TAMKANG JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 2000

ON THE FEKETE-SZEGÖ PROBLEM FOR ALPHA-QUASI-CONVEX FUNCTIONS

H. R. ABDEL-GAWAD

Abstract. Let $Q_\alpha(\alpha \geq 0)$ denote the class of normalized analytic alpha-quasi-convex functions f, defined in the unit disc, $D = \{z : |z| < 1\}$, by the condition

$$\mathrm{Re}\left[(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] > 0,$$

Where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is a convex univalent function in D. Sharp upper bounds are obtained for $|a_3 - \mu a_2^2|$, when $\mu \geq 0$.

Introduction

Denote by S the class of functions f which are analytic and univalent in $D = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Thus for $f \in S$ we may write

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Let C and K be those subsets of S, which are convex and close-to-convex respectively. Then $f \in K$ if, and only if, There exists $g \in C$ such that for $z \in D$, $\mathrm{Re}\frac{f'(z)}{g'(z)} > 0$. In [3] Noor considered a new subclass C^* of univalent function that is f given by (1), belongs to C^* if, and only if, f is analytic in D and is such that there exists $g \in C$ satisfying

$$\mathrm{Re}\frac{(zf'(z))'}{g'(z)} > 0 \quad \text{For} \quad z \in D.$$

The function in C^* are called quasi-convex and $C \subset C^* \subset K \subset S$. It is shown [3] that $f \in C^*$ if, and only if, $zf' \in K$. Recently the function f called α-quasi-convex function has been defined and its properties Studied in [4].

A function f, analytic in D, is said to be α-quasi-convex if, and only if, there exists function $g \in C$ such that, for α real and positive

$$\mathrm{Re}\left[(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] > 0. \tag{2}$$

Received July 16, 1999.
2000 *Mathematics Subject Classification*. Primary 30C45.
Key words and phrases. Univalent function, close to convex, quasi-convex, alpha quasi-convex functions.

The class of such functions will be denoted by Q_α. A classical theorem of Fekete and Szegö [1] states that for $f \in S$ given by (1)

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu & \text{if} \quad \mu \le 0. \\ 1 + 2e^{-2\mu(1-\mu)} & \text{if} \quad 0 \le \mu \le 1. \\ 4\mu - 3 & \text{if} \quad \mu \ge 1. \end{cases}$$

This inequality is sharp in the sense that for each μ there exists a function in S such that equality holds. The above inequalities can be improved [2]. In particular for $f \in K$ and given by (1), Keogh and Merkes [2] showed that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu & \text{if} \quad \mu \le \frac{1}{3}. \\ \frac{1}{3} + \frac{4}{9\mu} & \text{if} \quad \frac{1}{3} \le \mu \le \frac{2}{3}. \\ 1 & \text{if} \quad \frac{2}{3} \le \mu \le 1. \\ 4\mu - 3 & \text{if} \quad \mu \ge 1. \end{cases} \tag{3}$$

Again, for each μ, there is a function in K such that equality holds. In this paper we solve the Fekete-Szego problem for the class Q_α of α-quasi-convex functions.

Results

Theorem. Let $f \in Q_\alpha$ and be given by (1), then for $0 \le \alpha \le 1$

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{3}{1+2\alpha} - \frac{4\mu}{(1+\alpha)^2} & \text{if} \quad \mu \le \frac{(1+\alpha)^2}{3(1+2\alpha)}. \\ \frac{1}{3(1+2\alpha)} + \frac{4(1+\alpha)^2}{9(1+2\alpha)^2\mu} & \text{if} \quad \frac{(1+\alpha)^2}{3(1+2\alpha)} \le \mu \le \frac{2(1+\alpha)^2}{3(1+2\alpha)}. \\ \frac{1}{1+2\alpha} & \text{if} \quad \frac{2(1+\alpha)^2}{3(1+2\alpha)} \le \mu \le \frac{(1+\alpha)^2}{(1+2\alpha)}. \\ \frac{4\mu}{(1+\alpha)^2} - \frac{3}{(1+2\alpha)} & \text{if} \quad \mu \ge \frac{(1+\alpha)^2}{(1+2\alpha)}. \end{cases}$$

This inequality is sharp in the sense that for each α in $[0, 1]$ and each μ in the appropriate range of each part of the theorem, there is an f in Q_α such that the "$\le$" symbol can be replaced by the "$=$" symbol in the conclusion of that part of the theorem.

Special cases.

1) When $\alpha = 0$. Then $f \in K$ and we have a result given in [2].
2) Let $a = 1$. Then $f \in Q$, quasi-convex function given by (1). Then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 1 - \mu & \text{if} \quad \mu \le \frac{4}{9}. \\ \frac{1}{9} + \frac{16}{81\mu} & \text{if} \quad \frac{4}{9} \le \mu \le \frac{8}{9}. \\ \frac{1}{3} & \text{if} \quad \frac{8}{9} \le \mu \le \frac{4}{3}. \\ \mu - 1 & \text{if} \quad \mu \ge \frac{4}{3}. \end{cases}$$

Again for each μ, there is a function in Q such that equality holds.

