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Characterization of n-Jordan Homomorphisms on Rings

Abbas Zivari-Kazempour and Mohammad Valaei

Abstract. In this paper, we prove that if $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is an n-Jordan homomorphism, where $\mathcal{R}$ has a unit e, then the map $a \longmapsto \varphi(e)^{n-2}\varphi(a)$ is a Jordan homomorphism. As a consequence we show, under special hypotheses, that each n-Jordan homomorphism is an n-homomorphism.

1 Introduction and Preliminaries

The study of additive mappings from one ring $\mathcal{R}$ into another ring $\mathcal{R}'$ which preserve squares was initiated by Ancochea [1] in connection with problems arising in projective geometry. Among others, Kaplansky [8], Jacobson and Rickart [7] and Herstein [6] then proceeded to carry out an extensive study of such functions.

The additive mapping $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ between two rings is called an *n-homomorphism* if for all $a_1, a_2, \cdots, a_n \in \mathcal{R}$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n),$$

and it is called an n-Jordan homomorphism if $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{R}$.

The concept of an n-homomorphism was studied in [4, 5], and the notion of n-Jordan homomorphism was dealt with firstly by Herstein [6]. For the case $n = 2$, this concepts coincides the classical definitions of homomorphism and Jordan homomorphism, respectively.

It is clear that every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. There are plenty of known examples of n-Jordan homomorphism which are not n-homomorphism. For $n = 2$, it is proved in [7] that some Jordan homomorphism on the polynomial rings can not be homomorphism.

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Corresponding author: Abbas Zivari-Kazempour.

Obviously, each homomorphism is an n-homomorphism for every $n \geqslant 2$, but the converse does not hold in general. For instance, if $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is a homomorphism, then $\psi := -\varphi$ is a 3-homomorphism which is not a homomorphism [4].

Let $n \geqslant 2$ be an integer and $\mathcal{R}$ be an associative ring. Following [6], we say that $\mathcal{R}$ is of characteristic not n if $na = 0$ implies $a = 0$ for every $a \in \mathcal{R}$, and $\mathcal{R}$ is of characteristic greater than n if $n!a = 0$ implies $a = 0$ for every $a \in \mathcal{R}$. A ring $\mathcal{R}$ is a domain if $\mathcal{R} \neq 0$ and either $a = 0$ or $b = 0$ whenever $ab = 0$.

The Jacobson radical $\mathfrak{J}(\mathcal{R})$ of a ring $\mathcal{R}$ is the intersection of the primitive ideals of $\mathcal{R}$, and $\mathcal{R}$ is called *semisimple* whenever $\mathfrak{J}(\mathcal{R}) = \{0\}$.

Theorem 1.1. *[6, Theorem H] If φ is a Jordan homomorphism of a ring $\mathcal{R}$ onto a prime ring $\mathcal{R}'$ of characteristic different from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.*

Zelazko [10] has given a characterization of Jordan homomorphism, that we mention in the following (see also [9]).

Theorem 1.2. *Every Jordan homomorphism from Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$ is a homomorphism.*

This result has been proved by the author in [11] for 3-Jordan homomorphisms with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. In other words, he presented the next theorem.

Theorem 1.3. *Suppose that $\mathcal{A}$ is a unial Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a 3-homomorphism.*

After that, An [2] extended Theorem 1.3 for all $n \in \mathbb{N}$, and obtained the next result.

Theorem 1.4. *[2, Theorem 2.4] Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings, where $\mathcal{R}$ has a unit e and char$(\mathcal{R}') > n$. Then every n-Jordan homomorphism $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is an n-homomorphism provided that every Jordan homomorphism $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is a homomorphism.*

Some significant results concerning characterization of n-Jordan homomorphisms on Banach algebras obtained by the first author in [11, 12] and [13].

In this paper we prove that if $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is an n-Jordan homomorphism, where $\mathcal{R}$ has a unit e and char$(\mathcal{R}')> n$, then the map $a \longmapsto \varphi(e)^{n-2}\varphi(a)$ is a Jordan homomorphism. As a consequence we generalize Theorem 1.3 for all $n \in \mathbb{N}$, and obtain [2, Corollary 2.5].

2 Characterization of n-Jordan homomorphisms

The next example provided that we cannot assert that n-Jordan homomorphisms of rings are always n-homomorphisms.

Example 1. *Let*

$$\mathcal{R} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : \quad A, B \in M_2(\mathbb{C}) \right\}.$$

Then under the usual matrix operations, $\mathcal{R}$ is a unital and semisimple ring. Define additive map $\varphi : \mathcal{R} \longrightarrow \mathcal{R}$ by $\varphi\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} A & 0 \\ 0 & B^T \end{bmatrix}$, where B^T is the transpose of matrix B. Then, for all $X \in \mathcal{R}$, we have $\varphi(X^n) = \varphi(X)^n$. Thus, φ is n-Jordan homomorphism, but φ is not n-homomorphism.

