TAMKANG JOURNAL OF MATHEMATICS
Volume 40, Number 1, 67-75, Spring 2009

FUNCTIONS AND REGULARITY OF TOPOLOGICAL SPACES

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Abstract. Preservation of regularity, almost regularity, and s-regularity of topological spaces, is considered.

1. Introduction

It is known that regularity is preserved under continuous closed surjections with compact point inverses (i.e., under perfect functions). In 1978, Noiri [19] proved that if $f:(X,\tau)\to(Y,\sigma)$ is a continuous semi-closed surjection with compact point inverses, and (X,τ) is regular, then (Y,σ) is s-regular. Murdeshwar [15] showed that regularity is preserved under surjections that are open, continuous, and closed. In 1986, Greenwood and Reilly gave a certain generalization of the Murdeshwar's result. In this paper we, in turn, generalize the Greenwood and Reilly's result [7, Proposition 8]. Moreover, we propose a number of further theorems related to preservation of regularity, almost regularity, and s-regularity of spaces, under surjections.

2. Preliminaries

Throughout, (X,τ) (and (Y,σ)) means a topological space (briefly: a space). Let $S\subset X$ be a subset of a space (X,τ). The ***closure*** of S and the ***interior*** of S will be denoted by $\mathrm{cl}_\tau(S)$ and $\mathrm{int}_\tau(S)$ (or simply $\mathrm{cl}(S)$ and $\mathrm{int}(S)$), respectively. The set S is said to be *regular open* (resp. *α-open* [16]; *semi-open* [10]; *preopen* [13]) in (X,τ), if $S=\mathrm{int}(\mathrm{cl}(S))$ (resp. $S\subset\mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$; $S\subset\mathrm{cl}(\mathrm{int}(S))$; $S\subset\mathrm{int}(\mathrm{cl}(S))$). The S is said to be *α-closed* (resp. *semi-closed* [3]) in (X,τ), if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))\subset S$ (resp. $\mathrm{int}(\mathrm{cl}(S))\subset S$). The collection of all regular open (resp. α-open; semi-open; preopen) subsets of (X,τ) is denoted by $\mathrm{RO}(X,\tau)$ (resp. τ^α; $\mathrm{SO}(X,\tau)$; $\mathrm{PO}(X,\tau)$). It is known that $\tau\subset\tau^\alpha$ (the equality does not hold, in general), and that τ^α forms a topology on X. Obviously, S is semi-closed in (X,τ) iff $X\setminus S$ is semi-open in (X,τ). Moreover, S is semi-closed in (X,τ) iff $S=s\mathrm{cl}_\tau(S)$ [3, Theorem 1.4(2)], where $s\mathrm{cl}_\tau(S)$ (the semi-closed of S in (X,τ)) is defined analogously to the ordinary closrue. It is known that $\tau^\alpha=\mathrm{SO}(X,\tau)\cap\mathrm{PO}(X,\tau)$ [21, Lemma 2.1].

Received March 23, 2007; revised September 23, 2007.
2000 *Mathematics Subject Classification*. 54C08.
Key words and phrases. Almost regular, s-regular spaces, almost open, α-open, α-closed, semi-closed functions, almost continuity, $\mathscr{R}$-mapness, α-irresolutness.

A function $f:(X,\tau)\to(Y,\sigma)$ is called ***almost open in the sense of Wilansky*** (briefly ***a.o.W.***) [27, for injections], [20, Definition 1.3], if $f^{-1}(\mathrm{cl}_\sigma(V))\subset \mathrm{cl}_\tau(f^{-1}(V))$ for every $V\in\sigma$. In 1984, Rose [24, Theorem 11] has proved that f if a.o.W. iff $f(U)\in \mathrm{PO}(Y,\sigma)$ for every $U\in\tau$ (i.e., iff f is preopen [13]. A function $f:(X,\tau)\to(Y,\sigma)$ is called ***almost open in the sense of the Singals*** (briefly ***a.o.S.***) [24], if $f(U)\in\sigma$ for every $U\in \mathrm{RO}(X,\tau)$. A function f is said to be ***α-open*** [14] (resp. ***pre-α-open***; ***s-open*** [1]), if $f(U)\in\sigma^\alpha$ (resp. $f(U)\in\sigma^\alpha$; $f(U)\in\sigma$) for any set $U\in\tau$ (resp. $U\in\tau^\alpha$; $U\in \mathrm{SO}(X,\tau)$). A function f is called ***α-closed*** [14] (resp. ***pre-α-closed*** [5]; ***semi-closed*** [17]; ***pre-semi-closed*** [26]), if $f(U)$ is an α-closed (resp. an α-closed; a semi-closed; a semi-closed) subset of (Y,σ) for each closed (resp. α-closed; closed; semi-closed) subset U of (X,τ).

