



Existence and Stability of the Solution to a Coupled System of Fractional-order Differential with a p -Laplacian Operator under Boundary Conditions

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Abstract. This paper is devoted to studying the uniqueness and existence of the solution to a nonlinear coupled system of (FODEs) with p -Laplacian operator under integral boundary conditions (IBCs). Our problem is based on Caputo fractional derivative of orders σ, λ , where $k - 1 \leq \sigma, \lambda < k, k \geq 3$. For these aims, the nonlinear coupled system will be converted into an equivalent integral equations system by the help of Green function. After that, we use Leray-Schauder's and topological degree theorems to prove the existence and uniqueness of the solution. Further, we study certain conditions for the Hyers-Ulam stability of the solution to the suggested problem. We give a suitable and illustrative example as an application of the results.

1 Introduction

Recently, fractional calculus has proved to be more important in applied science than in the integer-order differential equations to obtain better explanations and better results. Therefore, fractional calculus is the generalization of classical calculus. For information about applications of fractional equations, we suggest to the readers see these papers ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 53]). Fractional calculus has got the attention of researchers in the various applied sciences due to the important applications, high profile accuracy, and usability in the different fields like image processing, fractals theory, control system, electromagnetic theorem, control theory ecology, metallurgy, plasma physics, aerodynamics, economics, and biology. For further details, see these papers ([11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]). We are interested in a study a nonlinear coupled system of fractional-order because this type is very important in various models such as blood flow phenomena, Chemical Kinetics, Irregular Heartbeats, chemical reaction design, etc ([24, 25, 26, 27, 28, 29, 30, 31, 32]). At first, we introduce some important and recent contributions to researchers about study different aspects of existence and uniqueness of solution (EUS) of a nonlinear coupled system of FODEs. Shah, K., Kumam [33] used the Perov-type fixed point

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theory to study existence of a solution for a nonlinear coupled system of FODEs under integral boundary conditions (IBCs).

$$\begin{cases} \mathcal{D}^\lambda \zeta(s) + \phi(s, \zeta(s), \chi(s)) = 0, & s \in (0, 1) \quad k-1 < \lambda < k, \\ \mathcal{D}^\alpha Q(s) + \eta(s, Q(s), \zeta(s)) = 0, & s \in (0, 1) \quad k-1 < \alpha < k, \\ \zeta(0) = \zeta'(0) = \zeta''(0) = \dots = \zeta^{(l-2)}(0) = 0, & \zeta(1) = \rho \int_0^1 \zeta(s) ds, \\ Q(0) = Q'(0) = Q''(0) = \dots = Q^{(l-2)}(0) = 0, & Q(1) = \varrho \int_0^1 Q(s) ds, \end{cases}$$

where $s \in [0, 1], k \geq 3, \rho, \varrho \in (0, 2), \phi, \eta : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions, \mathcal{D}^λ and \mathcal{D}^α are Caputo derivatives. Cheng et al.[34] studied the BVPs for a high order of FDEs with p-Laplacian operator at resonance by using degree theorem given by

$$\begin{cases} \mathcal{D}^\lambda \phi_p(\mathcal{D}^{\sigma_1} x(s) + Q_1(s, y(s))) = 0, \\ \mathcal{D}^\lambda \phi_p(\mathcal{D}^{\sigma_2} y(s) + Q_2(s, x(s))) = 0, \\ \mathcal{D}^{\sigma_1} x(0) = \mathcal{D}^{\sigma_1} x(1) = \mathcal{D}^{\sigma_2} y(0) = \mathcal{D}^{\sigma_2} y(1) = 0, \end{cases}$$

where \mathcal{D}^{σ_i} and \mathcal{D}^λ are Caputo derivatives, where $i=1,2, s \in [0, 1], l-1 < \sigma_1, \sigma_2 \leq l, 0 < \lambda \leq 1, Q_1, Q_2 : [0, 1] \times R \rightarrow R$ are continuous functions. By means topological degree theory, Hu and Zhang [35] showed existence solutions of a nonlinear coupled system of FDEs with infinite-point boundary conditions as follow

$$\begin{cases} \mathcal{D}^{\lambda_1} \phi_p(\mathcal{D}^{\sigma_1} \Omega(s)) = \chi(s, \omega(s), \mathcal{D}^{\sigma_2-1} \Omega(s), \dots, \mathcal{D}^{\sigma_2-(n-1)} \Omega(s)), & s \in (0, 1), \\ \mathcal{D}^{\lambda_2} \phi_p(\mathcal{D}^{\sigma_2} \omega(s)) = Y(s, \Omega(s), \mathcal{D}^{\sigma_1-1} \Omega(s), \dots, \mathcal{D}^{\sigma_1-(n-1)} \Omega(s)), & s \in (0, 1), \\ \Omega'(0) = \dots = \Omega^{(n-1)}(0) = \mathcal{D}^{\sigma_1} \Omega(t) = 0, & \Omega(0) = \sum_{i=1}^{\infty} c_i \Omega(\mu_i), \\ \omega'(0) = \dots = \omega^{(n-1)}(0) = \mathcal{D}^{\sigma_2} \omega(t) = 0, & \omega(0) = \sum_{i=1}^{\infty} d_i \omega(\zeta_i), \end{cases}$$

where $\mathcal{D}^{\lambda_i}, \mathcal{D}^{\sigma_i}$ for $i = 1, 2$ are Caputo derivatives, $0 < \lambda_1, \lambda_2 < 1, n-1 < \sigma_1, \sigma_2 < n, 0 < \mu_1 < \mu_2 < \dots < \mu_i < \dots < 1, 0 < \zeta_1 < \zeta_2 < \dots < \zeta_i < \dots < 1, \sum_{i=1}^{+\infty} |c_i| < \infty, \sum_{i=1}^{+\infty} |d_i| < \infty, \sum_{i=1}^{+\infty} c_i = \sum_{i=1}^{+\infty} d_i = 1, \chi, Y$ are continuous functions. J. Tariboon, et al.[36] used the Riemann-Liouville to study EUS for a nonlinear coupled system of FDEs with Hadamard fractional conditions given by

$$\begin{cases} \mathcal{D}^\sigma v(s) = \xi(s, \omega(s), v(s)), & s \in [0, M], \quad 1 < \sigma \leq 2, \\ \mathcal{D}^\lambda \omega(s) = \eta(s, v(s), \omega(s)), & s \in [0, M], \quad 1 < \sigma \leq 2, \\ v(0) = 0, & v(M) = \sum_{i=1}^n \theta_i H I^{\zeta_i} \omega(\varepsilon_i), \\ \omega(0) = 0, & \omega(M) = \sum_{j=1}^m \delta_j H I^{\vartheta_j} v(\gamma_j), \end{cases}$$

where $\mathcal{D}^\sigma, \mathcal{D}^\lambda$ are the Riemann-Liouville derivative of orders $\sigma, \lambda, H I^{\zeta_i}, H I^{\vartheta_j}$ are the non-local Hadamard fractional of orders $\zeta^i, \vartheta^j > 0, \varepsilon_i, \gamma_j \in (0, M), \xi, \eta : [0, M] \times R^2 \rightarrow R$ and

