are obtained as intersection of a spacelike hyperplane of $\mathbb{R}_1^{n+2}$ with $\mathbb{S}_1^{n+1}$, and the causal character of the hyperplane determines the type of the hypersurface. More precisely, we obtain $< v, v >= -1$ and $x(M^n)$ is $\mathbb{S}^n \subset \mathbb{S}_1^{n+1}$, totally geodesic hypersurface in de Sitter space(see [17]), which contradicts with the assumption. Therefore, $\mathcal{B}$ is a linearly independent subset of $\mathcal{C}(M)$ and $\dim(\Lambda) = n+2$.

Similarly, Γ is the linear subspace of $\mathcal{C}^\infty(M)$, generated by $\hat{\mathcal{B}} := \{\gamma_{e_i}\}_{i=1}^{n+2}$. It is enough to show that $\hat{\mathcal{B}}$ is linearly independent. If not, there is a non-zero sequence $\{r_i\}_{i=1}^{n+2}$,of real numbers such that $\sum_{i=1}^{n+2} r_i\gamma_{e_i} \equiv 0$. Putting $u := \sum_{i=1}^{n+2} r_ie_i$ we have $\gamma_u \equiv 0$, which means that, $\mathbf{N}(M)$, the Gauss image of M , is contained in the hyperplane P with normal vector u. By the sign of P, its normal vector is positive definite. Without losing anything of generality, it is enough to consider the cases $< u, u >= 1$. So, $\mathbf{N}(M)$ lies in $P \cap \mathbb{H}_0^{n+1}$. So, as in [17], according to completeness of $\mathbf{N}(M)$ we obtain that, $\mathbf{N}(M)$ lies in a connected component of the hyperbolic space (i.e. $\mathbb{H}^n$). Similar to the well-known theorem of Nomizu-Smyth, one can see that in this case $\mathbf{N}(M)$ is a fixed vector and so $x(M)$ is a totally geodesic spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$. This contradiction with the assumptions implies that $\hat{\mathcal{B}}$ is linearly independent and then $\dim(\Gamma) = n+2$.

(ii) It is enough to show that $\mathcal{B} \cup \{1\}$ is linearly independent. Assume that it is linearly dependent. So, by independence of $\mathcal{B}$, there exists a finite sequence of real numbers as $\{r_i\}_{i=1}^{n+2}$ such that $\sum_{i=1}^{n+2} r_i \lambda_{e_i} \equiv 1$. Putting $v := \sum_{i=1}^{n+2} r_i e_i$ we have $\lambda_v \equiv 1$. Since $x(M)$ is spacelike, v may not be spacelike and it has to be timelike and then, we can assume that $< v, v >= -1$. Hence, $x(M^n)$ is a totally umbilical hypersurfaces in $\mathbb{S}_1^{n+1}$, which is in the contradiction with the assumption. Therefore, $\mathcal{B} \cup \{1\}$ is a linearly independent subset of $\mathcal{C}(M)$ and $\dim(\bar{\Gamma}) = n+3$.

In similar way, we show that $\hat{\mathcal{B}} \cup \{1\}$ is linearly independent. If not, because of linearly independence of $\hat{\mathcal{B}}$, there are real numbers $s_i (i = 1, 2, ..., n+2)$ such that $\sum_{i=1}^{n+2} s_i \gamma_{e_i} \equiv 1$. Putting $u := \sum_{i=1}^{n+2} s_i e_i$ we have $\gamma_u \equiv 1$, which means that, $\mathbf{N}(M)$, the Gauss image of M , is contained in the intersection of a connected component of the hyperbolic space $\mathbb{H}_0^{n+1}$ with the hyperplane $Q = \{p \in \mathbb{R}_1^{n+2} | < p, u >= 1\}$. So, according to [17] and completeness and connectedness of $\mathbf{N}(M)$, we obtain that $\mathbf{N}(M)$ lies in a connected component of the hyperbolic space. Similar to the well-known theorem of Nomizu-Smyth, one can see that in this case, $x(M)$ is a totally umbilical spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$. This contradiction with the assumptions implies that $\bar{\mathcal{B}} \cup \{1\}$ is linearly independent and then $\dim(W \cup \{1\}) = n+3$. □