A function $f:(X,\tau)\to(Y,\sigma)$ is ***almost continuous in the sense of the Singals*** (briefly ***a.c.S.***) [24] (resp. an ***$\mathscr{R}$-map*** [2]; *α-irresolute* [12]), if the preimage $f^{-1}(V)\in\tau$ [24, Theorem 2.2(b)] (resp. $f^{-1}(V)\in \mathrm{RO}(X,\tau)$; $f^{-1}(V)\in\tau^\alpha$) for every set $V\in \mathrm{RO}(Y,\sigma)$ (resp. $V\in \mathrm{RO}(Y,\sigma)$; $V\in\sigma^\alpha$). Every $\mathscr{R}$-map is a.c.S., but the converse is false in general [22, Example 4.14]. It is known that α-irresoluteness and continuity are independent notions [12, Examples 1 & 2].

A space (X,τ) is called ***extremally disconnected*** (briefly ***e.d.***) if $\mathrm{cl}(S)\in\tau$ for every $S\in\tau$. A space (X,τ) is e.d. iff $\tau^\alpha=\mathrm{SO}(X,\tau)$ [8, Theorem 2.9].

A space (X,τ) is called ***almost regular*** [25, Theorem 2.2(b)] (resp. ***s-regular*** [11, Theorem 2(b)] if for each point $x\in X$ and each set $V\in \mathrm{RO}(X,\tau)$) (resp. $V\in\tau$) containing x, there exists a set $U\in \mathrm{RO}(X,\tau)$ (resp. $U\in \mathrm{SO}(X,\tau)$) such that $x\in U\subset \mathrm{cl}(U)\subset V$ (resp. $x\in U\subset \mathrm{scl}(U)\subset V$). Every regular space is almost regular (resp. s-regular) and the converse is not always true [18, Remark 1] (resp. [11, Example 1]). Conditions of almost regularity and s-regularity are independent of each other [11, Examples 1 and 2]. Recall that (X,τ) is called ***semi-regular*** if for each $x\in X$ and each $V\in\tau$ with $x\in V$, there exists a set $U\in \mathrm{RO}(X,\tau)$ such that $x\in U\subset V$.

3. Almost regular spaces

Theorem 1. [7, Proposition 8]. *If $f:(X,\tau)\to(Y,\sigma)$ is an open, continuous, and α-closed surjection, and (X,τ) is regular, then (Y,σ) is regular.*

It is clear that for any bijection $f:(X,\tau)\to(Y,\sigma)$, the properties of being α-closed and being α-open are equivalent. Thus, Theorem 1 leads to the well known result: *every homeomorphism preserves the regularity property.*

In order to generalize Theorem 1 we need the following lemma, which is, in turn, a generalization of [7, Lemma 2].

Lemma 1. *Let (X,τ) be a space, $U\in \mathrm{PO}(X,\tau)$ and $U\subset V$. Then* $\mathrm{cl}(U)\subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(V)))$.

Proof. We have $U\subset \mathrm{int}(\mathrm{cl}(U))$, whence as $\mathrm{int}(\mathrm{cl}(U))\subset \mathrm{int}(\mathrm{cl}(V))$ we get $\mathrm{cl}(U)\subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(V)))$.

Theorem 2. *Let a functon $f:(X,\tau)\to(Y,\sigma)$ be an a.o.W., continuous, and α-closed surjection. If (X,τ) is regular, then (Y,σ) is regular.*

Proof. Let $y \in Y$ and $U \in \sigma$, $U \ni y$, be arbitrary, and let $x \in X$ be such that $f(x) = y$. By regularity of (X,τ) there exists a set $G \in \tau$ such that $x \in G \subset \mathrm{cl}_\tau(G) \subset f^{-1}(U)$. Then $y \in f(G) \subset f(\mathrm{cl}_\tau(G)) \subset U$. But, f is a.o.W. and so, by Lemma 1, $f(G) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(G))) \subset \mathrm{cl}_\sigma(f(G)) \subset \mathrm{cl}_\sigma(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(\mathrm{cl}_\sigma(G)))))$. Put $V = \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(G))) \in \mathrm{RO}(Y,\sigma)$. We obtain $y \in V \subset \mathrm{cl}_\sigma(V) \subset U$. This shows that (Y,σ) is regular.