$\theta_i, \delta_j \in R, i = 1, 2, \dots, n, j = 1, 2, \dots, m$ are real constants. Using classical fixed point theory one needs strong conditions to establish conditions for EUS of solutions to fractional order differential equations and therefore restrict the applicability to certain classes of FODEs and their coupled systems. Various degree theories play excellent roles for the existence of solutions to FODEs and their most systems. Brouwer and Leray-Schauder degree theories were established to deal with the existence theorem of FODEs. In 1979 introduced Mawhin [37] an important degree theory which known as topological degree theory and later on extended by Isaia [38], has been used to establish the existence theorem for solutions to nonlinear fractional differential and integral equations. The mentioned method is called prior-estimate method which needs no compactness of the operator and relaxes much the condition for existence and uniqueness of solutions to fractional differential and integral equations. In recent years, the mentioned degree theory has been applied to investigate certain classes of FODEs with Integral boundary conditions, for further ([39],[40],[41]). Inspired by the aforesaid works, in this literature, we use the topological degree theory to study EUS of a nonlinear coupled system of FODEs with IBCs and p-Laplacian operator and also, we study the Hyers-Ulam stability technique to the suggested problem given by

$$\begin{cases} {}^c\mathcal{D}^{\lambda_1}(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) + \mathcal{Q}_1(\zeta, \nu(\zeta)) = 0, & {}^c\mathcal{D}^{\lambda_2}(\phi_p({}^c\mathcal{D}^{\sigma_2}\nu(\zeta))) + \mathcal{Q}_2(\zeta, \mu(\zeta)) = 0, \\ (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))^{(i)}|_{\zeta=0} = 0, & (\phi_p({}^c\mathcal{D}^{\sigma_2}\nu(\zeta)))^{(i)}|_{\zeta=0} = 0 \quad for \quad i = 0, 3, 4, \dots, k-1, \\ (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))|_{\zeta=1} = 0, & (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\lambda_1-2}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}, \\ (\phi_p({}^c\mathcal{D}^{\sigma_2}\nu(\zeta)))''|_{\zeta=1} = 0, & (\phi_p({}^c\mathcal{D}^{\sigma_2}\nu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\lambda_2-2}\mathcal{Q}_2(\zeta, \mu(\zeta))|_{\zeta=1}, \\ (\mu(0))^{(j)} = 0 & (\nu(0))^{(j)} = 0 \quad for \quad j = 2, 3, 4, \dots, k-1, \\ \mu'(1) = 0, & \mu(1) = -\mathcal{I}^{\sigma_1-1}(\phi_q(\int_0^1 \mathcal{H}^{\lambda_1}(\zeta, \vartheta)\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)), \\ \nu'(1) = 0, & \nu(1) = -\mathcal{I}^{\sigma_2-1}(\phi_q(\int_0^1 \mathcal{H}^{\lambda_2}(\zeta, \vartheta)\mathcal{Q}_2(\vartheta, \mu(\vartheta))d\vartheta)), \end{cases} \quad (1.1)$$

where $k-1 < \lambda_i, \sigma_i < k$, $\mathcal{Q}_1, \mathcal{Q}_2 \in L[0, 1]$, and $\phi_p(\vartheta) = |\vartheta|^{(p-2)}\vartheta$ is the p-Laplacian operator and $\phi_p^{-1} = \phi_q$, such that $1/p + 1/q = 1$, ${}^c\mathcal{D}^{\lambda_i}$ and ${}^c\mathcal{D}^{\sigma_i}$ where $i = 1, 2$ for Caputo derivatives. We prove necessary conditions for EUS and HU-stability of a nonlinear coupled system (1.1) with the help of topological degree theorem and nonlinear functional analysis greatly developed by Deimling [42]. Our problem is more general and complicated than previous studies before and aforementioned.

2 Preliminaries

Here we recall some fundamental and necessary definitions and theories from fractional calculus and functional analysis which have a role in the results throughout this paper.

Definition 2.1 For $\mathcal{Q}(t) : (0, +\infty) \rightarrow R$, the fractional integral Riemann-Liouville of order $\sigma > 0$, is defined by

$$\mathcal{I}^\sigma \mathcal{Q}(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - \eta)^{\sigma-1} \mathcal{Q}(\eta) d\eta,$$

where that integral pointwise on the right side on $(0, +\infty)$.

Definition 2.2 for $\mathcal{Q}(t) : (0, +\infty) \rightarrow R$, the fractional Caputo derivative of order $\sigma > 0$ is defined by

$${}^c\mathcal{D}^\sigma \mathcal{Q}(t) = \frac{1}{\Gamma(k - \sigma)} \int_0^t (t - \eta)^{k-\sigma-1} \mathcal{Q}^{(k)}(\eta) d\eta,$$

provided that the integral pointwise on the right side on $(0, \infty)$, $\mathcal{Q}(t)$ is continuous function and $-1 < \sigma < k$.

Lemma 2.1. Let $\sigma > 0$ and $\vartheta \in C(0, 1) \cap L^1(0, 1)$, then

$\mathcal{D}^\sigma \vartheta(t) = \mathcal{Q}(t)$, is given by

$$\vartheta(t) = \mathcal{Q}(t) + d_0 + d_1 t + d_2 t^2 + \dots + d_{m-1} t^{m-1},$$

where m is integer and for some $d_j \in R, j = 0, 1, 2, \dots, m - 1$ such that $m \geq \sigma$.

Lemma 2.2. Let $\sigma \in (m - 1, m]$, $\mathcal{Q} \in AC^{(m-1)}$. Then

$$\mathcal{I}^\sigma \mathcal{D}^\sigma \mathcal{Q}(t) = \mathcal{Q}(t) + d_0 + d_1 t + d_2 t^2 + \dots + d_{m-1} t^{m-1}.$$

For $d_j \in R$ for $i = 0, 1, 2, \dots, m - 1$.

let $\mathfrak{F} = C([0, 1], R)$ be the space of all continuous functions $\vartheta : [0, 1] \rightarrow R$ endowed with a norm $\sup_{t \in [0, 1]} \{|\vartheta(t)| : \vartheta \in C[0, 1]\}$ which is a Banach space. Then the product space denoted by $\varpi^* = \mathfrak{F}_1 \times \mathfrak{F}_2$ under the norms $\|(\vartheta, \omega)(t)\| = \|\vartheta(t)\| + \|\omega(t)\|$ is also a Banach space which will need throughout this manuscript. For the topological degree theorem and functional analysis, we recall some definitions, from the available literature papers ([43, 44, 45, 46, 47, 48, 49, 50, 51, 52]).

Definition 2.3 Let the class of all bounded sets of $P(\mathfrak{F})$ be denoted by ζ . Then the mapping $\mathcal{L} : \zeta \rightarrow (0, +\infty)$ for Kuratowski measure of non-compactness is defined as

$$\mathcal{L}(u) = \inf\{a > 0 : u \text{ is the finite cover for sets of diameter } \leq a\},$$

where $u \in \zeta$.

