TAMKANG JOURNAL OF MATHEMATICS
Volume 53, Number 3, 267-275, September 2022
DOI:10.5556/j.tkjm.53.2022.3729

Doubly warped product submanifolds of a Riemannian manifold of nearly quasi-constant curvature

Mehraj Ahmad Lone, Mohamd Saleem Lone and Mohammad Hasan Shahid

Abstract. In the present paper, we form a sharp inequality for a doubly warped product submanifold of a Riemannian manifold of nearly quasi-constant curvature.

Keywords. Chen inequality, doubly warped product manifold, nearly quasi-constant curvature tensor

1 Introduction

In [7] B.-Y Chen and K. Yano introduced the notion of quasi-constant curvature. A Riemannian manifold $(\mathcal{M}, g)$ is called a Riemannian manifold of quasi-constant curvature if its curvature tensor R satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +b[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ &\quad +g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)], \end{aligned}$$

where a, b are scalar functions and A is a 1-form given by

$$g(X,P) = A(X),$$

P is a fixed unit vector field. It is straightforward to see that if $b = 0$, then $(\mathcal{M}, g)$ reduces to a Riemannian manifold of constant curvature.

For $n > 2$, a non-flat Riemannian manifold $(\mathcal{M}^n, g)$ is said to be a quasi-Einstein manifold if its Ricci tensor satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where a, b are scalar functions and A is 1-form acting same as above. It can be easily verified that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

In 2009, the notion of quasi-constant curvature was generalized to nearly quasi-constant by A. K. Gazi and U. C. De (see [8]). It is a Riemannian manifold whose curvature tensor satisfies

$$R(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

Received date: May 28, 2020; Published online: April 13, 2021.
2010 *Mathematics Subject Classification*. 53C05, 53C40.
Corresponding author: Mehraj Ahmad Lone.

$$+q[g(X,W)B(Y,Z) - g(X,Z)B(Y,W) + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)], \tag{1.1}$$

where a, b are scalar functions and B is a non-vanishing $(0,2)$ type symmetric tensor.

For $n > 2$, a non-flat Riemannian manifold $(\mathcal{M}^n, g)$ is said to be a nearly quasi-Einstein manifold if its Ricci tensor satisfy

$$S(X,Y) = ag(X,Y) + bB(X,Y).$$

It can be easily verified that every Riemannian manifold of nearly quasi-constant curvature is a nearly quasi-Einstein manifold.

We know that the outer product of two convariant vectors is a covariant $(0,2)$ tensor, but not conversely true always. Hence nearly quasi-constant Riemannian manifolds act as a bigger class of Riemannian manifolds in the sense that every Riemannian manifold of quasi-constant curvature is nearly quasi-constant Riemannian manifold, but there are plenty of examples where the converse is not true.

Example 1. ([8]) Let $(\mathbb{R}^4, g)$ be a Riemannian manifold with the metric g defined as follows

$$ds^2 = (x^4)^{\frac{4}{5}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

This is a nearly quasi-constant Riemannian manifold but not a quasi-constant Riemannian manifold.

In an attempt to construct Riemannian manifolds with negative sectional curvature, O' Neill and Bishop introduced the notion of singly warped products (see [3]). The warped product manifold model has plenty of applications in relativity. In [1], Beem, Ehrlich and Powell showed the exact solutions to Einstein's field equation are expressible in terms of Lorentzian warped products. For more applications, see [2, 6].

Definition 1. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian manifolds, the warped product $\mathcal{M} = \mathcal{M}_1 \times_\alpha \mathcal{M}_2$ is the product manifold equipped with the metric

$$g = \pi_1^*(g_1) + (\alpha \circ \pi_1)^2 \pi_2^*(g_2),$$

where $\alpha : \mathcal{M}_1 \to (0, \infty)$ is a smooth function on $\mathcal{M}_1$, $\pi_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$, $i = 1, 2$ are the projections and $*$ is the pullback. The Riemannian manifold $(\mathcal{M}_1, g_1)$ is called as the base, $(\mathcal{M}_2, g_2)$ is called as fibre and α is called as the warping function of the warped product.

A very prominent example of warped product manifold is a generalized Robertson-Walker space-time, which is a Lorentzian warped product of the form $\mathcal{M} = (a, b) \times_\alpha \mathcal{N}$, where (a, b) is an open interval, $\mathcal{N}$ is a three dimensional space form and the metric on $\mathcal{M}$ is given by $g = -dt^2 + \alpha^2 g_{\mathcal{N}}$ (cf. [2, 9]).

The notion of doubly warped manifold can be considered as a natural generalization of singly warped product manifold.