Corollary 1. *Let a bijection $f : (X,\tau) \to (Y,\sigma)$ be α-open and continuous. If (X,τ) is regular, then so is (Y,σ).*

Proof. Follows from Theorem 2. Just use the fact that each α-open mapping is a.o.W.

An analysis of the proof of Theorem 2 leads to Theorems 3 and 4 below. The details are left to the reader.

Theorem 3. *Let $f : (X,\tau) \to (Y,\sigma)$ be an a.o.W., a.c.S., and α-closed surjection. If (X,τ) is regular, then (Y,σ) is almost regular.*

Corollary 2. *Let a bijection $f : (X,\tau) \to (Y,\sigma)$ be α-open and a.c.S. If (X,τ) is regular, then (Y,σ) is almost regular.*

Theorem 4. *Let a surjection $f : (X,\tau) \to (Y,\sigma)$ be an a.o.S. and α-closed $\mathscr{R}$-map. If (X,τ) is almost regular, then (Y,σ) is almost regular.*

Corollary 3. *Assume a bijection $f : (X,\tau) \to (Y,\sigma)$ is an open $\mathscr{R}$-map. If (X,τ) is almost regular, then (Y,σ) is almost regular.*

Noiri has stated [20, Examples 1.6 & 1.7] that a.o.W. and a.o.S. are independent notions. The following examples show that these notions remain independent if considered within the class of bijective mappings.

Example 1. (a). Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, $\sigma = \{\emptyset, X\}$. The identity id: $(X,\tau) \to (X,\sigma)$ is a.o.W. and not a.o.S.

(b). Let $X = \{a,b\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{b\}\}$. Then: id: $(X,\tau) \to (X,\sigma)$ and id^{-1} are a.o.S., but not a.o.W.

Now, we provide an example of bijection $f : (X,\tau) \to (X,\sigma)$ which is continuous (and so a.c.S.), but is not an $\mathscr{R}$-map (compare this example to [22, Example 4.14] for the non-bijective case).

Example 2. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be defined as follows: $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is not an $\mathscr{R}$-map singce $f^{-1}(\{a\}) \notin \mathrm{RO}(X,\tau)$.

$\mathscr{R}$-mapness and continuity are independent notions, even in the class of bijections. Because of Example 2, it is enough to indicate an $\mathscr{R}$-map which is not continuous.

Example 3. Let $X = \{a,b\}$, $\tau = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$. The identity id: $(X,\tau) \to (X,\sigma)$ is an $\mathscr{R}$-map which is not continuous.

Remark 1. An a.c.S. functon (even a bijection) need not be continuous (compare the following example to [24, Example 2.2]) nor an $\mathscr{R}$-map. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\sigma = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{b, c, d\}\}$, and let $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$.

Theorem 5. *Let (X, τ) be a space. If (X, τ^α) is regular, then (X, τ) is regular.*

Proof. Assume (X, τ^α) is regular. Suppose $x \in X$ and $U \in \tau$ such that $x \in U$, are arbitrary. By the hypothesis, there is a set $V \in \tau^\alpha$ such that $x \in V \subset \mathrm{cl}_{\tau^\alpha}(V) \subset U$. But, $\mathrm{cl}_{\tau^\alpha}(V) = \mathrm{cl}_\tau(V)$ [6, Lemma 1(i)] and as $V \in \mathrm{PO}(X, \tau)$ we get $x \in V \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(V)) \subset \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(V))) \subset \mathrm{cl}_\tau(V)$. This shows (X, τ) is regular.

The converse to Theorem 5 may not be true as the following example shows.

Example 4. Consider the real line $\mathbb{R}$ endowed with the Euclidean topology τ_e. The space $(\mathbb{R}, \tau_e^\alpha)$ is not regular: take into account $x = 0$ and its α-neighbourhood $V = [-1, 1] \setminus \bigcup_{i=1}^{\infty} \{-\frac{1}{i}, \frac{1}{i}\}$.