Proposition 2.1 The following are the characteristics of the measure $\mathcal{L} :$

- (ι_1) For relative compact u , the Kuratowski measure $\mathcal{L}(u) = 0$;

(ι_2) For seminorm \mathcal{L} , $\mathcal{L}(\mathcal{J}u) = |\mathcal{J}|\mathcal{L}(u)$, $\mathcal{J} \in R$, and $\mathcal{L}(u_1 + u_2) \leq \mathcal{L}(u_1) + \mathcal{L}(u_2)$;

(ι_3) $u_1 \subset u_2$ yields $\mathcal{L}(u_1); \mathcal{L}(u_1 \cup u_2) = \sup\{\mathcal{L}(u_1), \mathcal{L}(u_2)\}$;

(ι_4) $\mathcal{L}(\text{conv } u) = \mathcal{L}(u)$;

(ι_5) $\mathcal{L}(\bar{u}) = \mathcal{L}(u)$.

Definition 2.4 Suppose that mapping $\psi : \theta \rightarrow \mathfrak{F}$ is a continuous and bounded such that $\theta \subset \mathfrak{F}$. Then ψ is an \mathcal{L} -Lipschitz, where $\eta \geq 0$ such that

$$\mathcal{L}(\psi(u)) \leq \eta k(u) \quad \forall \text{ bounded } u \subset \theta.$$

Then ψ is a strict \mathcal{L} -contraction under the condition $\eta < 1$.

Definition 2.5 The function continuous ψ is \mathcal{L} -condensing if

$\mathcal{L}(\psi(u)) \leq \mathcal{L}(u)$, for all bounded $u \subset \theta$ such that $\mathcal{L}(u) > 0$.

Therefore $\mathcal{L}(\psi(u)) \geq k(u)$ yields $\mathcal{L}(u) = 0$.

Further we have $\psi : \theta \rightarrow \mathfrak{F}$ is Lipschitz for $\eta > 0$, such that

$\|\psi(\omega) - \psi(\bar{\omega})\| \leq \eta \|\omega - \bar{\omega}\|$, for all $\omega, \bar{\omega} \in \theta$. The condition $\eta < 1$, yields that ψ is strict contraction.

Proposition 2.2 The mapping ψ is \mathcal{L} -Lipschitz with constant $\eta = 0$ if and only if $\psi : \theta \rightarrow \mathfrak{F}$ is said to be compact.

Proposition 2.3 The operator ψ is \mathcal{L} -Lipschitz for some constant η if and only if $\psi : \theta \rightarrow \mathfrak{F}$ is Lipschitz with constant η .

Theorem 2.1. Let $\psi : \mathfrak{F} \rightarrow \mathfrak{F}$ is a \mathcal{L} -contraction and

$$E = \{v \in \mathfrak{F} : \text{there exist } 0 \leq \mu \leq 1 \text{ such that } v = \mu\psi(v)\}.$$

if $E \subset v_h(0)$, is bounded in \mathfrak{F} there exists $h > 0$ and $E \subset u_h(0)$ with degree

$\deg(I - \mu\psi, v_h(0), 0) = 1$, for very $\mu \in [0, 1]$.

Then, ψ has at least one fixed point.

Theorem 2.2. ⁵³ Let $\psi : \varpi^* \cap (\tilde{\mathfrak{F}}_2 \setminus \tilde{\mathfrak{F}}_1) \rightarrow \varpi^*$; we say ψ is compact operator iff it is uniformly bounded and is equicontinuous. Whereas $\tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2$ are two bounded subsets of \mathfrak{F} such that $0 \in \tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2 \subset \tilde{\mathfrak{F}}_2$, and $\psi : \varpi^* \cap (\tilde{\mathfrak{F}}_2 \setminus \tilde{\mathfrak{F}}_1) \rightarrow \varpi^*$ is an operator.

Lemma 2.3. Let ϕ_p be p -Laplacian. Then we have

(ξ_1) If $1 < p \leq 2$, $\varrho_1, \varrho_2 > 0$ and $|\varrho_1|, |\varrho_2| \geq \rho > 0$, then

$$|\phi_p(\varrho_1) - \phi_p(\varrho_2)| \leq (p-1)\rho^{(p-2)}|\varrho_1 - \varrho_2|.$$

(ξ_2) If $p > 2$ and $|\varrho_1|, |\varrho_2| \leq \rho^*$, then

$$|\phi_p(\varrho_1) - \phi_p(\varrho_2)| \leq (p-1)\rho^{*p-2}|\varrho_1 - \varrho_2|.$$

3 Existence the solution

Theorem 3.1. *Let the function $\mathcal{Q} \in C[0, 1]$ be an integrable satisfying (1.1) for $\lambda_1, \sigma_1 \in (3, k]$ and integer $k \geq 4$. Then the solution of*

$$\begin{cases} {}^c\mathcal{D}^{\lambda_1}(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) + \mathcal{Q}_1(\zeta, \nu(\zeta)) = 0, \\ (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))^{(i)}|_{\zeta=0} = 0, & \text{for } i = 0, 3, 4, \dots, k-1, \\ (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))''|_{\zeta=1} = 0, & (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\lambda_1-2}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}, \\ (\mu(0))^{(j)} = 0 & \text{for } j = 2, 3, 4, \dots, k-1, \\ \mu'(1) = 0, & \mu(1) = -\mathcal{I}^{\sigma_1-1}(\phi_q(\int_0^1 \mathcal{H}^{\lambda_1}(\zeta, \vartheta)\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)), \end{cases} \quad (3.1)$$

is given by integral equation

$$\mu(\zeta) = \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta)\phi_q\left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta)\mathcal{Q}_1(\eta, \nu(\eta))d\eta\right)d\vartheta. \quad (3.2)$$

Where $\mathcal{G}^{\sigma_1}(\zeta, \vartheta), \mathcal{H}^{\lambda_1}(\vartheta, \eta)$ are Green functions defined by

$$\mathcal{G}^{\sigma_1}(\zeta, \vartheta) = \begin{cases} \frac{(\zeta-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \frac{(1-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \zeta \frac{(1-\vartheta)^{\sigma_1-2}}{\Gamma(\sigma_1-1)}, & 0 \leq \vartheta \leq \zeta \leq 1, \\ -\frac{(1-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \zeta \frac{(1-\vartheta)^{\sigma_1-2}}{\Gamma(\sigma_1-1)}, & 0 \leq \zeta \leq \vartheta \leq 1, \end{cases} \quad (3.3)$$

$$\mathcal{H}^{\lambda_1}(\zeta, \vartheta) = \begin{cases} \frac{-(\zeta-\vartheta)^{\lambda_1-1}}{\Gamma(\lambda_1)} + \zeta \frac{(1-\vartheta)^{\lambda_1-2}}{\Gamma(\lambda_1-1)} + \zeta^2 \frac{(1-\vartheta)^{\lambda_1-3}}{2\Gamma(\lambda_1-2)}, & 0 \leq \vartheta \leq \zeta \leq 1, \\ \zeta \frac{(1-\vartheta)^{\lambda_1-2}}{\Gamma(\lambda_1-1)} + \zeta^2 \frac{(1-\vartheta)^{\lambda_1-3}}{2\Gamma(\lambda_1-2)}, & 0 \leq \zeta \leq \vartheta \leq 1, \end{cases} \quad (3.4)$$