Definition 2. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian manifolds, the doubly warped product $\mathcal{M} = {}_{\alpha_2}\mathcal{M}_1 \times_{\alpha_1} \mathcal{M}_2$ is the product manifold equipped with the metric

$$g = (\alpha_2 \circ \pi_2)^2 \pi_1^*(g_1) + (\alpha_1 \circ \pi_1)^2 \pi_2^*(g_2),$$

where $\alpha_i : \mathcal{M}_i \to (0, \infty)$ is a smooth function on $\mathcal{M}_i$, $\pi_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$, $i = 1, 2$ are the usual projections and $*$ is the pullback. If one of $\alpha_i = 1$, but not both, then we get warped product manifold. If both $\alpha_i = 1$, we get a Riemannian product manifold. If neither of α_i is constant, we get a non-trivial doubly product manifold (see [12]).

The structure of the paper is as following. In section 2, we compile the basic definitions and all the prerequisites needed afterwards. In section 3, we prove our main result.

2 Preliminaries

Let $\mathcal{M}$ be an n-dimensional Riemannian submanifold of a Riemannian m-dimensional manifold $\mathcal{N}$ and let ∇ and $\hat{\nabla}$ be the Levi-Civita connection of $\mathcal{M}$, $\mathcal{N}$, respectively. Then the Gauss and Weingarten formula are given respectively by

$$\begin{aligned} \hat{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \hat{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi \end{aligned}$$

for all $X, Y \in \Gamma(T\mathcal{M})$ and $\xi \in \Gamma(T^\perp \mathcal{M})$, where $\nabla^\perp$ is the normal connection, A is the shape operator and h is the second fundamental form and are related by the relation

$$g(h(X, Y), \xi) = g(A_\xi X, Y).$$

The Gauss equation is given by

$$\hat{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z) + g(h(X, Z), h(Y, W)) \tag{2.1}$$

for all $X, Y, Z, W \in \Gamma(T\mathcal{M})$, where $\hat{R}$ is the curvature tensor of $\mathcal{N}$ and R is the induced curvature tensor on $\mathcal{M}$.

Let $\{e_1, e_2, \cdots, e_n\}$, $\{e_{n+1}, \cdots, e_m\}$ be orthonormal basis of the tangent space $T_p(\mathcal{M})$ and $T_p^\perp(\mathcal{M})$, respectively. Then the mean curvature field is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n} \sum_{r=n+1}^{m} \left(\sum_{i=1}^{n} h^r_{ii} \right) e_r, \;\; h^r_{ij} = g(h(e_i, e_j), e_r),$$

$1 \leq i, j \leq n$, $n + 1 \leq r \leq m$.

Suppose α is a differentiable function on $\mathcal{M}$, then the Laplacian $\Delta\alpha$ is defined as

$$\Delta\alpha = \sum_{i=1}^{n} \left[(\nabla_{e_i} e_i)\alpha - e_i e_i \alpha \right].$$

Let $\pi \subset T_p\mathcal{M}$ be a 2-plane section and $K(\pi)$ be the sectional curvature of $\mathcal{M}$. Then for an orthonomal basis $\{e_1, \cdots, e_n\}$ of the tangent space, the scalar curvature is defined as

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

Now for a doubly warped product manifold, assuming $\mathcal{D}_1, \mathcal{D}_2$ the distributions obtained from the vectors tangent to leaves and fibres, respectively. Let $s : {}_{\alpha_2}\mathcal{M}_1 \times {}_{\alpha_1}\mathcal{M}_2 \to \mathcal{N}$ be an isometric immersion, then we have

$$H_i = \frac{1}{n_i} tr(h_i),$$

the partial mean curvature, where $tr(h_i)$ is the trace of h restricted to $\mathcal{M}_i$ and n_i =dim$\mathcal{M}_i$. The doubly warped product manifold is called as mixed totally geodesic if $h(X,Y)=0$ for any X,Y tangent to $\mathcal{D}_1,\mathcal{D}_2$, respectively.

For a warped product submanifold of a Riemannian manifold of constant sectional curvature, B.-Y Chen proved the following:

Theorem 2.1. *[5] Let $\mathcal{M}=\mathcal{M}_1\times_\alpha\mathcal{M}_2$ be an n-dimensional warped product submanifold of a Riemannian manifold $\mathcal{N}(c)$. Then, we have*

$$\frac{\Delta\alpha}{\alpha}\leq\frac{n^2}{4n_2}\|H\|^2+n_1c,$$

where $n_i=dim\mathcal{M}_i$, $n=n_1+n_2$. The equality holds if and only if $\mathcal{M}$ is a mixed totally geodesic and $n_1H_1=n_2H_2$, H_i is the partial mean curvature vectors, $i=1,2$.