Lemma 2. [9, Corollary 2.4(a)]. $\mathrm{RO}(X, \tau) = \mathrm{RO}(X, \tau^\alpha)$ *for each space* (X, τ).

Theorem 6. *A space (X, τ) is almost regular if and only if (X, τ^α) is alnost regular.*

Proof. Clear by Lemma 2 and the fact that $\mathrm{cl}_\tau(S) = \mathrm{cl}_{\tau^\alpha}(S)$ for each $S \in \mathrm{SO}(X, \tau)$ [6, Lemma 1(i)].

Definition 1. A function $f : (X, \tau) \to (Y, \sigma)$ is α-***a.o.W.*** if $f(U) \in \mathrm{PO}(Y, \sigma)$ for every $U \in \tau^\alpha$.

Obviously, each α-a.o.W. function is a.o.W. The converse implication does not hold.

Example 5. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The identity id: $(X, \tau) \to (X, \sigma)$ is open (and hence a.o.W.), but it is not α-a.o.W. because $\mathrm{id}(\{a, c\}) \notin \mathrm{PO}(X, \sigma)$.

Lemma 3. *Let (X, τ) be a space. Then,*
① [9, Corollary 2.5(a)]. $\mathrm{PO}(X, \tau) = \mathrm{PO}(X, \tau^\alpha)$;
② [16, Proposition 10]. $\tau^\alpha = (\tau^\alpha)^\alpha$.

Theorem 7. *Let a surjection $f : (X, \tau) \to (Y, \sigma)$ be α-a.o.W. α-irresolute, pre-α-closed. If (X, τ^α) is regular then (Y, σ^α) is regular.*

Proof. By Lemma 3, the surjection $f_\star : (X, \tau^\alpha) \to (Y, \sigma^\alpha)$, where $f_\star(x) = f(x)$ for each $x \in X$ (see [9, page 86, line 11]), is also a.o.W., continuous, and α-closed. Thus by Theorem 2 we get (Y, σ^α) is regular.

Since every bijection is pre-α-closed if and only if it is pre-α-open, one obtains the well known result for homeomorphisms: $(X, \tau^\alpha) \to (Y, \sigma^\alpha)$ (the one stated right under Theorem 1).

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is a semi-homeomorphism [4] if it is bijective, pre-semi-open ($f(U) \in \mathrm{SO}(Y, \sigma)$ for every $U \in \mathrm{SO}(X, \tau)$), and irresolute ($f^{-1}(V) \in \mathrm{SO}(X, \tau)$ for every $V \in \mathrm{SO}(Y, \sigma)$). The following result is known: *a function $f : (X, \tau) \to (Y, \sigma)$ is a semi-homeomorphism if and only if $f_\star : (X, \tau^\alpha) \to (Y, \sigma^\alpha)$ is a semi-homeomorphism* [6, Theorem 1].

Corollary 4. *Let a function $f:(X,\tau)\to(Y,\sigma)$ be a semi-homeomorphism. If (X,τ^{α}) is regular, then (Y,σ^{α}) is also regular.*

This corollary is interesting in regard for Theorem 5 and Example 4 above, and the fact that the image of a regular space under a semi-homeomorphism is not necessarily regular [4, Example 1.5]. Moreover, by Corollary 4 and Theorem 5, [4, Example 1.5] is another illustration that regularity of (X,τ) need not imply regularity of (X,τ^{α}).

The next theorem is a simple consequence of Theorems 3, 5, and 6.

Theorem 8. *Let $f:(X,\tau)\to(Y,\sigma)$ be an a.o.W., a.c.S., and α-closed surjection. If (X,τ^{α}) is regular, then (Y,σ^{α}) is almost regular.*

Corollary 5. *Let a bijection $f:(X,\tau)\to(Y,\sigma)$ be α-open and a.c.S. If (X,τ^{α}) is regular, then (Y,σ^{α}) is almost regular.*

Proof. Follows from Theorem 8.

In the class of bijective mappings, α-irresoluteness and continuity are independent of each other [12, Examples 1 & 2].

Example 6. (a). Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,X,\{a\},\{b,c\},\{a,b,c\}\}$, $\sigma=\{\emptyset,X,\{a\}\}$. The identity id: $(X,\tau)\to(X,\sigma)$ is continuous (hence a.c.S.) and not α-irresolute, since $\mathrm{id}^{-1}(\{a,b\})\notin\tau^{\alpha}$.