Proof. By using Lemma 2.2 and applying integral \mathcal{I}^{λ_1} on (3.1), we get from the problem (3.1) as follows

$$(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) = -\mathcal{I}^{\lambda_1}(\mathcal{Q}_1(\zeta, \nu(\zeta))) + a_0 + a_1\zeta + a_2\zeta^2 + \dots + a_{k-1}\zeta^{k-1}. \quad (3.5)$$

For the coefficients $i = 0, 3, 4, \dots, k-1$, by using the conditions $(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))^{(i)}|_{\zeta=0} = 0$ in (3.5), we obtain $a_0 = a_3 = a_4 = \dots = a_{k-1} = 0$, we get

$$(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) = -\mathcal{I}^{\lambda_1}(\mathcal{Q}_1(\zeta, \nu(\zeta))) + a_1\zeta + a_2\zeta^2. \quad (3.6)$$

Applying the condition $(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))''|_{\zeta=1} = 0$ in the equation (3.6), we get

$$a_2 = \frac{1}{2}\mathcal{I}^{\lambda_1-2}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}. \quad (3.7)$$

By using the condition $(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\lambda_1-2}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}$ in the equation (3.6), we obtain

$$a_1 = \mathcal{I}^{\lambda_1-1}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}. \quad (3.8)$$

Putting the values a_1, a_2 in (3.6), we get

$$\begin{aligned} (\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) &= -\mathcal{I}^{\lambda_1}(\mathcal{Q}_1(\zeta, \nu(\zeta))) + \zeta\mathcal{I}^{\lambda_1-1}(\mathcal{Q}_1(\zeta, \nu(\zeta)))|_{\zeta=1} + \frac{\zeta^2}{2}\mathcal{I}^{\lambda_1-2}(\mathcal{Q}_1(\zeta, \nu(\zeta)))|_{\zeta=1} \\ &= -\int_0^\zeta \frac{(\zeta-\vartheta)^{\lambda_1-1}}{\Gamma(\lambda_1)}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta + \zeta\int_0^1 \frac{(1-\vartheta)^{\lambda_1-2}}{\Gamma(\lambda_1-1)}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta \\ &\quad + \frac{\zeta^2}{2}\int_0^1 \frac{(1-\vartheta)^{\lambda_1-3}}{\Gamma(\lambda_1-2)}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta \end{aligned}$$

$$(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) = \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta. \quad (3.9)$$

Where $\mathcal{H}^{\lambda_1}(\zeta, \vartheta)$ is a Green function given in(3.4). From(3.9), we further have

$$(\phi_p({}^c\mathcal{D}^{\sigma_1}\mu(\zeta))) = \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta. \quad (3.10)$$

Applying $\phi_p^{-1} = \phi_q$ on sides of (3.10), we get

$$({}^c\mathcal{D}^{\sigma_1}\mu(\zeta)) = \phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta. \quad (3.11)$$

Applying the fractional integral \mathcal{I}^{σ_1} and using Lemma 2.2 again on both sides of (3.11), we obtain

$$\mu(\zeta) = \mathcal{I}^{\sigma_1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta) + b_0 + b_1\zeta + b_2\zeta^2 + \dots + b_{k-1}\zeta^{k-1}. \quad (3.12)$$

For the coefficients $j = 2, 3, 4, \dots, k-1$, by the conditions $(\mu(\zeta))^{(j)}|_{\zeta=0}$ in (3.12), then $b_2 = b_3 = b_4 = \dots = b_{k-1} = 0$, we get

$$\mu(\zeta) = \mathcal{I}^{\sigma_1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta) + b_0 + b_1\zeta. \quad (3.13)$$

Applying the condition $(\mu(1))' = 0$ in the equation (3.13), we have

$$b_1 = -\mathcal{I}^{\sigma_1-1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)|_{\zeta=1}. \quad (3.14)$$

By using the condition $\mu(1) = -\mathcal{I}^{\sigma_1-1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)|_{\zeta=1}$ in the equation (3.13), we have

$$b_o = -\mathcal{I}^{\sigma_1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1}\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)|_{\zeta=1}. \quad (3.15)$$

Putting the values b_0, b_1 in (3.13), we get

$$\begin{aligned}
\mu(\zeta) &= \mathcal{I}^{\sigma_1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1} \mathcal{Q}_1(\vartheta, \nu(\vartheta)) d\vartheta) - \mathcal{I}^{\sigma_1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1} \mathcal{Q}_1(\vartheta, \nu(\vartheta)) d\vartheta)|_{\zeta=1} \\
&\quad - \zeta \mathcal{I}^{\sigma_1-1}(\phi_q \int_0^1 \mathcal{H}^{\lambda_1} \mathcal{Q}_1(\vartheta, \nu(\vartheta)) d\vartheta)|_{\zeta=1}. \\
&= \int_0^\zeta \frac{(\zeta - \vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \\
&\quad - \int_0^1 \frac{(1 - \vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \\
&\quad - \zeta \int_0^1 \frac{(1 - \vartheta)^{\sigma_1-2}}{\Gamma(\sigma_1 - 1)} \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \\
&= \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta.
\end{aligned} \tag{3.16}$$

□

Where $\mathcal{G}^{\sigma_1}(\zeta, \vartheta), \mathcal{H}^{\lambda_1}(\vartheta, \eta)$ are Green functions defined by (3.3),(3.4).

According to Theory 3.1, then system (1.1) is equivalent to the following a nonlinear coupled system of Hammerstein-type integral equations:

$$\mu(\zeta) = \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta. \tag{3.17}$$

$$\nu(\zeta) = \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta. \tag{3.18}$$

Where $\mathcal{G}^{\sigma_2}(\zeta, \vartheta), \mathcal{H}^{\lambda_2}(\vartheta, \eta)$ are the following Green functions:

$$\mathcal{G}^{\sigma_2}(\zeta, \vartheta) = \begin{cases} \frac{(\zeta - \vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)} - \frac{(1 - \vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)} - \zeta \frac{(1 - \vartheta)^{\sigma_2-2}}{\Gamma(\sigma_2-1)}, & 0 \leq \vartheta \leq \zeta \leq 1, \\ -\frac{(1 - \vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)} - \zeta \frac{(1 - \vartheta)^{\sigma_2-2}}{\Gamma(\sigma_2-1)}, & 0 \leq \zeta \leq \vartheta \leq 1, \end{cases} \tag{3.19}$$

$$\mathcal{H}^{\lambda_2}(\zeta, \vartheta) = \begin{cases} \frac{-(\zeta - \vartheta)^{\lambda_2-1}}{\Gamma(\lambda_2)} + \zeta \frac{(1 - \vartheta)^{\lambda_2-2}}{\Gamma(\lambda_2-1)} + \frac{\zeta^2}{2} \frac{(1 - \vartheta)^{\lambda_2-3}}{\Gamma(\lambda_2-2)}, & 0 \leq \vartheta \leq \zeta \leq 1, \\ \zeta \frac{(1 - \vartheta)^{\lambda_2-2}}{\Gamma(\lambda_2-1)} + \frac{\zeta^2}{2} \frac{(1 - \vartheta)^{\lambda_2-3}}{\Gamma(\lambda_2-2)}, & 0 \leq \zeta \leq \vartheta \leq 1, \end{cases} \tag{3.20}$$

Define $\mathcal{M}_1^* : \mathfrak{F} \rightarrow \mathfrak{F}$ for $(i = 1, 2)$ by

$$\mathcal{M}_1^* \mu(\zeta) = \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta. \tag{3.21}$$

$$\mathcal{M}_2^* \nu(\zeta) = \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta. \tag{3.22}$$

by Theorem 3.1, then the solution of the Hammerstein-type integral equations (3.17),(3.18) is equivalent to the fixed point, say (μ, ν) , of the operator equation

$$(\mu, \nu) = \mathcal{M}^*(\mu, \nu) = (\mathcal{M}_1^*(\mu), \mathcal{M}_2^*(\nu))(\zeta). \quad (3.23)$$

For $\mathcal{M}^* = (\mathcal{M}_1^*, \mathcal{M}_2^*)$.