Later in [10], A. Olteanu obtained a sharp inequality for a doubly warped product submanifold of an arbitrary Riemannian manifold. In [12], using the quasi-constant curvature tensor, S. Sular obtained a sharp inequality for a doubly warped product submanifold of a Riemannian manifold. Motivated by the above studies, we discuss a sharp inequality for a doubly warped product submanifold of a Riemannian manifold of nearly quasi-constant curvature.

We shall use the following Chen's lemma while proving our main result.

Lemma 2.1. *[4] For $m\geq 2$ and $b_1,b_2,\cdots,b_m,\mu$ be reals, such that*

$$\left(\sum_{j=1}^{m}b_i\right)^2=(m-1)\left(\sum_{j=1}^{m}b_i^2+\mu\right). \tag{2.2}$$

Then $2b_1b_2\geq\mu$, with equality if and only if $b_1+b_2=b_3=\cdots=b_m$.

3 Doubly warped product submanifolds

In this section, we derive a sharp relationship between a doubly warped product submanifold $\mathcal{M}={}_{\alpha_2}\mathcal{M}_1\times_{\alpha_1}\mathcal{M}_2$, its warping functions and the squared mean curvature.

Theorem 3.1. *Let $\mathcal{M}={}_{\alpha_2}\mathcal{M}_1\times_{\alpha_1}\mathcal{M}_2$ be an n-dimensional doubly warped product submanifold of an m-dimensional Riemannian manifold $\mathcal{N}$. Then we have*

$$n_2\frac{\Delta_1\alpha_1}{\alpha_1}+n_1\frac{\Delta_2\alpha_2}{\alpha_2}\leq\frac{n^2}{4}\|H\|^2+n_1n_2p-q(n-1)tr(B), \tag{3.1}$$

where $n_1+n_2=n$, $n_i=dim\mathcal{M}_i$ and Δ_i is the Laplacian of $\mathcal{M}_i$. The equality in (3.13) holds if and only if $\mathcal{M}$ is totally geodesic with $tr(h_1)=tr(h_2)$.

Proof. Let $\mathcal{M}$ be a doubly warped product submanifold of a Riemannian manifold $\mathcal{N}$ of nearly quasi-constant curvature. Then, we have

$$\nabla_XY=\nabla_X^{\mathcal{M}_1}-\frac{\alpha_2^2}{\alpha_1^2}g_1(X,Y)\nabla^{\mathcal{M}_2}(\ln\alpha_2)$$
$$\nabla_XZ=Z(\ln\alpha_2)X+X(\ln\alpha_1)Z$$

for any $X, Y \in \Gamma(T\mathcal{M}_1)$, $Z \in \Gamma(T\mathcal{M}_2)$, where $\nabla^{\mathcal{M}_i}$ is the Levi-Civita connection of the Riemannian metric g_i, $i = 1, 2$.

The sectional curvature of the $\{X, Y\}$ plane is given by

$$K(X \wedge Y) = \frac{1}{\alpha_1}[(\nabla_X^{\mathcal{M}_1} X)\alpha_1 - X^2\alpha_1] + \frac{1}{\alpha_2}[(\nabla_Y^{\mathcal{M}_1} Y)\alpha_2 - Y^2\alpha_2].$$

Fix an orthonormal basis $\{e_1 \cdots, e_{n_1} e_{n_1+1}, \cdots, e_n\}$, such that first n_1 tuples acts as basis of $T_p\mathcal{M}_1$ and the remaining of $T_p\mathcal{M}_2$ and $e_{n_1+1} \parallel H$, we get

$$n_2\frac{\Delta_1\alpha_1}{\alpha_1} + n_1\frac{\Delta_2\alpha_2}{\alpha_2} = \sum_{1\leq s_1\leq n_1<s_2\leq n} K(e_{s_1} \wedge e_{s_2}) \tag{3.2}$$

for each $s_2 \in \{n_1 + 1, \cdots, n\}$.

Using Gauss equation for $X = W = e_i$ and $Y = Z = e_j$, $i \neq j$, we have

$$2\tau = n^2\|H\|^2 - \|h\|^2 + 2q(n-1)tr(B) + (n^2 - n)p, \tag{3.3}$$

where

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), (h(e_i, e_j))$$

is the squared norm of the second fundamental form h and τ is the scalar curvture.