Throughout this paper $\mathcal{R}$ denotes a unital ring with unit e. Recall that the Lie product of element $a, b \in \mathcal{R}$, is $[a, b] = ab - ba$, and it is easy to check that if $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is a Jordan homomorphism, then $[\varphi(a), \varphi(e)] = 0$, for all $a \in \mathcal{R}$.

Lemma 2.1. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings, char$(\mathcal{R}') \neq 2$ and $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be a 3-Jordan homomorphism. Then $[\varphi(a), \varphi(e)] = 0$, for all $a \in \mathcal{R}$.*

Proof. By assumption $\varphi(x^3) = \varphi(x)^3$, for all $x \in \mathcal{R}$. Replacing x by $y + e$, we get

$$\begin{aligned} 3\varphi(y^2 + y) &= \varphi(y)^2\varphi(e) + \varphi(e)\varphi(y)^2 + \varphi(y)\varphi(e)\varphi(y) \\ &\quad + \varphi(e)^2\varphi(y) + \varphi(y)\varphi(e)^2 + \varphi(e)\varphi(y)\varphi(e). \end{aligned} \tag{2.1}$$

Interchanging y by $-y$ in (2.1), gives

$$\begin{aligned} 3\varphi(y^2 - y) &= \varphi(y)^2\varphi(e) + \varphi(e)\varphi(y)^2 + \varphi(y)\varphi(e)\varphi(y) \\ &\quad - \varphi(e)^2\varphi(y) - \varphi(y)\varphi(e)^2 - \varphi(e)\varphi(y)\varphi(e). \end{aligned} \tag{2.2}$$

The equalities (2.1) and (2.2) imply that

$$3\varphi(y^2) = \varphi(y)^2\varphi(e) + \varphi(e)\varphi(y)^2 + \varphi(y)\varphi(e)\varphi(y), \tag{2.3}$$

for all $y \in \mathcal{R}$. Replacing y by $a + e$ in (2.3), we get

$$3\varphi(a) = \varphi(e)^2\varphi(a) + \varphi(a)\varphi(e)^2 + \varphi(e)\varphi(a)\varphi(e). \tag{2.4}$$

Multiplying $\varphi(e)$ from the right in (2.4), we obtain

$$2\varphi(a)\varphi(e) = \varphi(e)^2\varphi(a)\varphi(e) + \varphi(e)\varphi(a)\varphi(e)^2. \tag{2.5}$$

Similarly,

$$2\varphi(e)\varphi(a) = \varphi(e)\varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a)\varphi(e). \tag{2.6}$$

It follows from (2.5) and (2.6) that $\varphi(a)\varphi(e) = \varphi(e)\varphi(a)$, for each $a \in \mathcal{R}$. $\square$

Theorem 2.1. *Suppose that $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is a 3-Jordan homomorphism. Then the map $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ defined by $\psi(a) = \varphi(e)\varphi(a)$ is a Jordan homomorphism.*

Proof. It follows from Lemma 2.1, that

$$\psi(a^3) = \varphi(e)\varphi(a^3) = \varphi(e)^3\varphi(a)^3 = (\varphi(e)\varphi(a))^3 = \psi(a)^3,$$

and hence for all $a \in \mathcal{R}$,

$$\psi(a^3) = \psi(a)^3. \tag{2.7}$$

Therefore ψ is a 3-Jordan homomorphism. Replacing a by $a + e$ in (2.7), we get

$$\psi(a^2 + a) = \psi(a)^2\psi(e) + \psi(a)\psi(e)^2. \tag{2.8}$$

From (2.8), we have

$$\begin{aligned} \psi(a^2 + a) &= \psi(a)^2\psi(e) + \psi(a)\psi(e)^2 \\ &= (\varphi(e)\varphi(a))^2\varphi(e)^2 + \varphi(e)\varphi(a)(\varphi(e)^2)^2 \\ &= \varphi(e)^4\varphi(a)^2 + \varphi(e)^5\varphi(a) \\ &= \varphi(e)^2\varphi(a)^2 + \varphi(e)\varphi(a) \\ &= \psi(a)^2 + \psi(a). \end{aligned}$$

Thus, $\psi(a^2) = \psi(a)^2$, for all $a \in \mathcal{R}$ and so ψ is a Jordan homomorphism. □

Next we generalize Theorem 2.1 for all $n \in \mathbb{N}$. For the proof we need the following two useful lemma.