To proceed further, we need some the propositions:

(A₁) With positive constant value $a, b, \mathcal{T}_{\mathcal{Q}_1}^*, \mathcal{T}_{\mathcal{Q}_2}^*$ and $\varepsilon_1, \varepsilon_2 \in [0, 1]$, the functions $\mathcal{Q}_1, \mathcal{Q}_2$ satisfy the growth conditions

$$|\mathcal{Q}_1(\zeta, \mu)| \leq \phi_p(a\|x\|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*),$$

$$|\mathcal{Q}_2(\zeta, \nu)| \leq \phi_p(b\|\nu\|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*),$$

(A₂) There exist real valued constants $\gamma_{\mathcal{Q}_1}, \gamma_{\mathcal{Q}_2}$, for all $\mu, \nu, \xi, \omega \in \mathfrak{F}$,

$$|\mathcal{Q}_1(\zeta, \nu) - \mathcal{Q}_1(\zeta, \xi)| \leq \gamma_{\mathcal{Q}_1}|\nu - \xi|,$$

$$|\mathcal{Q}_2(\zeta, \mu) - \mathcal{Q}_2(\zeta, \omega)| \leq \gamma_{\mathcal{Q}_2}|\mu - \omega|,$$

For simplicity, we define these symbols:

$$\nabla_1 = \left(\frac{2}{\Gamma(\sigma_1 + 1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} + \frac{1}{2\Gamma(\lambda_1 - 1)} \right)^{q-1}$$

$$\nabla_2 = \left(\frac{2}{\Gamma(\sigma_2 + 1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} + \frac{1}{2\Gamma(\lambda_2 - 1)} \right)^{q-1}$$

Theorem 3.2. Under the supposition (A₁), The operator $\mathcal{M}^* : \varpi^* \rightarrow \varpi^*$ is continuous and satisfies the growth condition given by:

$$\|\mathcal{M}^*(\mu, \nu)\| \leq \delta\|(\mu, \nu)\|^\varepsilon + \alpha. \quad (3.24)$$

where

$$\alpha = (\nabla_1 + \nabla_2)(\mathcal{T}_1^* + \mathcal{T}_2^*) \quad \delta = (a + b)(\nabla_1 + \nabla_2)$$

for each $(\mu, \nu) \in \Omega_r \subset \varpi^*$.

Proof. Assume a bounded set $\Omega_r = \{(\mu, \nu) \in \varpi : \|(\mu, \nu)\| \leq r\}$ with sequence (μ_n, ν_n) converging to (μ, ν) in Ω_r . To show that $\|\mathcal{M}^*(\mu_n, \nu_n) - \mathcal{M}^*(\mu, \nu)\| \rightarrow 0$ as $n \rightarrow +\infty$, let us consider

$$\begin{aligned}
|\mathcal{M}_1^* \mu_n(\zeta) - \mathcal{M}_1^* \mu(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu_n(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| |\phi_q \left[\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu_n(\eta)) d\eta \right]| \\
&\quad - \phi_q \left[\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right]| d\vartheta \\
&\leq (q-1) \rho_1^{q-2} \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| \left[\int_0^1 |\mathcal{H}^{\lambda_1}(\vartheta, \eta)| \right. \\
&\quad \left. \times |\mathcal{Q}_1(\eta, \nu_n(\eta)) - \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right]| d\vartheta,
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
|\mathcal{M}_2^* \nu_n(\zeta) - \mathcal{M}_2^* \nu(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu_n(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| |\phi_q \left[\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu_n(\eta)) d\eta \right]| \\
&\quad - \phi_q \left[\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right]| d\vartheta \\
&\leq (q-1) \rho_2^{q-2} \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| \left[\int_0^1 |\mathcal{H}^{\lambda_2}(\vartheta, \eta)| \right. \\
&\quad \left. \times |\mathcal{Q}_2(\eta, \mu_n(\eta)) - \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right]| d\vartheta.
\end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we have

$$\begin{aligned}
|\mathcal{M}^*(\mu_n, \nu_n)(\zeta) - \mathcal{M}^*(\mu, \nu)(\vartheta)| &\leq (q-1) \rho_1^{q-2} \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| \left[\int_0^1 |\mathcal{H}^{\lambda_1}(s, \eta)| \right. \\
&\quad \left. \times |\mathcal{Q}_1(\eta, \nu_n(\eta)) - \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right]| d\vartheta \\
&\quad - (q-1) \rho_2^{q-2} \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| \left[\int_0^1 |\mathcal{H}^{\lambda_2}(\vartheta, \eta)| \right. \\
&\quad \left. \times |\mathcal{Q}_2(\eta, \mu_n(\eta)) - \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right]| d\vartheta.
\end{aligned} \tag{3.27}$$

From the continuity of the functions \mathcal{Q}_1 , \mathcal{Q}_2 and (3.27), we get $|\mathcal{M}^*(\mu_n, \nu_n)(t) - \mathcal{M}^*(\mu, \nu)(\zeta)| \rightarrow 0$ as $n \rightarrow +\infty$. This that implies \mathcal{M}^* is a continuous. Further, By using (3.21) and (3.22) , we

obtain as follows

$$\begin{aligned}
|\mathcal{M}_1^* \mu(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) |\mathcal{Q}_1(\eta, \nu(\eta))| d\eta \right) d\vartheta \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(1, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \phi_p(a \|\nu\|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*) d\eta \right) d\vartheta \\
&\leq \left(\frac{2}{\Gamma(\sigma_1 + 1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} + \frac{1}{2\Gamma(\lambda_1 - 1)} \right)^{q-1} \\
&\quad \times (a \|\nu\|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*) \\
&\leq \nabla_1 (a \|\nu\|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*),
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
|\mathcal{M}_2^* \nu(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(s, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) |\mathcal{Q}_2(\eta, \mu(\eta))| d\eta \right) d\vartheta \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(1, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \phi_p(b \|\mu\|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*) d\eta \right) d\vartheta \\
&\leq \left(\frac{2}{\Gamma(\sigma_2 + 1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} + \frac{1}{2\Gamma(\lambda_2 - 1)} \right)^{q-1} \\
&\quad \times (b \|\mu\|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*) \\
&\leq \nabla_2 (b \|\mu\|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*).
\end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we obtain

$$\begin{aligned}
\mathcal{M}^*(\mu, \nu)(\zeta) &\leq \nabla_1 (a \|\nu\|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*) + \nabla_2 (b \|\mu\|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*) \\
&\leq (a + b) (\nabla_1 + \nabla_2) (\|\nu\|^{\varepsilon_1} + \|\mu\|^{\varepsilon_2}) + (\nabla_1 + \nabla_2) (\mathcal{T}_1^* + \mathcal{T}_2^*) \\
&= \alpha \|(\mu, \nu)\|^\varepsilon + \delta.
\end{aligned} \tag{3.30}$$