We shall require the following

Lemma 1 ([5], p.166). *Let $h \in p$, i.e., let h be analytic in D and satisfy $Re\, h(z) > 0$ for $z \in D$, with $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ Then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2 ([2]). *Let $g \in C$, with $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, then for μ real,*

$$|b_3 - \mu b_2^2| \leq \max(1/3, |1-\mu|).$$

Proof of Theorem. It follows from (2) that we can write

$$(1-\alpha) f'(z) + \alpha (z f'(z))' = g'(z) h(z). \tag{4}$$

For $g \in C$ and $h \in p$. Equating coefficients in (4) we obtain

$$2(1+\alpha) a_2 = c_1 + 2b_2.$$

and

$$3(1+2\alpha) a_3 = c_2 + 2c_1 b_2 + 3b_3.$$

so that

$$\begin{aligned} a_3 - \mu a_2^2 = {} & \frac{1}{(1+2\alpha)} \left[b_3 - \frac{(1+2\alpha)\mu}{(1+\alpha)^2} b_2^2 \right] + \frac{1}{3(1+2\alpha)} \left[c_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2} c_1^2 \right] \\ & + \left[\frac{2}{3(1+2\alpha)} - \frac{\mu}{(1+\alpha)^2} \right] c_1 b_2. \end{aligned} \tag{5}$$

We consider first the case $\frac{(1+\alpha)^2}{3(1+1\alpha)} \leq \mu \leq \frac{2(1+\alpha)^2}{3(1+2\alpha)}$.

Equation (5) gives

$$\begin{aligned} \left| a_3 - \mu a_2^2 \right| \leq {} & \frac{1}{(1+2\alpha)} \left| b_3 - \frac{(1+2\alpha)\mu}{(1+\alpha)^2} b_2^2 \right| + \frac{1}{3(1+2\alpha)} \left| c_2 - \frac{3(1+2\alpha)u}{4(1+\alpha)^2} c_1^2 \right| \\ & + \left[\frac{2}{3(1+2\alpha)} - \frac{\mu}{(1+\alpha)^2} \right] |c_1||b_2|' \\ \leq {} & \frac{1}{(1+2\alpha)} \left[1 - \frac{(1+2\alpha)}{(1+\alpha)^2} \mu \right] + \frac{1}{3(1+2\alpha)} \left| c_2 - \frac{1}{2} c_1^2 \right| \\ & + \frac{1}{3(1+2\alpha)} \left| \frac{1}{2} - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2} \right| |c_1|^2 + \left[\frac{2}{3(1+2\alpha)} - \frac{\mu}{(1+\alpha)^2} \right] |c_1|' \\ \leq {} & \frac{1}{(1+2\alpha)} \left[1 - \frac{(1+2\alpha)}{(1+\alpha)^2} \mu \right] + \frac{1}{3(1+2\alpha)} \left[2 - \frac{|c_1|^2}{2} \right] \\ & + \frac{1}{3(1+2\alpha)} \left| \frac{1}{2} - \frac{3(1+2\alpha)}{4(1+\alpha)^2} \mu \right| |c_1|^2 + \left[\frac{2}{3(1+2\alpha)} - \frac{\mu}{(1+\alpha)^2} \right] |c_1|' \\ = {} & \Phi(t), \text{ say, with } t = |c_1|. \end{aligned}$$

Where we have used lemmas 1 and 2 and the fact that $|b_2| \leq 1$. Since the function Φ attains its maximum at $t_0 = \frac{2[2(1+\alpha)^2-3(1+2\alpha)\mu]}{3(1+2\alpha)\mu}$, it follows that the second inequality in the theorem is established, if $\mu \leq \frac{2(1+\alpha)^2}{3(1+2\alpha)}$. Choosing $b_2 = b_3 = 1$, $c_2 = 1$, $c_1 = \frac{2[2(1+\alpha)^2-3(1+2\alpha)\mu]}{3(1+2\alpha)\mu}$ in (5), shows that the result is sharp if $\mu \geq \frac{(1+\alpha)^2}{3(1+2\alpha)}$, since $|c_1| \leq 2$. Next, let us that $\mu \leq \frac{(1+\alpha)^2}{3(1+2\alpha)}$. Then

$$\begin{aligned}|a_3 - \mu a_2^2| &\leq \frac{3(1+2\alpha)}{(1+\alpha)^2}\mu\left|a_3 - \frac{(1+\alpha)^2}{3(1+2\alpha)}a_2^2\right| + \left[1 - \frac{3(1+2\alpha)}{(1+\alpha)^2}\right]|a_3|' \\ &\leq \frac{3}{(1+2\alpha)} - \frac{4\mu}{(1+\alpha)^2}.\end{aligned}$$

Where we have used the result already proved in the case $\mu = \frac{(1+\alpha)^2}{3(1+2\alpha)}$, and the fact that for $f \in Q_\alpha$, the inequality $|a_3| \leq \frac{3}{1+2\alpha}$ holds [4]. Equality is attained by the function f_0, which is defined by

$$f_0(z) = \frac{z}{\alpha z^{\frac{1}{\alpha}}}\int_0^z \frac{t^{\frac{1}{\alpha}}}{t(1-t)^2}dt = z + \frac{2}{1+\alpha}z^2 + \frac{3}{1+2\alpha}z^3 + \cdots,$$

So that $a_3 - \mu a_2^2 = \frac{3}{1+2\alpha} - \frac{4\mu}{(1+\alpha)^2}$ (see e.g. [4]). Suppose next that $\frac{2(1+\alpha)^2}{3(1+2\alpha)} \leq \mu \leq \frac{(1+\alpha)^2}{(1+2\alpha)}$.