(b). Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\}\}$, $\sigma=\tau^{\alpha}=\tau\cup\{\{a,b,c\}\}$. Then id: $(X,\tau)\to(X,\sigma)$ is discontinuous but α-irresolute.

Problem 1. *Find an α-irresolute bijection (or just a surjection) which is not a.c.S.*

Theorem 9. *Let $f:(X,\tau)\to(Y,\sigma)$ be a surjective α-closed and a.o.S. $\mathscr{R}$-map. If (X,τ^{α}) is regular, then (Y,σ^{α}) is almost regular.*

Proof. We apply Theorems 5, 4, and 6, respectively.

Corollary 6. *Let a bijection $f:(X,\tau)\to(Y,\sigma)$ be an open $\mathscr{R}$-map. If (X,τ^{α}) is regular then (Y,σ^{α}) is almost regular.*

Comparing Theorems 7 and 9 it is worthy of notice that α-a.o.W. and a.o.S. are independent notions, even within the class of bijections.

Example 7. (a). Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$, $\sigma=\{\emptyset,X\}$. Then, id: $(X,\tau)\to(X,\sigma)$ is α-a.o.W. and not a.o.S., because for $S=\{a\}$ (or $S=\{b\}$), $\mathrm{id}(S)\notin\sigma$.

(b). Let $X=\{a,b\}$, $\tau=\{\emptyset,X,\{a\}\}$, $\sigma=\{\emptyset,X,\{b\}\}$. The identity id: $(X,\tau)\to(X,\sigma)$ is a.o.S. and not α-a.o.W.

In order to see that α-irresoluteness and $\mathscr{R}$-mapness are independent of each other, it is enough to apply Examples 2 and 3.

4. *s*-regular spaces

It is well-known that regularity implies semi-regularity. Examples of spaces which are semi-regular and not regular are also known [25, Remark 3.2] and [11, Example 1].

Theorem 10. *If a space (X,τ) is semi-regular, then it is s-regular.*

Proof. We shall use [9, Proposition 2.7(a)]: $s\text{cl}(S) = \text{int}(\text{cl}(S))$ for any $S \in \text{PO}(X,\tau)$. Let $x \in X$ and $U \in \tau$ be such that $x \in U$. By hypothesis there exists a set $V \in \text{RO}(X,\tau)$ with $x \in V \subset U$. So, $x \in \text{int}(\text{cl}(V)) = s\text{cl}(V) \subset U$.

Problem 2. One does not know an example of s-regular space that fails to be semi-regular.

Theomre 11. *Let an (X,τ) be e.d. The following conditions are equivalent:*
(a) *(X,τ) is regular,*
(b) *(X,τ) is semi-regular,*
(c) *(X,τ) is s-regular.*

Proof. Since the implication (b)⇒(a) is clear by the definition of e.d. space, only (c)⇒(b) requires a proof. Let $x \in X$ and $U \in \tau$ be such that $x \in U$. By hypothesis, there exists a set $G \in \text{SO}(X,\tau)$ with $x \in G \subset s\text{cl}(G) \subset U$. Since $\text{int}(\text{cl}(S)) \subset s\text{cl}(S)$ for any $S \subset X$ [21, Lemma 4.14] and $\text{SO}(X,\tau) \subset \text{PO}(X,\tau)$ [9, Proposition 4.1], one obtains $x \in V \subset \text{int}(\text{cl}(V)) \subset U$ for $V = \text{int}(\text{cl}(G))$. Therefore (X,τ) is semi-regular.

Notice that s-regularity and e.d. are independent one of the other [11, Examples 1 and 2]. Recall also, that there is an almost regular and e.d. space which is not s-regular [11, Example 2], and a semi-regular not e.d. space which is almost regular [11, Example 1].