This completes the proof. \square

Theorem 3.3. *Under supposition that (A_1) hold. Then the operator $\mathcal{M}^* : \varpi^* \rightarrow \varpi^*$ is compact and \mathcal{L} -Lipschitz with constant zero.*

Proof. By using Theorem 3.2, we conclude that $\mathcal{M}^* : \omega \rightarrow \omega$ is bounded. Next, by supposition

(A₁), Lemma 2.3, and equations (3.17), (3.18), for any $\zeta_1, \zeta_2 \in [0, 1]$, we get

$$\begin{aligned}
|\mathcal{M}_1^* \mu(\zeta)_1 - \mathcal{M}_1^* \mu(\zeta_2)| &= \left| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta_1, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta_2, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta_1, \vartheta) - \mathcal{G}^{\sigma_1}(\zeta_2, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) |\mathcal{Q}_1(\eta, \nu(\eta))| d\eta \right) d\vartheta \\
&\leq \frac{|\zeta_1^{\sigma_1} - \zeta_2^{\sigma_1}|}{\Gamma(\sigma_1 + 1)} + \frac{|\zeta_1 - \zeta_2|}{\Gamma(\sigma_1)} \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} + \frac{1}{2\Gamma(\lambda_1 - 1)} \right)^{q-1} \\
&\quad \times (a|\nu|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*),
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
|\mathcal{M}_2^* \nu(\zeta)_1 - \mathcal{M}_2^* \nu(\zeta_2)| &= \left| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta_1, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta_2, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta_1, \vartheta) - \mathcal{G}^{\sigma_2}(\zeta_2, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) |\mathcal{Q}_2(\eta, \mu(\eta))| d\eta \right) d\vartheta \\
&\leq \frac{|\zeta_1^{\sigma_2} - \zeta_2^{\sigma_2}|}{\Gamma(\sigma_2 + 1)} + \frac{|\zeta_1 - \zeta_2|}{\Gamma(\sigma_2)} \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} + \frac{1}{2\Gamma(\lambda_2 - 1)} \right)^{q-1} \\
&\quad \times (b|\mu|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*).
\end{aligned} \tag{3.32}$$

With the help of (3.31), (3.32), we have

$$\begin{aligned}
|\mathcal{M}^*(\mu, \nu)(\zeta_1) - \mathcal{M}^*(\mu, \nu)(\zeta_2)| &\leq \frac{|\zeta_1^{\sigma_1} - \zeta_2^{\sigma_1}|}{\Gamma(\sigma_1 + 1)} + \frac{|\zeta_1 - \zeta_2|}{\Gamma(\sigma_1)} \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} + \frac{1}{2\Gamma(\lambda_1 - 1)} \right)^{q-1} \\
&\quad \times (a|\nu|^{\varepsilon_1} + \mathcal{T}_{\mathcal{Q}_1}^*) \\
&\quad + \frac{|\zeta_1^{\sigma_2} - \zeta_2^{\sigma_2}|}{\Gamma(\sigma_2 + 1)} + \frac{|\zeta_1 - \zeta_2|}{\Gamma(\sigma_2)} \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} + \frac{1}{2\Gamma(\lambda_2 - 1)} \right)^{q-1} \\
&\quad \times (b|\mu|^{\varepsilon_2} + \mathcal{T}_{\mathcal{Q}_2}^*).
\end{aligned} \tag{3.33}$$

As $\zeta_1 \rightarrow \zeta_2$, the right hand side of (3.33) approaches zero. Thus, the operator $\mathcal{M}^* = (\mathcal{M}_1^*, \mathcal{M}_2^*)$ is an equicontinuous on \mathcal{S} . By theory 2.2 implies that $\mathcal{M}^*(\mathcal{S})$ is compact. Subsequently, \mathcal{S} is \mathcal{L} -Lipschitz with constant zero. \square

Theorem 3.4. *Under the suppositions (A₁) and (A₂) hold and $\alpha < 1$. Then the nonlinear coupled system (1.1) of FODEs has at least one solution with condition that the set of the solutions ϱ is bounded in ϖ^* .*

Proof. For EUS of the nonlinear coupled system of FODEs (1.1), with the help of Theory 2.1. Let us consider that $\varrho = \{(\mu, \nu) \in \varpi^* : \text{there exist } \Upsilon \in [0, 1], \text{ where } (\mu, \nu) = \Upsilon\chi(\mu, \nu)\}$, To proof that ϱ is bounded we assume that $(\mu, \nu) \in \varrho$, with that $(\mu, \nu) = \mathcal{N} \rightarrow \infty$, from theorem 3.2, we get

$$\|(\mu, \nu)\| = \|\Upsilon\chi(\mu, \nu)\| \leq \|\chi(\mu, \nu)\| \leq \alpha\|(\mu, \nu)\|^\epsilon + \delta \quad (3.34)$$

as $\|(x, y)\| = \mathcal{N}$ then (3.34) implies that

$$\|(\mu, \nu)\| \leq \alpha\|(\mu, \nu)\|^\epsilon + \delta$$

$$1 \leq \alpha \frac{\|(\mu, \nu)\|^\epsilon}{\|(\mu, \nu)\|} + \frac{\delta}{\|(\mu, \nu)\|}$$

$$1 \leq \alpha \frac{1}{\mathcal{N}^{1-\epsilon}} + \frac{\delta}{\mathcal{N}} \rightarrow 0, \quad \text{as } \mathcal{N} \rightarrow \infty.$$

This is a contradiction. In the end, $\|(\mu, \nu)\| < \infty$ which means that ϱ is bounded, by theory 2.1, then the ϱ has at least one the solution to our problem (1.1). Thus, the set of the solutions ϱ of the nonlinear coupled system is bounded. \square