We fix

$$\epsilon = 2\tau - \frac{n^2}{2}\|H\|^2 - (n^2 - n)p - 2q(n-1)tr(B). \tag{3.4}$$

Then, from (3.3) and (3.4), we get

$$n^2\|H\|^2 = 2(\|h\|^2 + \epsilon). \tag{3.5}$$

For a suitable local orthonormal frame, the above relation can be written as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left[\epsilon + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right].$$

Put $b_1 = h_{11}^{n+1}$, $b_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $b_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the previous equation is equivalent to

$$\begin{aligned}\left(\sum_{i=1}^{3} b_i\right)^2 &= 2\Bigg[\epsilon + \sum_{i=1}^{3} b_i^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 \\ &\quad - \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n+1\leq s\neq t\leq n} h_{ss}^{n+1}h_{tt}^{n+1}\Bigg].\end{aligned} \tag{3.6}$$

For b_1, b_2, b_3, we see that, (3.6) satisfy lemma 2.1, implying

$$\left(\sum_{i=1}^{3} b_i\right)^2 = 2\left(\sum_{i=1}^{3} b_i^2 + \mu\right),$$

where

$$\mu = \epsilon + \sum_{1\le i\ne j\le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}.$$

Then, we obtain $2b_1b_2 \ge \mu$, with equality if and only if $b_1 + b_2 = b_3$, or

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\epsilon}{2} + \sum_{1\le\alpha<\beta\le n}(h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^{r})^2. \tag{3.7}$$

The equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.8}$$

Using Gauss equation again, we have

$$\begin{aligned} n_2\frac{\Delta_1\alpha_1}{\alpha_1} + n_1\frac{\Delta_2\alpha_2}{\alpha_2} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t) \\ &= \tau - \frac{1}{2}n_1p(n_1-1) - \sum_{r=n+1}^{m}\sum_{1\le j<k\le n_1}(h_{jj}^{r}h_{kk}^{r} - (h_{jk}^{r})^2) \\ &\quad -q(n_1-1)tr(B) - \frac{1}{2}n_2p(n_2-1) \\ &\quad - \sum_{r=n+1}^{m}\sum_{n_1+1\le s<t\le n_1}(h_{ss}^{r}h_{tt}^{r} - (h_{st}^{r})^2) - q(n_2-1)tr(B). \end{aligned} \tag{3.9}$$

Using (3.2), (3.7) and (3.9), we get

$$\begin{aligned} & n_2\frac{\Delta_1\alpha_1}{\alpha_1} + n_1\frac{\Delta_2\alpha_2}{\alpha_2} \le \tau - \frac{1}{2}np(n-1) + n_1n_2p - \frac{\epsilon}{2} - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta}^{n}(h_{\alpha\beta}^{r})^2 \\ & + \sum_{r=n+2}^{m}\sum_{1\le j<k\le n_1}((h_{jk}^{r})^2 - h_{jj}^{r}h_{kk}^{r}) + \sum_{r=n+2}^{m}\sum_{n_1+1\le s<t\le n_1}((h_{st}^{r})^2 - h_{ss}^{r}h_{tt}^{r}) \\ & -q(n_1-1)tr(B) - q(n_2-1)tr(B). \\ & = \tau - \frac{1}{2}np(n-1) + n_1n_2p - \frac{\epsilon}{2} - \sum_{r=n+1}^{m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(h_{jt}^{r})^2 \\ & -\frac{1}{2}\sum_{r=n+2}^{m}\left(\sum_{j=1}^{n_1}h_{jj}^{r}\right)^2 - \frac{1}{2}\sum_{r=n+2}^{m}\left(\sum_{t=n_1+1}^{n}h_{tt}^{r}\right)^2 \\ & -q(n_1-1)tr(B) - q(n_2-1)tr(B) \\ & \le \tau - \frac{1}{2}np(n-1) + n_1n_2p - \frac{\epsilon}{2} - q(n_1-1)tr(B) - q(n_2-1)tr(B) \end{aligned}$$

$$= \frac{n^2}{4}\|H\|^2 + n_1 n_2 p - q(n-1)tr(B).$$

This proves our claim.

It is straightforward to check that the equality holds in (3.13), if and only if

$$h^r_{jl} = 0, \ n+1 \le r \le m \tag{3.10}$$

and

$$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{l=n_1+1}^{n} h^r_{ll} = 0, \tag{3.11}$$

where $1 \le j \le n_1$, $n_1 + 1 \le l \le n$ and $n + 2 \le r \le m$.
From (3.10) vanishing of the second fundamental form of ${}_{\alpha_2}\mathcal{M}_1 \times_{\alpha_1} \mathcal{M}_2$ in $\mathcal{N}$ implies $h(\mathcal{D}_1, \mathcal{D}_2) = \{0\}$, or we can say that the immersion s is totally geodesic. Again from (3.8) and (3.11), we see that

$$\sum_{s_1=1}^{n_1} h(e_{s_1}, e_{s_2}) = \sum_{s_2=n_1+1}^{n} h(e_{s_2}, s_{s_2}),$$

implying $tr(h_1) = tr(h_2)$.