Lemma 2.2. *Let $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be an n-Jordan homomorphism between rings. Then*

$$\varphi(e)^{(n-1)(n-2)}\varphi(e)^{2n-4} = \left(\varphi(e)^{n-2}\right)^2.$$

Proof. By assumption $\varphi(e) = \varphi(e)^n$, hence we get

$$\begin{aligned} \varphi(e)^{(n-1)(n-2)}\varphi(e)^{2n-4} &= (\varphi(e)^{(n-2)})^{(n-1)}(\varphi(e)^{n-2})^2 \\ &= (\varphi(e)^{(n-2)})^{(n+1)} \\ &= (\varphi(e)^{(n-2)})^n\varphi(e)^{(n-2)} \\ &= (\varphi(e)^n)^{(n-2)}\varphi(e)^{(n-2)} \\ &= \varphi(e)^{(n-2)}\varphi(e)^{(n-2)} \\ &= \left(\varphi(e)^{n-2}\right)^2, \end{aligned}$$

as claimed. □

Lemma 2.3. *Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are two rings, char$(\mathcal{R}') > n$ and let $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be an n-Jordan homomorphism. Then $\varphi(e)$ commute with $\varphi(a)$, for all $a \in \mathcal{R}$.*

Proof. For $a, b \in \mathcal{R}$, define $f : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}'$ by

$$f(a,b) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(\varphi[(a + (n-1-k)b)^n] - [\varphi(a) + (n-1-k)\varphi(b)]^n\big).$$

Then for all $a, b \in \mathcal{R}$, $f(a,b) = 0$. In particular, $f(a,e) = 0$. Using $\varphi(e) = \varphi(e)^n$, we get

$$n!\varphi(a) = (n-1)!\big(\varphi(a)\varphi(e)^{n-1} + \varphi(e)\varphi(a)\varphi(e)^{n-2} + \ldots + \varphi(e)^{n-1}\varphi(a)\big), \tag{2.9}$$

and since the characteristic of $\mathcal{R}'$ exceeds n, this simplifies to

$$n\varphi(a) = \big(\varphi(a)\varphi(e)^{n-1} + \varphi(e)\varphi(a)\varphi(e)^{n-2} + \ldots + \varphi(e)^{n-1}\varphi(a)\big). \tag{2.10}$$

Multiplying $\varphi(e)$ from the right in (2.10), we obtain

$$(n-1)\varphi(a)\varphi(e) = \varphi(e)\varphi(a)\varphi(e)^{n-1} + \ldots + \varphi(e)^{n-1}\varphi(a)\varphi(e). \tag{2.11}$$

Similarly,

$$(n-1)\varphi(e)\varphi(a) = \varphi(e)\varphi(a)\varphi(e)^{n-1} + \ldots + \varphi(e)^{n-1}\varphi(a)\varphi(e). \tag{2.12}$$

The equalities (2.11) and (2.12) lead to $(n-1)\varphi(a)\varphi(e) = (n-1)\varphi(e)\varphi(a)$. As char$(\mathcal{R}')> n$, this forces $[\varphi(a), \varphi(e)] = 0$, for each $a \in \mathcal{R}$. $\square$

Our main theorem is the following.

Theorem 2.2. *Let $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be an n-Jordan homomorphism and let char$(\mathcal{R}') > n$. Then the map $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ defined by $\psi(a) = \varphi(e)^{n-2}\varphi(a)$ is a Jordan homomorphism.*

Proof. Let $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be an n-Jordan homomorphism. From Lemma 2.3, for all $a \in \mathcal{R}$, we have

$$\psi(a^n) = \varphi(e)^{n-2}\varphi(a^n) = \varphi(e)^{n-2}\varphi(a)^n = \big(\varphi(e)^n\big)^{n-2}\varphi(a)^n = (\varphi(e)^{n-2}\varphi(a))^n = \psi(a)^n,$$

and hence ψ is an n-Jordan homomorphism. Replacing a by $a + me$, where m is an integer with $1 \leqslant m \leqslant n$, we obtain

$$\psi((a+me)^n) = \psi(a+me)^n, \tag{2.13}$$

for all $a \in \mathcal{R}$. It follows from the equality (2.13) that

$$\sum_{i=1}^{n-1} m^{n-i} \binom{n}{i} [\psi(a^i) - \psi(e)^{n-i}\psi(a)^i] = 0, \qquad (1 \leqslant m \leqslant n), \tag{2.14}$$

for all $a \in \mathcal{R}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We can rewrite the equalities in (2.14) as follows