Theorem 3.5. *Let suppositions (A_1) and (A_2) hold. Then the nonlinear coupled system (1.1) has a unique solution if and only if $\kappa^* < 1$, such that*

$$\begin{aligned} \kappa_1 &= (q-1)\rho_1^{q-2} \left(\frac{2}{\Gamma(\sigma_1+1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1+\lambda_1}{\Gamma(\lambda_1+1)} + \frac{1}{2\Gamma(\lambda_1-1)} \right) \gamma_{\mathcal{Q}_1}, \\ \kappa_2 &= (q-1)\rho_2^{q-2} \left(\frac{2}{\Gamma(\sigma_2+1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1+\lambda_2}{\Gamma(\lambda_2+1)} + \frac{1}{2\Gamma(\lambda_2-1)} \right) \gamma_{\mathcal{Q}_2}, \end{aligned}$$

$$\kappa^* = \kappa_1 + \kappa_2.$$

Proof. From (3.21),(3.22),and suppositions (A_1) and (A_2) , we have

$$\begin{aligned}
|\mathcal{M}_1^* \mu(\zeta) - \mathcal{M}_1^* \bar{\mu}(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \bar{\nu}(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) \\
&\quad - \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(s, \eta) \mathcal{Q}_1(\eta, \bar{\nu}(\eta)) d\eta \right) | d\vartheta \\
&\leq (q-1) \rho_1^{q-2} \int_0^1 |\mathcal{G}^{\sigma_1}(1, \vartheta)| \int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \\
&\quad \times |\mathcal{Q}_1(\eta, \nu(\eta)) - \mathcal{Q}_1(\eta, \bar{\nu}(\eta))| d\eta d\vartheta \\
&\leq (q-1) \rho_1^{q-2} \left(\frac{2}{\Gamma(\sigma_1 + 1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_1 - 1)} \right) \gamma_{\mathcal{Q}_1} |\nu(\zeta) - \bar{\nu}(\zeta)| = \kappa_1 |\nu(\zeta) - \bar{\nu}(\zeta)|,
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
|\mathcal{M}_2^* \nu(\zeta) - \mathcal{M}_2^* \bar{\nu}(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(s, \eta) \mathcal{Q}_2(\eta, \bar{\mu}(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) \\
&\quad - \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(s, \eta) \mathcal{Q}_2(\eta, \bar{\mu}(\eta)) d\eta \right) | d\vartheta \\
&\leq (q-1) \rho_2^{q-2} \int_0^1 |\mathcal{G}^{\sigma_2}(1, \vartheta)| \int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \\
&\quad \times |\mathcal{Q}_2(\eta, \mu(\eta)) - \mathcal{Q}_2(\eta, \bar{\mu}(\eta))| d\eta d\vartheta \\
&\leq (q-1) \rho_2^{q-2} \left(\frac{2}{\Gamma(\sigma_2 + 1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_2 - 1)} \right) \gamma_{\mathcal{Q}_2} |\mu(t) - \bar{\mu}(\zeta)| \\
&= \kappa_2 |\mu(t) - \bar{\mu}(\zeta)|,
\end{aligned} \tag{3.36}$$

with the help of (3.35), (3.36), we get

$$\begin{aligned}
|\mathcal{M}^*(\mu, \nu)(\zeta) - \mathcal{M}^*(\bar{\mu}, \bar{\nu})(\zeta)| &\leq (q-1)\rho_1^{q-2} \left(\frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{2}{\Gamma(\sigma_1+1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1+\lambda_1}{\Gamma(\lambda_1+1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_1-1)} \right) \gamma_{\mathcal{Q}_1} |\nu(\zeta) - \bar{\nu}(\zeta)| \\
&\quad + (q-1)\rho_2^{q-2} \left(\frac{2}{\Gamma(\sigma_2+1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1+\lambda_2}{\Gamma(\lambda_2+1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_2-1)} \right) \gamma_{\mathcal{Q}_2} |\mu(\zeta) - \bar{\mu}(\zeta)| \\
&= \kappa_1 |\nu(\zeta) - \bar{\nu}(\zeta)| + \kappa_2 |\mu(\zeta) - \bar{\mu}(\zeta)| \\
&= \kappa^* (\|(\mu, \nu)(\zeta) - (\bar{\mu}, \bar{\nu})(\zeta)\|).
\end{aligned} \tag{3.37}$$

With the help of Banach's fixed point theory and $\kappa^* < 1$, the contraction \mathcal{M}^* has a unique fixed point. Thus, the nonlinear coupled system of FODEs (1.1) has a unique solution. \square

4 Hyers-Ulam stability of system

Here we study HU-stability of a solution for the nonlinear coupled system of FODEs with p-Laplacian operator (1.1) and boundary conditions.

Definition 4.1 The nonlinear coupled system of Hammerstein-type integral Eqs (3.17),(3.18) is HU-stable if there exist positive constants $\mathcal{W}_1^*, \mathcal{W}_2^*$ achieves the these conditions:

For every $\beta_1, \beta_2 > 0$, if

$$\begin{aligned}
|\mu(\zeta) - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta| &\leq \beta_1, \\
|\nu(\zeta) - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta| &\leq \beta_2.
\end{aligned} \tag{4.1}$$

There exists a pair, say $(\mu^*(\zeta), \nu^*(\zeta))$, satisfying

$$\begin{aligned}
\mu^*(\zeta) - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu^*(\eta)) d\eta \right) d\vartheta, \\
\nu^*(\zeta) - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu^*(\eta)) d\eta \right) d\vartheta,
\end{aligned} \tag{4.2}$$

such that

$$\begin{aligned}
|\mu(\zeta) - \mu^*(\zeta)| &\leq \mathcal{W}_1^* \beta_1, \\
|\nu(t) - \nu^*(\zeta)| &\leq \mathcal{W}_2^* \beta_2.
\end{aligned} \tag{4.3}$$

Theorem 4.1. . With the suppositions (A_1) and (A_2) hold, the solution of the nonlinear coupled system of FODEs with p -Laplacian operator(1.1) is HU-stable.

Proof. From Theory 3.5 and definition 4.1 let $(\mu(\zeta), \nu(\zeta))$ be a solution of the system (3.17),(3.18). Let $(\mu^*(\zeta), \nu^*(\zeta))$ be any other approximation achieves (4.2).Then we get

$$\begin{aligned}
|\mu(\zeta) - \mu^*(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu^*(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_1}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu(\eta)) d\eta \right) \\
&\quad - \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \mathcal{Q}_1(\eta, \nu^*(\eta)) d\eta \right) | d\vartheta \\
&\leq (q-1) \rho_1^{q-2} \int_0^1 |\mathcal{G}^{\sigma_1}(1, \vartheta)| \int_0^1 \mathcal{H}^{\lambda_1}(\vartheta, \eta) \\
&\quad \times |\mathcal{Q}_1(\eta, \nu(\eta)) - \mathcal{Q}_1(\eta, \nu^*(\eta))| d\eta d\vartheta \\
&\leq (q-1) \rho_1^{q-2} \left(\frac{2}{\Gamma(\sigma_1 + 1)} + \frac{1}{\Gamma(\sigma_1)} \right) \left(\frac{1 + \lambda_1}{\Gamma(\lambda_1 + 1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_1 - 1)} \right) \gamma_{\mathcal{Q}_1} |\nu(\zeta) - \nu^*(\zeta)| \\
&= \mathcal{W}_1^* |\nu(\zeta) - \nu^*(\zeta)| = \mathcal{W}_1^* \beta_1,
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
|\nu(\zeta) - \nu^*(\zeta)| &= \left| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) d\vartheta \right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta) \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu^*(\eta)) d\eta \right) d\vartheta \right| \\
&\leq \int_0^1 |\mathcal{G}^{\sigma_2}(\zeta, \vartheta)| \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu(\eta)) d\eta \right) \\
&\quad - \phi_q \left(\int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \mathcal{Q}_2(\eta, \mu^*(\eta)) d\eta \right) | d\vartheta \\
&\leq (q-1) \rho_2^{q-2} \int_0^1 |\mathcal{G}^{\sigma_2}(1, \vartheta)| \int_0^1 \mathcal{H}^{\lambda_2}(\vartheta, \eta) \\
&\quad \times |\mathcal{Q}_2(\eta, \mu(\eta)) - \mathcal{Q}_2(\eta, \mu^*(\eta))| d\eta d\vartheta \\
&\leq (q-1) \rho_2^{q-2} \left(\frac{2}{\Gamma(\sigma_2 + 1)} + \frac{1}{\Gamma(\sigma_2)} \right) \left(\frac{1 + \lambda_2}{\Gamma(\lambda_2 + 1)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(\lambda_2 - 1)} \right) \gamma_{\mathcal{Q}_2} |\mu(\zeta) - \mu^*(\zeta)| \\
&= \mathcal{W}_2^* |\mu(\zeta) - \mu^*(\zeta)| = \mathcal{W}_2^* \beta_2,
\end{aligned} \tag{4.5}$$