Proposition 4.3. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a non-totally geodesic r-maximal closed spacelike hypersurface isometrically immersed into $\mathbb{S}_1^{n+1}$ with $H_r > 0$ and $rank(S) > r$. Then, for any non-zero vectors $u, v \in \mathbb{R}_1^{n+2}$, the set $\{\lambda_u, \gamma_v, 1\}$ is linearly independent.

Proof. First we show that $\{\lambda_u, \gamma_v\}$ is linearly independent. Suppose that $\lambda_u = s\gamma_v$ for some non-zero vectors $u, v \in \mathbb{R}_1^{n+2}$ and some real number $s \in \mathbb{R}$. If $\lambda_u = 0$, then M^n is totally geodesic in $\mathbb{S}_1^{n+1}$ which is a contradiction. Assume that $\lambda_u \neq 0$. Then from $\lambda_u = s\gamma_v$ we get $s \neq 0$ and $L_r(\lambda_u) = sL_r(\gamma_v)$. So, by Lemma 4.2 we have $-c_r H_r \lambda_u = 0$ which is a contradiction again. Therefore, $\{\lambda_u, \gamma_v\}$ is linearly independent.

In the second stage, we show the linearly independence of $\{\lambda_u, \gamma_v, 1\}$. It is enough to consider the case where $\lambda_u^2 + \gamma_v^2 > 0$. Suppose that $\lambda_u = s_1\gamma_v + s_2$ for some real numbers $s_1, s_2 \in \mathbb{R}$. By the first part of proof, we know that $s_2 \neq 0$. So, by Lemma 4.2 we have $-c_r H_r \lambda_u = 0$, which is a contradiction. □

Theorem 4.3. Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a r-maximal closed spacelike hypersurface isometrically immersed into $\mathbb{S}_1^{n+1}$ with $H_r > 0$ and $rank(S) > r$. Then:
(i) If M^n is totally geodesic in $\mathbb{S}_1^{n+1}$, then $Ind^r(M^n) = 1$;
(ii) If M^n is not totally geodesic in $\mathbb{S}_1^{n+1}$, then $Ind^r(M^n) \geq n+2$,
(iii) If M^n is not totally umbilical in $\mathbb{S}_1^{n+1}$, then $Ind^r(M^n) \geq n+3$.

Proof. Part (i) is derived from a similar version of Theorem 5.1.1 in ([20]). For other parts, since $H_{r+1} = H_{r+2} = 0$, we have $B_r(f) = \int_M (J_r f) f = \int_M (L_r f + c_r H_r f) f dM$ for any $f \in \mathcal{C}^\infty(M)$. So, choosing $f = t + \lambda_u + \gamma_v$ to obtain

$$J_r f = L_r \lambda_u + L_r \gamma_v + t c_r H_r + c_r H_r \lambda_u + c_r H_r \gamma_v.$$

Then, by Lemma 4.2, we get $J_r f = c_r H_r (t + \gamma_v)$ and therefore we obtain

$$f J_r f = c_r H_r (t + \gamma_v)^2 + c_r H_r \lambda_u (t + \gamma_v).$$

Hence, we have

$$B_r(f) = c_r \int_M H_r (\gamma_v + t)^2 dM + c_r \int_M H_r \lambda_u (t + \gamma_v) dM.$$

Taking into account that $H_r > 0$, by putting $u = 0$, we have

$$B_r(f) = c_r \int_M H_r (\gamma_v + t)^2 dM > 0. \tag{4.1}$$

Therefore, $B_r(f) > 0$ for all $f \in span\{1, \gamma_{e_1}, ..., \gamma_{e_{n+2}}\}$. So, by parts (i) and (ii) of Theorem 4.2, we obtain (respectively) parts (ii) and (iii). □

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