Conversely assuming $\mathcal{N}$ is the required Riemannian manifold, such that $\mathcal{M}$ is totally geodesic with $tr(h_1) = tr(h_2)$, then the equality in (3.13) follows easily. □

Corollary 3.2. *Let $\mathcal{M} = {}_{\alpha_2}\mathcal{M}_1 \times_{\alpha_1} \mathcal{M}_2$ be a compact, orientable n-dimensional doubly warped product submanifold of an m-dimensional Riemannian manifold $\mathcal{N}$. Then we have*

$$\|H\|^2 \ge \frac{4}{n^2}[q(n-1)tr(B) - n_1 n_2 p]. \tag{3.12}$$

Proof. Suppose $\mathcal{M}$ be a compact orientable Riemannian manifold without boundary satisfying (3.13), then we have

$$n_2 \frac{\Delta_1 \alpha_1}{\alpha_1} + n_1 \frac{\Delta_2 \alpha_2}{\alpha_2} \le \frac{n^2}{4}\|H\|^2 + n_1 n_2 p - q(n-1)tr(B).$$

Therefore, from the definition of volume element

$$\int_{\mathcal{M}} \Delta_i \alpha_i dV = 0, \quad i = 1, 2.$$

Thus, we get

$$0 \le \int_{\mathcal{M}} \left(\frac{n^2}{4}\|H\|^2 + n_1 n_2 p - q(n-1)tr(B) \right) dV = 0.$$

□

Corollary 3.3. *Let $\mathcal{M} = {}_{\alpha_2}\mathcal{M}_1 \times_{\alpha_1} \mathcal{M}_2$ be an n-dimensional doubly warped product submanifold of an m-dimensional Riemannian manifold $\mathcal{N}$ satisfying*

$$n_2 \frac{\Delta_1 \alpha_1}{\alpha_1} + n_1 \frac{\Delta_2 \alpha_2}{\alpha_2} > n_1 n_2 p - q(n-1)tr(B), \tag{3.13}$$

then $\mathcal{M}$ is non-minimal.

Acknowledgments

We would like to thank the referees for careful reading of the paper and pointing out some errors which improved the paper.

References

[1] J. Beem, P. Ehrlich and T. G. Powell, Warped product manifolds in relativity, Selected Studies: A volume dedicated to the memory of Albert Einstein (1982), 4156.

[2] J. K. Beem, Global Lorentzian geometry, Routledge, 2017.

[3] R. L. Bishop and B. ONeill, Manifolds of negative curvature, Transactions of the American Mathematical Society **145** (1969), 149.

[4] B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, Archiv der Mathematik **60** (1993), 568578.

[5] B. Y. Chen, On isometric minimal immersions from warped products into real space forms, Proceedings of the Edinburgh Mathematical Society **45** (2002), 579587.

[6] B. Y. Chen, Differential geometry of warped product manifolds and submanifolds, World Scientific Singapore, 2017.

[7] B. Y. Chen and K. Yano, Hypersurfaces of a conformally flat space, Tensor(NS) **26** (1972), 318322.

[8] A. K. Gazi and U. C. De On the existence of nearly quasi-Einstein manifolds, Novi Sad J. Math **39** (2009), 111117.

[9] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, vol. 1. Cambridge university press, 1973.

[10] A. Olteanu, A general inequality for doubly warped product submanifolds, Mathematical Journal of Okayama University **52** (2010), 133-142.

[11] S. Sular, Doubly warped product submanifolds of a Riemannian manifold of quasi-constant curvature, Annals of the Alexandru Ioan Cuza University- Mathematics **61** (2015), 235244.

[12] B. Unal, Doubly warped products [ph. d. thesis], University of Missouri Columbia (2000).

Mehraj Ahmad Lone National Institute of Technology Srinagar, J&K, India

E-mail: mehrajlone@nitsri.ac.in, mehraj.jmi@gmail.com

Mohammad Saleem Lone International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, 560089, Bengaluru, India.

E-mail: saleemraja2008@gmail.com

Mohammad Hasan Shahid Department of Mathematics, Jamia Millia Islamia , 110025, New Delhi, India.

E-mail: hasan_jmi@yahoo.com