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & 2 \\ 3^{n-1} & 3^{n-2} & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ n^{n-1} & n^{n-1} & \cdots & n \end{bmatrix} \begin{bmatrix} \Gamma_1(a) \\ \Gamma_2(a) \\ \Gamma_3(a) \\ \cdots \\ \Gamma_n(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \tag{2.15}$$

where

$$\Gamma_i(a) = \binom{n}{i} [\psi(a^i) - \psi(e)^{n-i}\psi(a)^i],$$

for all $a \in \mathcal{R}$ and for each $1 \leqslant i \leqslant n$. It is shown in [3, Lemma 2.1] that the above square matrix is invertible. This implies that $\Gamma_i(a) = 0$ for every $1 \leqslant i \leqslant n$ and all $a \in \mathcal{R}$. In particular, $\Gamma_2(a) = 0$. Hence for all $a \in \mathcal{R}$,

$$\psi(a^2) = \psi(e)^{n-2}\psi(a)^2. \tag{2.16}$$

It follows from definition of ψ, (2.16), and Lemma 2.2 that

$$\begin{aligned} \psi(a^2) &= \psi(e)^{n-2}\psi(a)^2 \\ &= \big(\varphi(e)^{n-1}\big)^{n-2}\big(\varphi(e)^{n-2}\varphi(a)\big)^2 \\ &= \varphi(e)^{(n-1)(n-2)}\varphi(e)^{2n-4}\varphi(a)^2 \\ &= \big(\varphi(e)^{n-2}\varphi(a)\big)^2 \\ &= \psi(a)^2. \end{aligned}$$

Consequently, ψ is a Jordan homomorphism. This finishes the proof. □

Corollary 2.3. *[2, Corollary 2.5] Let $\mathcal{A}$ be unital Banach algebra and $\mathcal{B}$ be a semisimple commutative Banach algebra. Then, every n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-homomorphism.*

Proof. By Theorem 2.2 the map $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ defined by $\psi(a) = \varphi(e)^{n-2}\varphi(a)$ is a Jordan homomorphism. Hence by Theorem 1.2, ψ is a homomorphism. Therefore

$$\varphi(e)^{n-2}\varphi(ab) = \psi(ab) = \psi(a)\psi(b) = \varphi(e)^{2(n-2)}\varphi(a)\varphi(b),$$

for all $a, b \in \mathcal{A}$. Hence

$$\varphi(e)^{n-2}\big(\varphi(ab) - \varphi(e)^{n-2}\varphi(a)\varphi(b)\big) = 0.$$

Since $\varphi(e)^{n-2} \neq 0$ for every $n \in \mathbb{N}$, and $\mathcal{B}$ is semisimple, we have

$$\varphi(ab) = \varphi(e)^{n-2}\varphi(a)\varphi(b), \tag{2.17}$$

Replacing b by e in (2.17), we get $\varphi(a) = \varphi(e)^{n-1}\varphi(a)$, for all $a, \in \mathcal{A}$. Using (2.17) to obtain

$$\begin{aligned}\varphi(a_1a_2...a_n) &= \big(\varphi(e)^{n-2}\big)^{n-1}\varphi(a_1)\varphi(a_2)...\varphi(a_n)\\ &= \big(\varphi(e)^{n-1}\big)^{n-2}\varphi(a_1)\varphi(a_2)...\varphi(a_n)\\ &= \big(\varphi(e)^{n-1}\varphi(a_1)\big)...\big(\varphi(e)^{n-1}\varphi(a_{n-2})\big)\varphi(a_{n-1})\varphi(a_n)\\ &= \varphi(a_1)\varphi(a_2)...\varphi(a_n).\end{aligned}$$

Thus, φ is an n-homomorphism. □

Corollary 2.4. *[12, Corollary 2.8] A unital linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between Banach algebras is a Jordan homomorphism if and only if it is an n-Jordan homomorphism.*

Corollary 2.5. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings, where $\mathcal{R}'$ is domain. If $\varphi(e)$ is an idempotent in $\mathcal{R}'$, then every n-Jordan homomorphism $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is a Jordan homomorphism.*

Proof. By Theorem 2.2 the map $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ defined by $\psi(a) = \varphi(e)^{n-2}\varphi(a)$ is a Jordan homomorphism. Therefore

$$\varphi(e)^{n-2}\varphi(a^2) = \psi(a^2) = \psi(a)^2 = \varphi(e)^{2(n-2)}\varphi(a)^2,$$

for all $a \in \mathcal{R}$. Since $\varphi(e)$ is an idempotent, we conclude that $\varphi(e)\big(\varphi(a^2) - \varphi(a)^2\big) = 0$. As $\varphi(e) \neq 0$ and $\mathcal{R}'$ is domain, we get $\varphi(a^2) = \varphi(a)^2$, for all $a \in \mathcal{R}$. □

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Abbas Zivari-Kazempour Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.

E-mail: zivari@abru.ac.ir, zivari6526@gmail.com

Mohammad Valaei Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.

E-mail: Mohammad.valaei@abru.ac.ir