We deal first with the case $\mu = \frac{(1+\alpha)^2}{(1+2\alpha)}$. Thus (5) gives

$$a_3 - \frac{(1+\alpha)^2}{(1+2\alpha)}a_2^2 = \frac{1}{(1+2\alpha)}(b_3 - b_2^2) + \frac{1}{3(1+2\alpha)}(c_2 - \frac{3}{4}c_1^2) - \frac{c_1 b_2}{3(1+2\alpha)},$$

and so,

$$\begin{aligned}\left|a_3 - \frac{(1+\alpha)^2}{(1+2\alpha)}a_2^2\right| &\leq \frac{1}{(1+2\alpha)}\left|b_3 - b_2^2\right| + \frac{1}{3(1+2\alpha)}\left|c_2 - \frac{1}{2}c_1^2\right| + \frac{|c_1|^2}{12(1+2\alpha)} + \frac{|c_1||b_2|}{3(1+2\alpha)}, \\ &\leq \frac{1}{3(1+2\alpha)}(1-|b_2|^2) + \frac{1}{3(1+2\alpha)}(2 - \frac{|c_1|^2}{2}) + \frac{|c_1|^2}{12(1+2\alpha)} + \frac{|c_1||b_2|}{3(1+2\alpha)}'. \\ &= \frac{1}{1+2\alpha} - \frac{1}{3(1+2\alpha)}\left[|b_2| - \frac{|c_1|}{2}\right]^2 \leq \frac{1}{1+2\alpha}.\end{aligned}$$

Where we have used lemma 1, the fact $|b_2| \leq 1$ and the inequality $|b_3 - b_2^2| \leq 1/3[1-|b_2|^2]$. See e.g. [6]. Now write,

$$\begin{aligned}a_3 - \mu a_2^2 &= \frac{(1+2\alpha)}{(1+\alpha)^2}\left[3\mu - \frac{2(1+\alpha)^2}{(1+2\alpha)}\right]\left[a_3 - \frac{(1+\alpha)^2}{(1+2\alpha)}a_2^2\right] \\ &\quad + \frac{3(1+2a)}{(1+\alpha)^2}\left[\frac{(1+\alpha)^2}{(1+2\alpha)} - \mu\right]\left[a_3 - \frac{2(1+\alpha)^2}{3(1+2\alpha)}a_2^2\right].\end{aligned}$$

and the result follows at once on using the theorem already proved for $\mu = \frac{(1+\alpha)^2}{(1+2\alpha)}$ and $\mu = \frac{2(1+\alpha)^2}{3(1+2\alpha)}$. Equality is attained when f_1 is given by

$$f_1(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1}(1-t^2)^{-1}dt = z + \frac{1}{1+2\alpha}z^3 + \frac{1}{1+4\alpha}z^5 + \cdots,$$

So that $a_3 - \mu a_2^2 = \frac{1}{1+2\alpha}$. We finally assume that $\mu \geq \frac{(1+\alpha)^2}{(1+2\alpha)}$. Write

$$a_3 - \mu a_2^2 = \left[a_3 - \frac{(1+\alpha)^2}{(1+2\alpha)}a_2^2\right] + \left[\frac{(1+\alpha)^2}{(1+2\alpha)} - \mu\right] a_2^2.$$

and the result follows at once on using again the theorem already proved for $\mu = \frac{(1+\alpha)^2}{(1+2\alpha)}$ and the inequality $|a_2| \leq \frac{2}{1+\alpha}$, which was proved in [4]. Equality is attained by the function f_0, which is defined above.

References

[1] M. Fekete and G. Szego, *Eine Bermerkung Uber ungerade schlichte Funktionen*, J. London Math. Soc., **8**(1933), 85-89.

[2] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Pro. Amer. Math. Soc., **20**(1969), 8-12.

[3] K. I. Noor and D. K. Thomas, *On quasi-convex univalent functions*, Inter. J. Math. and Math. Sci, **3**(1980), 255-266.

[4] K. I. Noor and F. M. Alobudi, *Alpha-quasi-convex functions*, Caribb J. Math., **3**(1984), 1-8.

[5] Ch. Pommerenke, *Univalent functions*, Van denhoeck and Ruprecht, Göttingen, 1975.

[6] S. Y. Trimble, *A coefficient inequality for convex univalent functios*, Proc. Amer. Math. Soc., **48**(1975), 266-267.

Mathematics Department, Faculty of Science, Aswan-Egypt.