Theorem 12. *Let a surjection $f : (X,\tau) \to (Y,\sigma)$ be α-a.o.W., continuous, and semi-closed. If (X,τ) is s-regular and e.d., then (Y,σ) is semi-regular.*

Proof. Let $y \in Y$ and $U \in \sigma$ with $y \in U$. Let for an $x \in X$, $y = f(x)$. By s-regularity of (X,τ) we have $x \in V \subset s\text{cl}_\tau(V) \subset f^{-1}(U)$ for a certain $V \in \text{SO}(X,\tau)$. Hence $y \in f(V) \subset f(s\text{cl}_\tau(V)) \subset U$. But (X,τ) is e.d., so by [8, Theorem 2.9], $s\text{cl}_\tau(V) = \text{cl}_{\tau^\alpha}(V)$. On the other hand $\text{cl}_{\tau^\alpha}(V) = \text{cl}_\tau(V)$ [6, Lemma 1(i)]. As f is semi-closed, $\text{int}_\sigma(\text{cl}_\sigma(f(V))) \subset s\text{cl}_\sigma(f(\text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(V))))) \subset f(\text{cl}_\tau(V))$; [21, Lemma 4.14] and [8, Theorem 2.9]. Put $G = \text{int}_\sigma(\text{cl}_\sigma(f(V)))$. Thus $y \in G \subset \text{int}_\sigma(\text{cl}_\sigma(G)) \subset U$, because f is α-a.o.W. This completes the proof.

Definition 2. A mapping $f : (X,\tau) \to (Y,\sigma)$ is s-**a.o.W.** if $f(U) \in \text{PO}(Y,\sigma)$ for every $U \in \text{SO}(X,\tau)$.

Obviously, each s-a.o.W. mapping is α-a.o.W. The converse is false, even for bijections.

Example 8. Let $X = \{a,b,c,d\}$, $\sigma = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}$, $\tau = \sigma \cup \{\{b\}, \{a,b\}\}$. Then, id: $(X,\tau) \to (X,\sigma)$ is α-a.o.W. and not s-a.o.W., because $\text{id}(\{a,d\}) \notin \text{PO}(x,\sigma)$.

We notice that α-a.o.W. with openness and s-a.o.W. with openness, are couples of independent notions; see the examples below.

Example 9. (a). Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, $\sigma = \{\emptyset, X\}$. The identity from (X,τ) to (X,σ) is s-a.o.W., but it is not open.

(b). Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. The identity from (X,τ) to (X,σ) is open and not α-a.o.W., since $\mathrm{id}(\{a,c\}) \notin \mathrm{PO}(X,\sigma)$.

Theorem 13. *Let a surjection $f : (X,\tau) \to (Y,\sigma)$ be s-a.o.W., continuous, and pre-semi-closed. If (X,τ) is s-regular, then (Y,σ) is semi-regular.*

Proof. Let $y \in Y$ and $U \in \sigma$ with $y \in U$, and let $x \in X$ be such that $f(x) = y$. As (X,τ) is s-regular, there exists a set $V \in \mathrm{SO}(X,\tau)$ such that $x \in V \subset s\mathrm{cl}_\tau(V) \subset f^{-1}(U)$. So, $y \in f(V) \subset f(s\mathrm{cl}_\tau(V)) \subset U$. From pre-semi-closedness of f and from [21, Lemma 4.14] it follows that

$$\mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(\mathrm{int}_\tau(\mathrm{cl}_\tau(V))))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(s\mathrm{cl}_\tau(V)))) \subset f(s\mathrm{cl}_\tau(V)).$$

Continuity of f implies $f(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(V)))) \subset \mathrm{cl}_\sigma(f(\mathrm{int}_\tau(\mathrm{cl}_\tau(V))))$. But, f is s-a.o.W. and thus

$$f(V) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(V))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(V)))))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(s\mathrm{cl}_\tau(V)))).$$

Put $G = \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(s\mathrm{cl}_\tau(V))))$. This shows (Y,σ) is semi-regular.

Deinition 3. A function $f : (X,\tau) \to (Y,\sigma)$ is $s\alpha$-**open** if $f(U) \in \sigma^\alpha$ for every $U \in \mathrm{SO}(X,\tau)$.

Corollary 7. *Let a bijection $f : (X,\tau) \to (Y,\sigma)$ be continuous and $s\alpha$-open. If (X,τ) is s-regular, then (Y,σ) is semi-regular.*

Proof. By Theorem 13 and [21, Lemma 3.1].

Each s-open mapping is $s\alpha$-open, but the reverse implications does not hold in general, even for bijections.

Example 10. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,c\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. The identity from (X,τ) to (X,σ) is $s\alpha$-open and not s-open (consider $\{a,b\} \in \mathrm{SO}(X,\tau)$, for instance).

Acknowledgement

The author is grateful to referee for his/her helpful suggestions and comments.

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