where $\mathcal{W}_1^* = ((q-1)\rho_1^{q-2}(\frac{2}{\Gamma(\sigma_1+1)} + \frac{1}{\Gamma(\sigma_1)})(\frac{1+\lambda_1}{\Gamma(\lambda_1+1)} + \frac{1}{2\Gamma(\lambda_1-1)})\gamma_{\mathcal{Q}_1})$,
 $\mathcal{W}_2^* = ((q-1)\rho_2^{q-2}(\frac{2}{\Gamma(\sigma_2+1)} + \frac{1}{\Gamma(\sigma_2)})(\frac{1+\lambda_2}{\Gamma(\lambda_2+1)} + \frac{1}{2\Gamma(\lambda_2-1)})\gamma_{\mathcal{Q}_2})$ Hence, with the help of (4.4) and (4.5), the system (3.17) and (3.18) is HU- stable. Consequently, the system with p-Laplacian operator (1.1) is HU-stable \square

5 Illustrative examples

Here we will introduce an example to prove our results of proposed problem in sections 3 and 4.

Example 5.1. let the system with p-Laplacian operator with FODE and IBCs as following:

$$\left\{ \begin{array}{l} {}^c\mathcal{D}^{\frac{4}{3}}(\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\mu(\zeta))) + \mathcal{Q}_1(\zeta, \nu(\zeta)) = 0, \quad {}^c\mathcal{D}^{\frac{4}{3}}(\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\nu(\zeta))) + \mathcal{Q}_2(\zeta, \mu(\zeta)) = 0, \\ (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\mu(\zeta)))^{(i)}|_{\zeta=0} = 0, \quad (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\nu(\zeta)))^{(i)}|_{\zeta=0} = 0 \quad \text{for } i = 0, 3, 4, \dots, k-1, \\ (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\mu(\zeta)))|_{\zeta=1} = 0, \quad (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\mu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\frac{5}{2}-2}\mathcal{Q}_1(\zeta, \nu(\zeta))|_{\zeta=1}, \\ (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\nu(\zeta)))''|_{\zeta=1} = 0, \quad (\phi_5({}^c\mathcal{D}^{\frac{5}{2}}\nu(\zeta)))'|_{\zeta=1} = \mathcal{I}^{\frac{5}{2}-2}\mathcal{Q}_2(\zeta, \mu(\zeta))|_{\zeta=1}, \\ (\mu(0))^{(j)} = 0 \quad (\nu(0))^{(j)} = 0 \quad \text{for } j = 2, 3, 4, \dots, k-1, \\ \mu'(1) = 0, \quad \mu(1) = -\mathcal{I}^{\frac{5}{2}-1}(\phi_q(\int_0^1 \mathcal{H}^{\frac{4}{3}}(\zeta, \vartheta)\mathcal{Q}_1(\vartheta, \nu(\vartheta))d\vartheta)), \\ \nu'(1) = 0, \quad \nu(1) = -\mathcal{I}^{\frac{5}{2}-1}(\phi_q(\int_0^1 \mathcal{H}^{\frac{4}{3}}(\zeta, \vartheta)\mathcal{Q}_2(\vartheta, \mu(\vartheta))d\vartheta)), \end{array} \right. \quad (5.1)$$

where $\zeta \in [0, 1]$, $p = 5$, $\rho_i = 3$, $\sigma_i = \frac{5}{3}$, $\lambda_i = \frac{4}{3}$, for $i = 1, 2$, $\mathcal{Q}_1(\zeta, \mu(\zeta)) = -21/13 + 1/17\sin(\nu)$, $\mathcal{Q}_2 = 28/16 + 1/19\cos(\mu)$, which implies $\gamma_{\mathcal{Q}_1}^* = \gamma_{\mathcal{Q}_2}^* = \frac{1}{5}$. By simple mathematical computations, we get

$$\begin{aligned} \kappa_1 &= (q-1)\rho_1^{q-2}(\frac{2}{\Gamma(\sigma_1+1)} + \frac{1}{\Gamma(\sigma_1)})(\frac{1+\lambda_1}{\Gamma(\lambda_1+1)} + \frac{1}{2\Gamma(\lambda_1-1)})\gamma_{\mathcal{Q}_1} \\ &= \frac{4}{5}3^3(\frac{2}{\Gamma(\frac{8}{3})} + \frac{1}{\Gamma(\frac{5}{3})})(\frac{1+\frac{4}{3}}{\Gamma(\frac{7}{3})} + \frac{1}{2\Gamma(\frac{1}{3})})\frac{1}{5}, \\ \kappa_2 &= (q-1)\rho_2^{q-2}(\frac{2}{\Gamma(\sigma_2+1)} + \frac{1}{\Gamma(\sigma_2)})(\frac{1+\lambda_2}{\Gamma(\lambda_2+1)} + \frac{1}{2\Gamma(\lambda_2-1)})\gamma_{\mathcal{Q}_2} \\ &= \frac{4}{5}3^3(\frac{2}{\Gamma(\frac{8}{3})} + \frac{1}{\Gamma(\frac{5}{3})})(\frac{1+\frac{4}{3}}{\Gamma(\frac{7}{3})} + \frac{1}{2\Gamma(\frac{1}{3})})\frac{1}{5}, \\ \kappa^* &= \kappa_1 + \kappa_2 = 2(\frac{4}{5}3^3(\frac{2}{\Gamma(\frac{8}{3})} + \frac{1}{\Gamma(\frac{5}{3})})(\frac{1+\frac{4}{3}}{\Gamma(\frac{7}{3})} + \frac{1}{2\Gamma(\frac{1}{3})})\frac{1}{5}) < 1. \end{aligned} \quad (5.2)$$

By Theory 3.5 and Eq (5.2), we deduce that (5.1) has positive a unique solution. Thus, the conditions of Theory 4.1 may be verified simply. Similarly, the system (5.1) is HU-stable.

6 Conclusion

In this literature, we applied the topological degree theory successfully to investigate sufficient conditions for existence and uniqueness of solutions to a nonlinear coupled system of FODEs with IBCs and a p -Laplacian operator. For these goals, we used Green functions to convert the proposed problem (1.1) into an integral equation and then to topological degree theory. Further, we had investigated the conditions of the Hyers-Ulam stability to the proposed problem. For application, we included an illustrative example to verify from the results. For future work, we suggest to readers study the problem for multiple solutions. Also, the problem may be studied for the existence of the solution using various definitions of the fractional derivative.

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